Math 1B Discussion Section Problems

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You should work on the following problems in groups of 3 or 4. Try to get through as many as you can, but you aren't expected to finish everything. Instead, you should make sure everyone in your group knows **how** to solve all the problems, and not just the answers.

Modeling with Differential Equations

1. (If you didn't get to it last time) A certain curve in the plane has the property that **every** normal line (that is, a line perpendicular to the tangent line) to the curve passes through (2,0). Find the equation for this curve if you know it passes through (1,1). Hint: recall that two lines are perpendicular if the product of their slopes is -1.

2. Biologists stocked a lake with 400 fish of some species and estimate the species' carrying capacity in the lake to be 10,000.

   (a) Assuming the fish population satisfies the logistic equation, find an expression for the number of fish after t years.

   (b) Now suppose that we also know that after 1 year, there were 1200 fish in the lake. Use this information to find the value of the growth constant k.

3. The air in a room with volume 180 m^3 contains .15% CO_2. After opening a window, fresh air with only 0.05% CO_2 enters the room at a rate of $2m^3/min$ and mixed air flows out of the room at the same rate. Find the percentage of CO_2 in the air as a function of time.

4. Let's try to figure out what would happen if we were to drop a baseball off a really tall building.

   Some physics: From Newton's Second Law, we know that the $F = ma = mv'$, where F is the total force acting on the baseball, m is its mass, and $v = v(t)$ is its velocity. We also know that the total foce acting on the baseball is $F = F_{gravity} - F_{resistance} = mg - mbv^2$, where $g = 9.8$ and b is the "friction coefficient," which is approximately $.005m$ for a typical baseball.

   (a) Explain why the above information suggests that $v' = g - bv^2$ would be a good differential equation to model the fall of a baseball.

   (b) Find the equilibrium solutions to this equation and sketch a few solution curves. Are your equilibria stable or unstable?

   (c) Solve your differential equation to find a formula for the velocity of the ball at time t if we also know that $v(0) = 0$.

   (d) What is $\lim_{t\to\infty} v(t)$? Is this consistent with what you found in part (b)?

5. Suppose you just bought a room-temperature can of Coke and want to cool it as quickly as possible. In particular, you have two options: you could put it in the freezer, or you could put it in a bucket of icewater. On the one hand, the freezer is colder than the ice water, but on the other, ice water conducts heat much better than freezer air. This problem will investigate which method is more efficient.

   Some background physics: A reasonable differential equation for the temperature of the can is $mcT' = -hA(T - T_\infty)$, where m is the mass of the can, c is its heat capacity, h is the conduction coefficient, A is the surface area of the can, and T_∞ is the temperature of the ice water or freezer.

   (a) Solve this differential equation to get a model for the temperature of the can as a function of time.

   (b) It turns out that $c = 4000$ is a pretty good choice for a typical can of soda, that $h = 40$ for freezer air or 160 for ice water, and that $T_\infty = -20$ for the freezer and 0 for the ice water. Suppose the can was originally at 30^o and that $m = .13$, $A = .033$ for a typical can. Using this information, find equations for the temperature of the can as a function of time. Note: the units for all these constants are in terms of meters, seconds, kgs, and joules.

(c) How cold will the soda be after one hour (3600 seconds) in the freezer? In the ice water?

(d) If you put a can in the freezer and in the ice water at the same time, is there ever a time in the future where they will be the same temperature again? Hint: think about the Intermediate Value Theorem.

(e) Why is T_∞ an appropriate choice of variable name for the icewater/air temperature?

6. Find the family of curves orthogonal (perpendicular) to the family of curves $x^2+y^2=r^2$. What about $x^2-y^2=r^2$? Hint: given these equations, come up with an equation for y'. Then the slope of any orthogonal curve must be $-1/y'$

First Order ODEs of Homogeneous Type

1. Find the general solution to $x^2y'=y^2+3yx+x^2$

General Integration Review

In addition to knowing all the integration techniques from chapter 6, it's equally important to know when to use each one. Work through the following integrals and discuss as a group why you pick the techniques you pick. Also, if you get to a point where you know how to do the rest of the problem, feel free to skip any messy calculations.

1. $\int e^{x+e^x}dx$
2. $\int \frac{\sec^2(\sin\theta)}{\sec\theta}d\theta$
3. $\int \frac{1}{\sqrt{x+1}+\sqrt{x}}dx$
4. $\int \frac{\ln(x+1)}{x^2}dx$
5. $\int \frac{t^3+1}{t^3-t^2}dt$
6. $\int \cos^3 2x \sin 2x dx$
7. $\int \frac{dt}{\sqrt{e^t}}$
8. $\int \frac{1}{x\sqrt{x^2+4}}dx$
9. $\int \frac{1}{x\sqrt{x+4}}dx$
10. $\int \frac{x}{\sqrt{x^2+4}}dx$
11. $\int \frac{dx}{\sqrt[4]{x}+\sqrt[3]{x}}$
12. $\int \sqrt{\frac{x-1}{x+1}}dx$