Loop Groups and twisted K-theory III

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10 January 2008

Abstract

This is the third paper in a series relating the equivariant twisted K-theory of a compact Lie group G to the "Verlinde space" of isomorphism classes of projective lowest-weight representations of the loop groups. Here, we treat arbitrary compact Lie groups. We also discuss the relation to semi-infinite cohomology, the fusion product of Conformal Field theory, the rôle of energy and the topological Peter-Weyl theorem.

Introduction

In [FHT1, FHT2] the twisted equivariant K-theory of a compact Lie group was described in terms of positive energy representations of its loop group. There, we assumed that the group was connected, with torsion-free fundamental group. Here, we remove those restrictions; we also relax the constraints on the twisting, assuming only its *regularity*. Additional constraints allow the introduction of an *energy* operator, matching the rotation of loops, and lead to the *positive energy representations* relevant to conformal field theory. Finer restrictions on the twisting lead to a structure of 2-dimensional topological field theory, the "Verlinde TFT". While this is not fully discussed here — see instead [FHT4] for the topological story, and [FHT3, §8] for the relation to holomorphic bundles — we do prove two key underlying results: we identify the fusion product with the topological cup-product, and equate the bilinear form of the topological TFT with the duality pairing between irreducible representations at opposite levels.

Capturing the Verlinde ring topologically lets us revisit, via twisted K-theory, some constructions on representations that were hitherto assumed to rely on the algebraic geometry of loop groups. Thus, restriction to and induction from the maximal torus recover in twisted K-theory the *semi-infinite restriction* and *induction* on representations due to Feigin and Frenkel [FF]. The *energy* operator comes from the natural circle action on the quotient stack of G, under its own conjugation action. The numerator in the character formula can be obtained by dualising the Gysin inclusion of the identity in G. Next, the cup-product action of $R(G)$ on $K^{\tau}_{G}(G)$ corresponds to the *fusion* of Conformal Field Theory, defined via holomorphic induction. Finally, the Borel-Weil theorem for the "annular" flag variety of a product of two copies of the loop group is now interpreted as a topological Peter-Weyl theorem. This last result can also be interpreted as a computation of the bilinear form in the Verlinde TFT, but can also be further extended to an index theorem for generalised flag varieties of loop groups, in which twisted K-theory provides the topological side. We refer to [FHT3, §8] for a verification of this result in the special case of connected groups with free π_1, and to [T2] for further developments concerning higher twistings of K-theory.

The paper is organised as follows. Chapter I states the main theorems and describes the requisite technical specifications. Two examples are discussed in Chapter II: the first relates our theorem in the case of a torus to the classical spectral flow of a family of Dirac operators, while the second recalls the Dirac family associated to a compact group [FHT2], whose loop group analogue is the "non-abelian spectral flow" implementing our isomorphism. Chapter III computes the twisted K-theory $K^\tau_G(G)$ topologically, by reduction to the maximal torus and its normaliser in G. Chapter IV reviews the theory of loop groups and their lowest-weight representations; the classification of irreducibles in §10 reproduces the basis for $K^\tau_G(G)$ constructed in Chapter III. The Dirac family in Chapter V assigns a twisted K-class to any (admissible) representation of the loop group, and this is shown to recover the isomorphism already established by our classification. Chapter VI gives the topological interpretation of some known constructions on loop group representations as discussed above. Appendix A reviews the diagram automorphisms of simple Lie algebras and relates our definitions and notation to those in Kac's monograph [K].

Acknowledgements.

The authors would like to thank Graeme Segal for many useful conversations. We also thank G. Landweber for detailed comments on the early version of the manuscript and U. Bunke for discussions on the Göttingen seminar notes [BS]. During the course of this work, the first author was partially supported by NSF grants DMS-0072675 and DMS-0305505, the second partially by NSF grants DMS-9803428 and DMS-0306519, and the third partially by NSF grant DMS-0072675. We also thank the KITP of Santa Barbara (NSF Grant PHY99-07949) and the Aspen Centre for Physics for hosting their "topology and physics" programs, where various sections of this paper were revised and completed.

Index of Notation

Groups		
G, G_1	Compact Lie group and its identity component	
T, N	Maximal torus and its normaliser in G	
W, W_1	Weyl groups N/T, $N \cap G_1/T$ of G and G_1	
$G(f)$, $N(f)$	Centralisers in G and N of the class of f in $\pi_0 G$	§6
$\mathfrak{g}$, $\mathfrak{t}$, $\mathfrak{g}_\mathbb{C}$, $\mathfrak{t}_\mathbb{C}$	Lie algebras and their complexifications	
$\mathfrak{n} \subset \mathfrak{g}_\mathbb{C}$	Sub-algebra spanned by the positive root vectors	
$\langle \mid \rangle$, $\{\xi_a\}$	Basic inner product on $\mathfrak{g}$ (when semi-simple); orthonormal basis	§4
$\rho, \theta \in \mathfrak{t}^*$	Half-sum of positive roots, highest root	§4
$h^\vee$	(for simple $\mathfrak{g}$) Dual Coxeter number $\rho\theta + 1$	§4
ε; $\underline{\mathfrak{g}}$, $\underline{\mathfrak{t}}$	Diagram automorphism of $\mathfrak{g}$; ε-invariant sub-algebras of $\mathfrak{g}, \mathfrak{t}$	§7
$\underline{W}$; $\underline{T}$	Weyl group of $\underline{\mathfrak{g}}$; torus $\exp(\underline{\mathfrak{t}})$	§7
$\underline{\rho}, \underline{\theta} \in \underline{\mathfrak{t}}^*$	Half-sum of positive roots in $\underline{\mathfrak{g}}$, highest $\underline{\mathfrak{g}}$-weight of $\mathfrak{g}/\underline{\mathfrak{g}}$	§9
R, $R^\vee$	Root and co-root lattices	
Λ, $\underline{\Lambda}$; $\underline{\Lambda}^\tau$	Weight lattices of T and $\underline{T}$; lattice of τ-affine weights	
Loop Groups		
LG, $L_f G$	Smooth loop group and f-twisted loop group	§I
LG^τ	Central extension by $\mathbb{T}$ with cocycle τ	
$L\mathfrak{g}$, $L_f\mathfrak{g}$	Smooth loop Lie algebras	
$L'\mathfrak{g}$, $L'G$	Laurent polynomial Lie algebra, loop group	§8, §16
$N^e_{\text{aff}} = \Gamma_f N$	Group of (possibly f-twisted) geodesic loops in N	§6
W^e_{aff}	extended affine Weyl group $N^e_{\text{aff}}/\underline{T}$	
$W_{\text{aff}}(\mathfrak{g}, f)$	f-twisted affine Weyl group of $\mathfrak{g}$	§10.4, §A.4
$\mathfrak{a}$, $\underline{\mathfrak{a}}$	(simple $\mathfrak{g}$) Alcove of dominant $\xi \in \mathfrak{t}, \underline{\mathfrak{t}}$ with $\theta(\xi) \leq 1$, resp. $\underline{\theta}(\xi) \leq 1/r$	§8.3, §9.4
$\tau \cdot \underline{\mathfrak{a}}^* \subset \underline{\mathfrak{t}}$	Product of the centre of $\underline{\mathfrak{g}}$ and the $[\tau]$-scaled alcoves on simple factors	§10

Twistings

τ; $[\tau]$	(graded) 2-cocycle on LG; level in $H^1_G(G;\mathbb{Z}/2)\times H^3_G(G;\mathbb{Z})$	§2.1
κ^τ	Linear map $H_1(\underline{T})\to H^2(B\underline{T})$ given by contraction with $[\tau]$	§6
σ, $\underline{\sigma}$	LG-cocycle of the Spin modules for $L\mathfrak{g}$, $L_f\mathfrak{g}$	§1.6
$\sigma(\mathfrak{t})$, $\sigma(\underline{\mathfrak{t}})$	W-cocycle for the spinors on $\mathfrak{t}$ and $\underline{\mathfrak{t}}$	§6
τ', τ''	Twisting for the W^e_{aff}-action on $\underline{\Lambda}^\tau$; shifted twisting $\tau'-\sigma(\underline{\mathfrak{t}})$	§6

Contents

I Statements

Throughout the paper, cohomology and K-theory have integer coefficients, if no others are specified. K-theory has *compact supports*. For proper actions of non-compact groups, or for stacks in general, this refers to the quotient space. For a twisting τ on X, the twisted K-theory will be denoted $K^\tau(X)$. This is a $\mathbb{Z}/2$-graded group, whose two components are denoted $K^{\tau+0}(X), K^{\tau+1}(X)$. For a central extension G^τ of G, the Grothendieck group of τ-projective representations is denoted by $R^\tau(G)$; it is a module over the representation ring $R(G)$. Note that, when τ is graded, this module can have an *odd* component $R^{\tau+1}$, cf. §1.3.

1. Main theorems

(1.1) Simply connected case. The single most important special case of our result concerns a simple, simply connected compact Lie group G. Central extensions of its smooth loop group LG by the circle group $\mathbb{T}$ are classified by their *level*, the Chern class $c_1\in H^2(LG)=\mathbb{Z}$ of the underlying circle bundle. These extensions are equivariant under loop rotation. Among the projective rep-

resentations of LG with fixed level k are the *positive energy* ones: they are those which admit an intertwining action of the group of loop rotations, with spectrum bounded below. Working up to infinitesimal equivalence, as is customary with non-compact groups, these representations are semi-simple, with finitely many irreducibles, all of them unitarisable [PS]. The free abelian group on the irreducible isomorphism classes is the *Verlinde ring of G at level k*; the multiplication is the *fusion* of Conformal Field Theory. We denote this ring by $R^k(LG)$, by analogy with the representation ring $R(G)$ of G.

Theorem 1. *If $k + h^\vee > 0$, $R^k(LG)$ is isomorphic to the twisted K-theory $K_G^{\tau+\dim G}(G)$.*

Here, G act on itself by conjugation and τ is a twisting for the G-equivariant K-theory of G whose class $[\tau] \in H^3_G(G) \cong \mathbb{Z}$ equals $k + h^\vee$, with the *dual Coxeter number $h^\vee$* of G. Because $H^2_G(G) = 0$, the K-groups are canonically determined by the twisting class alone. It is part of our statement that the K-groups are supported in degree $\dim G \pmod 2$. The ring structure on K-theory is the convolution (Pontryagin) product. The isomorphism is established by realising both sides as quotient rings of $R(G)$, via *holomorphic induction* on the loop group side, and via the Thom push-forward from the identity in G, on the K-theory side.

(1.2) General groups. The isomorphism between the two sides and the relation between level and twisting cannot be described so concisely for general compact Lie groups. This is due to the presence of torsion in the group H^3 of twisting classes, to the additional type of twistings classified by $H^1_G(G; \mathbb{Z}/2)$, related to gradings of the loop group, and to the fact that the two sides need not be quotients of $R(G)$. (A simple statement can be given when G is connected and $\pi_1(G)$ is free, as in [FHT3, §6], precisely because both sides are quotients of $R(G)$). For a construction of the map via a correspondence induced by conjugacy classes, we refer to [F2] (see also §6.11 here). Ignoring the difficulties for a moment, there still arises a natural isomorphism between the twisted equivariant K-groups of G and those of the category of positive energy representations at a *shifted* level, provided that:

(i) we use $\mathbb{Z}/2$-*graded* representations;
(ii) we choose a central extension of LG which is equivariant under loop rotation;
(iii) the cocycle of the extension satisfies a positivity condition.

The energy operator cannot be defined without (ii), and without (iii), representations of positive energy do not exist. While (iii) is merely a question of choosing signs correctly, there are topological obstructions to equivariance in (ii) when G is not semi-simple (for instance, the absence of symmetry in the level, §15). The problem here is caused by tori, whose loop groups, ironically, have a straightforward representation theory.

This formulation is unsatisfactory in several respects. The loop group side involves the *energy*, with no counterpart in $K_G(G)$. Instead, a rotation-equivariant enhancement of the latter will give a better match. There is also the positivity restriction, whereas the topological side is well-behaved for *regular* twistings (§2). There is, finally, the unexplained "dual Coxeter" shift.

We now formulate the most canonical statement. This need not be the most comprehensible one (see Theorems 3 and 5 instead). However, it has the virtue of explaining the shift between level and twisting as the projective cocycle of the positive energy spinors on LG. Gradings in (i), if not originally present in the twisting τ, are also imposed upon us by the spinors whenever the Ad-representation of G does not spin.

(1.3) Untwisted loop groups.[1] Let G be any compact Lie group and LG^τ a smooth $\mathbb{T}$-central extension of its loop group. We allow LG to carry a *grading*, or homomorphism to $\mathbb{Z}/2$; this is classified by

[1] The term *twisting* for loop groups (§1.5) and for K-theory refers to different things, but both uses are well-entrenched.

an element of $H^1_G(G_1;\mathbb{Z}/2)$ and is notationally incorporated into τ. An Ad-invariant L^2 norm on $L\mathfrak{g}$ defines the graded Clifford algebra $\mathrm{Cliff}(L\mathfrak{g}^*)$, generated by odd elements $\psi(\mu)$, $\mu \in L\mathfrak{g}^*$, with relations $\psi(\mu)^2 = \|\mu\|^2$. (For a more canonical construction, this algebra must be based on *half-forms* on the circle, which carry a natural bilinear form.)

A *τ-representation* of LG is a graded representation of LG^τ on which the central circle acts by the natural character. We are interested in complex, graded τ-representations of the crossed product $LG \ltimes \mathrm{Cliff}(L\mathfrak{g}^*)$, with respect to the co-adjoint action. Graded modules for $\mathrm{Cliff}(L\mathfrak{g}^*)$ can be viewed as *b-projective representations* of the odd vector space $\psi(L\mathfrak{g}^*)$, where b is the L^2 inner product, so we are considering (τ, b)-representations of the *graded super-group* $LG_s := LG \ltimes \psi(L\mathfrak{g}^*)$. Subject to a *regularity restriction* on τ, an *admissibility condition* on representations will ensure their complete reducibility (§2).

A *super-symmetry* of a graded representation is an odd automorphism squaring to 1. Let $R^{\tau+0}(LG_s)$ be the free $\mathbb{Z}$-module of graded admissible representations, modulo the super-symmetric ones, and $R^{\tau+1}(LG_s)$ that of representations with a super-symmetry, modulo those carrying a second super-symmetry anti-commuting with the first. These should be regarded as the LG_s-equivariant K^τ-groups of a point. The reader should note that defining K-theory for graded algebras a delicate matter in general [Bl]; the shortcut above, also used in [FHT3, §4], relies on the semi-simplicity of the relevant categories of modules.

Since $K^\tau_G(G)$ is a $K_G(G)$-module, it carries in particular an action of the representation ring $R(G)$. *Fusion* with G-representations defines an $R(G)$-module structure on $R^\tau(LG_s)$; the definition is somewhat involved, and we must postpone it until §16. Here is our main result.

Theorem 2. *For regular τ, there is a natural isomorphism $R^\tau(LG_s) \cong K^\tau_G(G)$ of (graded) $R(G)$-modules, wherein K-classes arise by coupling the Dirac operator family of Chapter V to admissible LG_s-modules.*

1.4 Remark. For twistings that are *transgressed from BG* in a suitable sense [FHT4], both sides carry isomorphic Frobenius ring structures. The portion of the product that exists for any regular twisting is the $R(G)$-module structure, discussed in §16. A geometric construction of the duality pairing is described in §17.

(1.5) Twisted loop groups. When G is disconnected, there are twisted counterparts of these notions. For any $f \in G$, the *twisted loop group* L_fG of smooth maps $\gamma : \mathbb{R} \to G$ satisfying $\gamma(t + 2\pi) = f\gamma(t)f^{-1}$ depends, up to isomorphism, only on the conjugacy class in π_0G of the component fG_1 of f. Let $[fG_1] \subset G$ denote the union of conjugates of fG_1; the topological side of the theorem is $K^\tau_G([fG_1])$, while the representation side involves the admissible representations of $L_fG \ltimes \psi(L_f\mathfrak{g}^*)$.

(1.6) Removing the spinors. A lowest-weight spin module $\mathbf{S}$ for $\mathrm{Cliff}(L\mathfrak{g}^*)$ (see §2.9) carries an intertwining projective action of the loop group LG. Denoting by σ ($\underline{\sigma}$, in the f-twisted case) the projective cocycle of this action and by $\underline{d}$ the dimension of the centraliser G^f, a *Morita isomorphism*

$$R^\tau(L_fG_s) \cong R^{\tau-\underline{\sigma}-\underline{d}}(L_fG) \tag{1.7}$$

results from the fact that an admissible, graded τ-module of $L_fG \ltimes \mathrm{Cliff}(L_f\mathfrak{g}^*)$ has the form $\mathbf{H} \otimes \mathbf{S}$, for a suitable $(\tau - \underline{\sigma})$-representation $\mathbf{H}$ of L_fG, unique up to canonical isomorphism. Note in particular the dimension shift by $\underline{d}$, from the parity of the Clifford algebra. We obtain the following reformulation of Theorem 2.

Theorem 3. *For regular τ, there is a natural isomorphism $K^\tau_G([fG_1]) \cong R^{\tau-\underline{\sigma}-\underline{d}}(L_fG)$.*

The loop group may well acquire a grading from the spinor twist $\underline{\sigma}$, even if none was present in τ; if so, $R^{\tau-\underline{\sigma}}(L_fG)$ is built from graded representations, as in §1.3.

(1.8) Classifying representations. In proving the theorems, we compute both sides of the isomorphism in Theorem 3. More precisely, we compute the twisted K-theory by reduction to the torus and the Weyl group, and produce an answer which agrees with the classification of admissible representations in terms of their lowest weights. In fact, twisted K-theory allows for an attractive formulation of the lowest-weight classification for disconnected (loop) groups, as follows.

Choose a maximal torus $T \subset G$ which, along with a dominant chamber, is stable under f-conjugation. (Such tori always exists, see Proposition 7.2.) Recall that the *extended affine Weyl group* W^e_{aff} for $L_f G$ is π_0 of $L_f N$, the group of f-twisted loops in the normaliser N of T. Let $\underline{T} \subset T$ denote the subtorus centralised by f, and $\underline{\Lambda}^\tau$ the set of its τ-affine weights. The conjugation action of $L_f N$ on $\underline{T}$ descends to an action of W^e_{aff} on $\underline{\Lambda}^\tau$, which preserves the subset $\underline{\Lambda}^\tau_{reg}$ of *regular* weights. A tautological twisting τ' is defined for this action, because every weight defines a $\mathbb{T}$-central extension of its centraliser in W^e_{aff} (see §10.4 for details). Finally, after projection to the finite Weyl group $W = N/T$, W^e_{aff} also acts on the Lie algebra $\underline{\mathfrak{t}}$ of $\underline{T}$.

Theorem 4. *The category of graded, admissible τ-representations of $L_f G \ltimes \mathrm{Cliff}(L_f \mathfrak{g}^*)$ is equivalent to that of τ'-twisted $W^e_{\text{aff}} \ltimes \mathrm{Cliff}(\underline{\mathfrak{t}})$-modules on $\underline{\Lambda}^\tau_{reg}$.*

It follows that the corresponding K-groups agree. We reduce Theorem 4 in §10 to the well-known cases of simply connected compact groups and tori.

Computing both sides is a poor explanation for a natural isomorphism, and indeed we improve upon this in Chapter V by recalling the map from representations to K-classes using families of Dirac operators. The construction bypasses Theorem 4 and ties in beautifully with Kirillov's *orbit method*, recovering the co-adjoint orbit and line bundle that correspond to an irreducible representation. Another offshoot of this construction emerges in relation with the *semi-infinite cohomology* of Feigin and Frenkel [FF], for which we give a topological model (Theorem 14.11): for integrable representations, the Euler characteristic of semi-infinite $L\mathfrak{n}$-cohomology becomes the restriction from G to T, on the K-theory side. While this can also be checked by computing both sides, our Dirac family gives a more natural proof, providing the same rigid model for both.

(1.9) Loop rotation. Assume now that the extension LG^τ carries a lifting of the loop rotation action on LG. It is useful to allow *fractional* lifts, that is, actions on LG^τ of a *finite cover* $\mathbb{T}$ of the loop rotation circle; such a lift always exists when G is semi-simple (Remark 15.3). If so, admissible τ-representations carry an intertwining, semi-simple action of this new $\mathbb{T}$. Schur's lemma implies that the action is unique up to an overall shift on any irreducible representation. A further positivity condition (§15.5) on τ ensures that the spectrum of this action is bounded below, and its real infinitesimal generator is then called the *energy*.

In this favourable situation, we can incorporate the loop rotation into our results. The requisite object on the topological side is the *quotient stack* of the space $\mathcal{A}$ of $\mathfrak{g}$-valued smooth connections on the circle by the semi-direct product $\mathbb{T} \ltimes LG$, the loop group acting by gauge transformations and $\mathbb{T}$ by loop rotation. We denote the twisted K-theory of this stack by $K^\tau_{\mathbb{T}}(G_G)$. This notation, while abusive, emphasises that the $\mathbb{T}$-action makes it into an $R(\mathbb{T})$-module; its *fibre over* 1 is the quotient by the augmentation ideal of $R(\mathbb{T})$.

Theorem 5. *If the regular twisting τ is rotation-equivariant, $K^\tau_{\mathbb{T}}(G_G)$ is isomorphic to the R^τ-group of graded, admissible, representations of $\mathbb{T} \ltimes LG_s$ (cf. §1.3). It is a free module over $R(\mathbb{T})$, and its fibre over* 1 *is isomorphic to $K^\tau_G(G)$.*

The formulation, while a bit awkward, has the virtue of being canonical: there is no natural isomorphism of $K^\tau_{\mathbb{T}}(G_G)$ with $K^\tau_G(G) \otimes R(\mathbb{T})$. A noteworthy complement to Theorem 5 is that $K^\tau_{\mathbb{T}}(G_G)$ contains the Kac numerator formula for LG^τ-representations, §15.6. It would be helpful to understand this as a twisted Chern character, just as the the Kac numerator at $q = 1$ is the Chern character for $K^\tau_G(G)$, see [FHT3].

2. Technical definitions

In this section, we describe our regularity conditions on the central extension LG^τ and define the class of admissible representations. There is a *topological* and an *analytical* component to regularity.

(2.1) Topological regularity. The central extension LG^τ has a characteristic class $[\tau] \in H^3_G(G_1)$, the *level*.[2] It is an equivariant version of the *Dixmier-Douady invariant* of a gerbe, and arises from the connecting arrow in the exponential sequence for group cohomology with smooth circle coefficients, $H^2_{LG}(\mathbb{T}) \to H^3_{LG}(\mathbb{Z})$: the last group is purely topological, and equals $H^3(BLG) \cong H^3_G(G_1)$. When $\mathfrak{g}$ is semi-simple, the smooth-cochain group cohomology $H^2_{LG}(\mathbb{R})$ vanishes [PS, XIV], and $[\tau]$ then determines the central extension LG^τ, up to isomorphism. In any case, restricting to a maximal torus $T \subset G$ and writing H^2_T for $H^2(BT)$, we obtain a class in

$$H^3_T(T) = H^1(T) \otimes H^2_T \oplus H^3(T).$$

For classes arising from central extensions, it turns out that the $H^3(T)$ component vanishes. In view of the isomorphism $H^1(T) \cong H^2_T$, we make the following

2.2 Definition. We call τ *topologically regular* iff $[\tau]$ defines a non-singular bilinear form on $H_1(T)$.

For a twisted loop group L_fG, topological regularity is detected instead by the f-invariant subtorus $\underline{T} \subset T$ in an f-stable maximal torus T as in §1.8. Restricting $[\tau]$ there leads to a bilinear form on $H_1(\underline{T})$, and regularity refers to the latter. In the next section, we will see how the bilinear form captures the commutation in LT^τ of the constant loops T with the group of components $\pi_1 T$.

(2.3) Analytic regularity. This condition, which holds in the standard examples, concerns the centre $\mathfrak{z} \subset \mathfrak{g}$, and ensures that the topologically invisible summand $L\mathfrak{z}/\mathfrak{z}$ does not affect the classification of representations of LG^τ. Split $L\mathfrak{z}$ into the constants $\mathfrak{z}$ and their L^2-complement V, and observe that LG is the semi-direct product of the normal subgroup $\exp(V)$ by the subgroup Γ of loops γ whose velocity $d\gamma \cdot \gamma^{-1}$ has constant $\mathfrak{z}$-projection. Because the action of Γ on V factors through the *finite* group $\pi_0 G$, invariant central extensions of $\exp(V)$ have a preferred continuation to LG.

2.4 Definition. τ is *analytically regular* iff it is the sum of an extension of Γ and a Heisenberg extension of $\exp(V)$, and, moreover, the Heisenberg cocycle $\omega : \Lambda^2 V \to \mathrm{i}\mathbb{R}$ has the form $\omega(\xi, \eta) = b(S\xi, \eta)$, for some skew-adjoint Fredholm operator $\mathrm{i}S$ on V.

The standard example[3] has $S = \mathrm{i}d/dt$, an unbounded operator, so we really ask that $S/(1 + \sqrt{S^*S})$ should be Fredholm. We need to tame ω for the Dirac constructions in Chapter V. For twisted loop groups, the analytic constraints refer to $L_f\mathfrak{z}\,/\mathfrak{z}^f$.

(2.5) Linear splittings. Restricted to any simple summand in $\mathfrak{g}$, every extension class is a multiple of the *basic* one in §8.1, and is detected by the level $[\tau]$. However, the cocycle $\omega : \Lambda^2 L\mathfrak{g} \to \mathrm{i}\mathbb{R}$ depends on a linear splitting of the extension

$$0 \to \mathrm{i}\mathbb{R} \to L\mathfrak{g}^\tau \to L\mathfrak{g} \to 0. \tag{2.6}$$

For the unique $\mathfrak{g}$-invariant splitting, S is a multiple of $\mathrm{i}d/dt$. Preferred splittings for the twisted loop groups also exist; they are discussed in §9. Hence, subject to topological regularity, and using the preferred splittings, the second part of Condition (2.4) holds for the entire Lie algebra. Varying the splitting by a *representable* linear map $L\mathfrak{g} \to \mathrm{i}\mathbb{R}$, that is, one of the form $\eta \mapsto \omega(\xi, \eta)$ (with a fixed $\xi \in L\mathfrak{g}$) changes S by an *inner* derivation. We assume now that such a splitting has been chosen.

[2]For *graded* central extensions, $[\tau]$ has an additional component in $H^1_G(G; \mathbb{Z}/2)$, but this plays no role in the regularity conditions.

[3]This is the only possibility for $\mathrm{Diff}(S^1)$-equivariant extensions [PS, VIII].

2.7 Remark. (i) S must vanish on $\mathfrak{z}$, because the latter exponentiates to a torus, over which any $\mathbb{T}$-extension is trivial. The Heisenberg condition allows $\ker S \cap L\mathfrak{z}$ to be no larger. Combining this with the discussion of simple summands shows that, for regular τ, $\ker S$ is the Lie algebra of a full-rank compact subgroup of LG. This is the constant copy of G, for the standard splitting.
(ii) Assuming rotation-equivariance (§1.9), S commutes with d/dt and Γ with V; this justifies our first analytic constraint in (2.4). However, the constraint needed for our classification of admissible representations is weaker: namely, conjugation by Γ should implement a *representable* change in any splitting of (2.6) over V. Indeed, if $\mathrm{Ad}\gamma$ changes the splitting by $\eta \mapsto \omega(\xi, \eta)$ for some (γ-dependent) $\xi \in V$, then the alternate copy of Γ in $L_f G$, which replaces γ by $e^{-\xi}\gamma$, decomposes the latter as a semi-direct product of $\exp(V)^\tau$ by a central extension of the new Γ. The new Γ-extension may differ from the original, but it has the same topological level.

(2.8) Lowest-weight representations. The semi-positive spectral projection of S is an ω-isotropic subalgebra $\mathfrak{P} \subset L\mathfrak{g}_{\mathbb{C}}$; we call it the *positive polarisation*. The strictly positive part $\mathfrak{U} \subset \mathfrak{P}$ is a Lie ideal, and $\ker S \otimes \mathbb{C}$ is isomorphic to $\mathfrak{P}/\mathfrak{U}$. A linear splitting in §2.5 restricts to a *Lie algebra* splitting over $\mathfrak{P}$. A *lowest weight* τ-representation of $L\mathfrak{g}$ is one generated by an irreducible module of $\ker S$, which is killed by the lifted copy of $\overline{\mathfrak{U}}$ in (2.6).

The lowest-weight condition depends on S and on the splitting of (2.6) over the centre $\mathfrak{z}$. However, if we insist on *integrability* of the representation to the identity component of the loop group (see §8.5), lowest-weight modules are irreducible, unitarisable, and their Hilbert space completions are unchanged under a representable variation of that splitting.

(2.9) Admissible representations. A projective representation of LG is called *admissible* if it decomposes as a finite-multiplicity sum of Hilbert space completed lowest-weight representations of the Lie algebra. Assuming topological regularity, any integrable lowest-weight representation of $L\mathfrak{g}$ exponentiates to an action of the identity component of LG on the Hilbert space completion. This induces an admissible representation of the full loop group. Moreover, at fixed level, there are finitely many irreducibles, up to isomorphism; see §10.

There is a similar notion of lowest-weight and admissibility for $\mathrm{Cliff}(L\mathfrak{g}^*)$-modules, using the same polarisation. (Note that $\mathfrak{U}$ is b-isotropic.) As in the finite-dimensional case, there are one or two isomorphism classes of lowest-weight representations, according to whether $\dim \mathfrak{g}$ is even or odd, and they are irreducible. The numbers are switched if we ask for *graded* representations; any of the graded irreducibles is called a spin module. The K-theory of graded, admissible $\mathrm{Cliff}(L\mathfrak{g}^*)$-modules (as in §1.3) is $\mathbb{Z}$, in degrees $\dim G \pmod 2$. The two spin modules, in the even case, differ by parity-reversal, and represent opposite generators of K^0. (In the odd case, two opposite generators come from the two choices of a super-symmetry on the irreducible spin module.)

2.10 Remark. The algebraic approach to representations starts from the Laurent polynomial loop algebra $L'\mathfrak{g}$ and the finite-multiplicity sums of integrable lowest-weight modules of $L'\mathfrak{g} \ltimes \mathrm{Cliff}(L'\mathfrak{g}^*)$. These are the Harish-Chandra modules underlying our admissible representations. However, as our Dirac construction of K-classes involves the smooth loop group and its unitary representations, we must work analytically.

II Two examples

We recall from [FHT2] two examples relevant to the construction of the Dirac operator families in Chapter IV, which relate representations to K-theory classes. The first concerns the group LT of loops in a torus; the second is a finite-dimensional Dirac family, which leads to an interpretation of our theorem as an infinite-dimensional Thom isomorphism.

3. Spectral flow over a torus

(3.1) The circle [APS]. Let $\not{D} := d/d\theta$ be the one-dimensional Dirac operator on the complex Hilbert space $\mathbf{L} := L^2(S^1;\mathbb{C})$, acting as $\mathrm{i}n$ on the Fourier mode $e^{in\theta}$. For any $\xi \in \mathbb{R}$, the modified operator $\not{D}_\xi := \not{D} + \mathrm{i}\xi$ has the same eigenvectors, but with shifted spectrum $\mathrm{i}(n+\xi)$. Let $M : \mathbf{L} \to \mathbf{L}$ be the operator of multiplication by $e^{\mathrm{i}\theta}$. The relation $M^{-1}\not{D}_\xi M = \not{D}_{\xi+1}$ shows that the family $\not{D}_\xi$, parametrised by $\xi \in \mathbb{R}$, descends to a family of operators on the Hilbert bundle $\mathbb{R} \times_{\mathbb{Z}} \mathbf{L}$ over $\mathbb{R}/\mathbb{Z}$ (where we let M generate the $\mathbb{Z}$-action on $\mathbf{L}$).

Following the spectral decomposition of $\not{D}_\xi$, we find that one eigenvector crosses over from the negative to the positive imaginary spectrum as ξ passes an integer value. Thus, the dimension of the positive spectral projection, although infinite, changes by 1 as we travel once around the circle $\mathbb{R}/\mathbb{Z}$. This property of the family $\not{D}_\xi$ is invariant under continuous deformations and captures the following topological invariant. Recall [ASi] that the interesting component of the space $\mathrm{Fred}^{\mathrm{sa}}$ of skew-adjoint Fredholm operators on $\mathbf{L}$ has the homotopy type of the *small unitary group* $U(\infty)$; in particular, $\pi_1\mathrm{Fred}^{\mathrm{sa}} = \mathbb{Z}$. (The other two components, of essentially positive and essentially negative Fredholm operators, are contractible.) Weak contractibility of the *big* unitary group allows us to trivialise our Hilbert bundle on $\mathbb{R}/\mathbb{Z}$, uniquely up to homotopy; so our family defines a map from the circle to $\mathrm{Fred}^{\mathrm{sa}}$, up to homotopy. This map detects a generator of $\pi_1\mathrm{Fred}^{\mathrm{sa}}$.

(3.2) Generalisation to a torus. A metric on the Lie algebra $\mathfrak{t}$ of a torus T defines the Clifford algebra $\mathrm{Cliff}(\mathfrak{t}^*)$, generated by the dual $\mathfrak{t}^*$ of $\mathfrak{t}$. Denote by $\psi(\mu)$ the Clifford action of $\mu \in \mathfrak{t}^*$ on a complex, graded, irreducible spin module $\mathbf{S}(\mathfrak{t}) = \mathbf{S}^+(\mathfrak{t}) \oplus \mathbf{S}^-(\mathfrak{t})$ [ABS]. Let $\mathbf{L}^\pm = L^2(T) \otimes \mathbf{S}^\pm(\mathfrak{t})$, denote by $\not{D}$ the Dirac operator $\sum_a \partial/\partial\theta^a \otimes \psi^a$ on $\mathbf{L} := \mathbf{L}^+ \oplus \mathbf{L}^-$, and consider the family of operators parametrised by $\mu \in \mathfrak{t}^*$,

$$\not{D}_\mu = \not{D} + \mathrm{i}\psi(\mu) : \mathbf{L}^+ \to \mathbf{L}^-.$$

Let $\Pi = (2\pi)^{-1}\log 1$ be the *integer lattice* in $\mathfrak{t}$, isomorphic to $\pi_1 T$. For a weight $\lambda \in \Pi^* := \mathrm{Hom}(\Pi;\mathbb{Z})$, let $M_\lambda : \mathbf{L} \to \mathbf{L}$ be the operator of multiplication by the associated character $e^{\mathrm{i}\lambda} : t \mapsto t^{\mathrm{i}\lambda}$. The relation $M_{-\lambda} \circ \not{D}_\mu \circ M_\lambda = \not{D}_{\mu+\lambda}$ shows that $\not{D}_\mu$ descends to a family of fibre-wise operators on the Hilbert bundle $\mathfrak{t}^* \times_{\Pi^*} \mathbf{L}$ over the dual torus $T^* := \mathfrak{t}^*/\Pi^*$. Here, Π^* acts on $\mathfrak{t}^*$ by translation and on $\mathbf{L}$ via the M. As before, contractibility of the unitary group leads to a continuous family of Fredholm operators over T^*. When $\ell := \dim \mathfrak{t}$ is odd, we choose a self-adjoint volume form $\varpi \in \mathrm{Cliff}^1(\mathfrak{t}^*)$. This commutes with all the $\psi^\bullet$ and converts $\not{D}_\mu$ to a skew-adjoint family $\varpi \cdot \not{D}_\mu$ of operators acting on $\mathbf{L}^+$. Thus, in every case, we obtain a class in $K^\ell(T^*)$. This represents the K-theoretic volume form; more precisely, it is a Fredholm model for the Thom push-forward of the identity in T^*.

(3.3) Representations and twisted K-classes. Relating this construction to our concerns requires a bit more structure, in the form of a linear map $\tau : \Pi \to \Pi^*$ (not related to the metric). A *central extension* Γ^τ of the product $\Gamma := \Pi \times T$ by the circle group $\mathbb{T}$ is defined by the commutation rule

$$ptp^{-1} = t \cdot t^{\mathrm{i}\tau(p)} \qquad p \in \Pi, t \in T \;\text{ and }\; t^{\mathrm{i}\tau(p)} \in \mathbb{T}. \tag{3.4}$$

The group Γ^τ has a unitary representation on $L^2(T)$, with T acting by translation and Π by the $M_{\tau(p)}$'s. If τ has full rank, $L^2(T)$ splits into a finite sum of irreducible ${}^\tau\Gamma$-representations $\mathbf{F}_{[\lambda]}$, each of them comprising the weight spaces of T in a fixed residue class $[\lambda] \in \Pi^*/\tau(\Pi)$. Moreover, these are all the unitary τ-irreducibles of Γ, up to isomorphism. (This will be shown in §10.)

Now, τ also induces a map $T \to T^*$, where-under the pull-back of $\mathbf{L}$ splits, according to the splitting of $L^2(T)$ into the $\mathbf{F}_{[\lambda]}$. Each component carries the lifted Dirac family $\not{D}_\xi := \not{D} + \mathrm{i}\psi(\tau(\xi))$, descending to a spectral flow family over T. Except at the single value $\exp(\tau^{-1}[-\lambda]) \in T$ of the parameter, $\not{D}_\bullet$ is invertible on the fibres $\mathbf{F}_{[\lambda]} \otimes \mathbf{S}$.

All families $\mathbf{F}_{[\lambda]} \otimes \mathbf{S}$ have the same image in $K^{\ell}(T)$, but this problem is cured by remembering the T-action, as follows. Instead of viewing the $\not{D}_{\bullet}$ as families over T, we interpret them as ${}^{\tau}\Gamma$-equivariant Fredholm families parametrised by $\mathfrak{t}$. Now, $\mathfrak{t}$ is a principal Π-bundle over the torus T, equivariant for the trivial action of T on both, and the central extension ${}^{\tau}\Gamma$ defines a *twisting* for the T-equivariant K-theory of T: see [FHT1, §2]. Classes in $K_T^{\tau+0}(T)$ are described by Γ-equivariant families of Fredholm operators, parametrised by $\mathfrak{t}$, on τ-projective unitary representations of Γ; twisted K^1-classes are represented by skew-adjoint families. Thus, our families $\not{D}_{\xi} : \mathbf{F}_{[\lambda]} \otimes \mathbf{S}^{+} \to \mathbf{F}_{[\lambda]} \otimes \mathbf{S}^{-}$ (respectively $\varpi \cdot \not{D}_{\xi}$ on $\mathbf{F}_{[\lambda]} \otimes \mathbf{S}^{+}$ in odd dimensions) give classes in $K_T^{\tau+\ell}(T)$. A special case of our main theorem asserts that, when τ is regular, these classes form a $\mathbb{Z}$-basis of the twisted K-groups in dimension $\ell \pmod 2$, while the other K-groups vanish.

3.5 Remark. The inverse map from $K_T^{\tau+\ell}(T)$ to representations of Γ^{τ} ought to be "integration over $\mathfrak{t}$" from $K_\Gamma^{\tau+\ell}(\mathfrak{t})$ to $R^{\tau+0}(\Gamma)$. This is consistent with our interpretation of our main theorem as an infinite-dimensional Thom isomorphism, on the space of connections over the circle (Chapter V). However, we only know how to define $R^{\tau}(\Gamma)$ in terms of C^{*}-algebras.

(3.6) Direct image interpretation. Here, we give a topological meaning for the family $(\not{D}_{\bullet}, \mathbf{L})$; this will be used in §17. We claim it represents the image of the unit class $[1]$ under the Gysin map

$$p_{*} : K^{0}(T) \to K_T^{\tau-\ell}(T).$$

To define p_*, we must trivialise the lifted twisting $p^*\tau$. Recall that the twisting τ for the (trivial) T-action on T is the groupoid defined from the action of Γ^{τ} on $\mathfrak{t}$. The matching model for $p^*\tau$ on $T = \mathfrak{t}/\Pi$ comes from the restricted extension Π^{τ}, and this is trivialised by its construction (3.4). The class $[1]$ then corresponds to the trivial line bundle on $\mathfrak{t}$ with trivial Π-action.

We now give an equivalent, but more concrete model for p_*. Replace $K^*(T)$ by $K_T^*(T \times T)$, where T translates the second factor; the projection P to the first factor replaces p. If we represent T by the Γ-action groupoid on $\mathfrak{t} \times T$, where Π and T act by translation on $\mathfrak{t}$, resp. T, then the twisting $P^*\tau$ is represented by the action of Γ^{τ} on $\mathfrak{t} \times T$.

Call $\mathcal{O}(\tau)$ the trivial line bundle on $\mathfrak{t} \times T$, but with the translation action of T and with Π-action via the operators $M_{\tau(\bullet)}$. The two assemble to a τ-action of Γ, so $\mathcal{O}(\tau)$ gives a class in $K_T^{\tau+0}(T \times T)$. We claim that this is the image of $[1]$ under the trivialisation of $p^*\tau$. Indeed, our model for $p^*\tau$ as the action of Π^{τ} on $\mathfrak{t}$ maps to the model for $P^*\tau$ by inclusion at $\mathfrak{t} \times \{1\}$; thereunder, $\mathcal{O}(\tau)$ restricts to the trivial bundle with trivial Π-action.

The Gysin image $P_*[1]$ is now represented by any Γ^{τ}-invariant family of Dirac operators on the fibres of P, and $(\not{D}_{\bullet}, \mathbf{L})$ is an example of this.

(3.7) Relation to the loop group LT. Decompose LT as $\Gamma \times \exp(V)$, where $V = L\mathfrak{t} \ominus \mathfrak{t}$. Central extensions of $\exp(V)$ by the circle group $\mathbb{T}$ are classified by skew 2-forms ω on V. We choose a regular such form, in the sense of §2.4, together with a positive isotropic subspace $\mathfrak{U} \subset V_{\mathbb{C}}$. There exists then, up to isomorphism, a unique irreducible, unitary projective *Fock representation* $\mathbf{F}$ of $\exp(V)$ which contains a vector annihilated by $\overline{\mathfrak{U}}$. The sum of Γ^{τ} and our extension of $\exp(V)$ is a $\mathbb{T}$-central extension LT^{τ} of LT, whose irreducible admissible representations are isomorphic to the $\mathbf{F}_{[\lambda]} \otimes \mathbf{F}$. Our construction assigns to each of these a class in $K_T^{\tau+\ell}(T)$.

We will extend this construction, and the resulting correspondence between LG-representations and twisted K classes, to arbitrary compact groups G. Observe, by factoring out the space of based loops, that Γ^{τ}-equivariant objects over $\mathfrak{t}$ are in natural correspondence to LT^{τ}-equivariant ones over the space $\mathcal{A}$ of $\mathfrak{t}$-valued connection forms on the circle, for the gauge action; and it is in this form that our construction of the Dirac spectral flow generalises. The explicit removal of the Fock factor $\mathbf{F}$ has no counterpart for non-abelian groups, and the same effect is achieved instead by coupling the Dirac operator to the spinors on $L\mathfrak{t}/\mathfrak{t}$.

4. A finite-dimensional Dirac family

We now recall from [FHT2] the finite-dimensional version of our construction of twisted K-classes from loop group representations (Chapter V). For simplicity, we take G to be simple and simply connected. Choosing a dominant Weyl chamber in $\mathfrak{t}$ defines the nilpotent algebra $\mathfrak{n}$ spanned by positive root vectors, the *highest root* θ and the *Weyl vector* ρ, the half-sum of the positive roots. Roots and weights live in $\mathfrak{t}^*$, a weight λ defines the character $e^{\mathrm{i}\lambda} : T \to \mathbb{T}$, sending $e^{\xi} \in T$ to $e^{\mathrm{i}\lambda(\xi)}$.

The *basic* invariant bilinear form $\langle \mid \rangle$ on $\mathfrak{g}$ is normalised so that the long roots have square-length 2. Define the *structure constants* f_{ab}^c by $[\xi_a, \xi_b] = f_{ab}^c \xi_c$, in an orthonormal basis $\{\xi_a\}$ of $\mathfrak{g}$ with respect to this bilinear form.[4] Note that $f_{bc}^a f_{ad}^c = 2h^\vee \delta_{bd}$, where $h^\vee = \rho\theta + 1$ is the dual Coxeter number. Let $\mathrm{Cliff}(\mathfrak{g}^*)$ be the Clifford algebra generated by elements ψ^a dual to the basis ξ_a, satisfying $\psi^a\psi^b + \psi^b\psi^a = 2\delta^{ab}$, and let $\mathbf{S} = \mathbf{S}^+ \oplus \mathbf{S}^-$ be a graded, irreducible complex module for it. This is unique up to isomorphism and (if $\dim \mathfrak{g}$ is even) up to parity switch. There is a unique action of $\mathfrak{g}$ on $\mathbf{S}$ compatible with the adjoint action on $\mathrm{Cliff}(\mathfrak{g}^*)$; the action of ξ_a can be expressed in terms of Clifford generators as

$$\sigma_a = -\frac{1}{4} f_{bc}^a \cdot \psi^b \psi^c.$$

It follows from the Weyl character formula that $\mathbf{S}$ is a sum of $2^{\lceil \dim \mathfrak{t}/2 \rceil}$ copies of the irreducible representation $V_{-\rho}$ of $\mathfrak{g}$ of lowest weight $(-\rho)$. The lowest-weight space is a graded $\mathrm{Cliff}(\mathfrak{t}^*)$-module; for dimensional reasons, it is irreducible.

(4.1) The Dirac operator. Having trivialised the Clifford and Spinor bundles over G by left translation, consider the following operator on spinors, called by Kostant [K1] the "cubic Dirac operator":

$$\not{D} = R_a \otimes \psi^a + \frac{1}{3}\sigma_a \cdot \psi^a = R_a \otimes \psi^a - \frac{1}{12} f_{abc}\psi^a\psi^b\psi^c, \tag{4.2}$$

where R_a denotes the right translation action of ξ_a on functions. Let also $T_a = R_a + \sigma_a$ be the total right translation action of ξ_a on smooth spinors.

4.3 Proposition. $\left[\not{D}, \psi^b\right] = 2T_b$; $\left[\not{D}, T_b\right] = 0$.

Proof. The second identity expresses the right-invariance of the operator, while the first one follows by direct computation:

$$\begin{aligned} \left[\not{D}, \psi^b\right] &= R_a \otimes \left[\psi^a, \psi^b\right] + \frac{\sigma_a}{3}\left[\psi^a, \psi^b\right] - \frac{1}{3}\left[\sigma_a, \psi^b\right] \cdot \psi^a \\ &= 2R_b + \frac{2}{3}\sigma_b - \frac{1}{3} f_{ca}^b \cdot \psi^c \psi^a \\ &= 2(R_b + \sigma_b). \end{aligned}$$

□

(4.4) The Laplacian. The Peter-Weyl theorem decomposes $L^2(G; \mathbf{S})$ as $\bigoplus_\lambda V_{-\lambda}^* \otimes V_{-\lambda} \otimes \mathbf{S}$, where the sum ranges over the dominant weights λ of $\mathfrak{g}$. Left translation acts on the left, R_a on the middle and σ_a on the right factor. Hence, $\not{D}$ acts on the two right factors alone. As a consequence of (4.3), the Dirac Laplacian $\not{D}^2$ commutes with the operators $T_\bullet$ and $\psi^\bullet$. As these generate $V_{-\lambda} \otimes \mathbf{S}$ from its $-(\lambda + \rho)$-weight space, $\not{D}^2$ is determined from its action there. To understand this action, rewrite $\not{D}$ in a root basis of $\mathfrak{g}$,

$$\not{D} = R_j \otimes \psi^j + \frac{1}{3}\sigma_j\psi^j + R_\alpha \otimes \psi^{-\alpha} + R_{-\alpha} \otimes \psi^\alpha + \frac{1}{3}\left(\sigma_\alpha\psi^{-\alpha} + \sigma_{-\alpha}\psi^\alpha\right), \tag{4.5}$$

[4] We use the Einstein summation convention, but will also use the metric to raise or lower indexes as necessary, when no conflict arises.

where the j's label a basis of $\mathfrak{t}$ and α ranges over the positive roots. The commutation relation $[\sigma_{-\alpha}, \psi^\alpha] = \psi(-2\mathrm{i}\rho)$, where summation over α has been implied, converts (4.5) to

$$\not{D} = R_j \otimes \psi^j + \frac{1}{3}\,\sigma_j\psi^j - \frac{2\mathrm{i}}{3}\,\psi(\rho) + R_\alpha \otimes \psi^{-\alpha} + R_{-\alpha} \otimes \psi^\alpha + \frac{1}{3}\left(\sigma_\alpha\psi^{-\alpha} + \psi^\alpha\sigma_{-\alpha}\right),$$

and the vanishing of all α-terms on the lowest weight space leads to the following

4.6 Proposition. (i) $\not{D} = -\mathrm{i}\psi(\lambda+\rho)$ *on the* $-(\lambda+\rho)$*-weight space of* $V_{-\lambda} \otimes \mathbf{S}$.
(ii) $\not{D}^2 = -(\lambda+\rho)^2$ *on* $V_{-\lambda} \otimes \mathbf{S}$. □

(4.7) The Dirac family. Consider now the family $\not{D}_\mu := \not{D} + \mathrm{i}\psi(\mu)$, parametrised by $\mu \in \mathfrak{g}^*$. Conjugation by a suitable group element brings μ into the *dominant chamber* of $\mathfrak{t}^*$. From (4.6), we obtain the following relations, where $\langle T|\mu\rangle$ represents the contraction of μ with $T \in \mathfrak{g}^* \otimes \mathrm{End}\,(V \otimes \mathbf{S})$, in the basic bilinear form (the calculation is left to the reader).

4.8 Corollary. (i) $\not{D}_\mu = \mathrm{i}\psi(\mu-\lambda-\rho)$ *on the lowest weight space of* $V_{-\lambda} \otimes \mathbf{S}$.
(ii) $\not{D}_\mu^2 = -(\lambda+\rho-\mu)^2 + 2\mathrm{i}\langle T|\mu\rangle - 2\langle\lambda+\rho|\mu\rangle$. □

(4.9) The kernel. Because $\mathrm{i}\langle T|\mu\rangle \le \langle\lambda+\rho|\mu\rangle$, with equality only on the $-(\lambda+\rho)$-weight space, $\not{D}_\mu$ is invertible on $V_{-\lambda} \otimes \mathbf{S}$, except when μ is in the co-adjoint orbit $\mathfrak{O}$ of $(\lambda+\rho)$. In that case, the kernel at $\mu \in \mathfrak{g}^*$ is that very weight space, with respect to the Cartan sub-algebra $\mathfrak{t}_\mu$ and dominant chamber *defined by the regular element* μ. This is the lowest-weight line of $V_{-\lambda}$ tensored with the lowest-weight space of $\mathbf{S}$, and is an irreducible module for the Clifford algebra generated by the normal space $\mathfrak{t}_\mu^*$ to $\mathfrak{O}$ at μ. More precisely, the kernels over $\mathfrak{O}$ assemble to the normal spinor bundle to $\mathfrak{O} \subset \mathfrak{g}^*$, twisted by the natural line bundle $\mathcal{O}(-\lambda-\rho)$. Finally, at a nearby point $\mu+\nu$, with $\nu \in \mathfrak{t}_\mu^*$, $\not{D}_{\mu+\nu}$ acts on $\ker(\not{D}_\mu)$ as $\mathrm{i}\psi(\nu)$.

(4.10) Topological interpretation. The family of operators $\not{D}_\mu$ on $V_{-\lambda} \otimes \mathbf{S}$ is a compactly supported K-cocycle on $\mathfrak{g}^*$, equivariant for the co-adjoint action of G. As before, when $\dim\mathfrak{g}$ is odd, we use the volume form ϖ to produce the skew-adjoint family $\varpi\not{D}_\mu$, which represents a class in K_G^1. Our computation of the kernel identifies these classes with the Thom classes of $\mathfrak{O} \subset \mathfrak{g}^*$, with coefficients in the natural line bundle $\mathcal{O}(-\lambda-\rho)$. Sending $V_{-\lambda}$ to this class defines a linear map

$$R(G) \to K_G^{\dim\mathfrak{g}}(\mathfrak{g}^*). \tag{4.11}$$

There is another way to identify this map. Deform $\not{D}_\mu$ to $\mathrm{i}\psi(\mu)$ via the (compactly supported Fredholm) family $\varepsilon \cdot \not{D} + \mathrm{i}\psi(\mu)$. At $\varepsilon = 0$ we obtain the standard Thom class of the origin in $\mathfrak{g}^*$, coupled to $V_{-\lambda}$. Therefore, our construction is an alternative rigid implementation of the Thom isomorphism $K_G^0(0) \cong K_G^{\dim\mathfrak{g}}(\mathfrak{g}^*)$.

The inverse isomorphism is the push-forward from $\mathfrak{g}^*$ to a point. In view of our discussion, this expresses $V_{-\lambda}$ as the Dirac index of $\mathcal{O}(-\lambda-\rho)$ over $\mathfrak{O}$, leading to (the Dirac index version of) the Borel-Weil-Bott theorem. The affine analogue of the Thom isomorphism (4.11) is Theorem 3, equating the module of admissible projective representations with a twisted $K_G(G)$.

(4.12) Application to Dirac induction. For later use, we record here the following proposition; when combined with the Thom isomorphisms and the resulting twists, it gives the correct version of Dirac induction for *any compact Lie group* G (not necessarily connected). Let $N \subset G$ be the normaliser of the maximal torus T. We have a restriction map $K_G(\mathfrak{g}^*) \to K_N(\mathfrak{t}^*)$ and an "induction" $K_N(\mathfrak{t}^*) \to K_G(\mathfrak{g}^*)$ (Thom push-forward from $\mathfrak{t}^*$ to $\mathfrak{g}^*$, followed by Dirac induction from N to G).

4.13 Proposition. *The composition* $K_G(\mathfrak{g}^*) \to K_N(\mathfrak{t}^*) \to K_G(\mathfrak{g}^*)$ *is the identity.*

Proof. Express the middle term as $K_G(G \times_N \mathfrak{t}^*)$, with the left action of G on the induced space. The map from $G \times_N \mathfrak{t}^*$ to $\mathfrak{g}^*$ sends (g, μ) to $g\mu g^{-1}$. Since N meets every component of G (Prop. 7.2), this map is a diffeomorphism over regular points. Every class in $K_G(\mathfrak{g}^*)$ is the Thom push-forward of a class $[V] \in K_G(0)$. Deforming this to $\not{D} + i\psi(\mu)$ leads to a class supported on a *regular* orbit; *a fortiori*, our composition is the identity on such classes, hence on the entire K-group. $\square$

III Computation of twisted $K_G(G)$

In this chapter, we compute the twisted K-theory $K_G^\tau(G)$ by topological methods, for arbitrary compact Lie groups G and regular twistings τ. A key step is the reduction to the maximal torus, Proposition 7.8. Our answer takes the form of a twisted K-theory of the set of regular affine weights at level τ, equivariant under the *extended affine Weyl group* (§6.3, §6.4). This action has finite quotient and finite stabilisers, and the K^τ-theory is a free abelian group of finite rank. For foundational questions on twisted K-theory, we refer to [FHT1] and the references therein.

5. A "Mackey decomposition" lemma

The key step in our computation of $K_G(G)$ is a construction generalising Example 1.13 in [FHT1] and Lemma 2.14 in [FHT3]. It is a topological form of the Mackey decomposition of irreducible representations of a group, when restricted to a normal subgroup; this analogy will become essential in §10.6.

(5.1) Construction. Let H be a compact group acting on a compact Hausdorff space X, τ a twisting for H-equivariant K-theory, $M \subseteq H$ a normal subgroup acting trivially on X. We make the simplifying assumption that the H-action has contractible local slices: that is, each $x \in X$ has a closed H-neighbourhood of the form $H \times_{H_x} S_x$, with an H_x-contractible S_x. The following data can then be extracted from this:

(i) an H-equivariant family, parametrised by X, of $\mathbb{T}$-central extensions M^τ of M;

(ii) an H/M-equivariant covering space $p : Y \to X$, whose fibres label the isomorphism classes of irreducible, τ-projective representations of M;

(iii) an H-equivariant, tautological projective bundle $\mathbb{P}R \to Y$, whose fibre $\mathbb{P}R_y$ at $y \in Y$ is the projectivised τ-representation of M labelled by y;

(iv) a class $[R] \in K_H^{\mathbb{P}R}(Y)$, represented by R;

(v) a twisting τ' for the H/M-equivariant K-theory of Y, and an isomorphism of H-equivariant twistings $\tau' \cong p^*\tau - \mathbb{P}R$.

Items (iii) and (v) are only defined up to canonical isomorphism. Note that, if M^τ is abelian, as will be the case in our application, then $\mathbb{P}R = Y$, which can be taken to represent the zero twisting. However, $[R]$ is *not* the identity class $[1]$, because of the non-trivial M-action on the fibres R_y.

5.2 Lemma (Key Lemma). *The twisted K-theories $K_{H/M}^{\tau'}(Y)$ and $K_H^\tau(X)$ are naturally isomorphic.*

Proof. We claim that the following composition is an isomorphism:

$$K_{H/M}^{\tau'}(Y) \longrightarrow K_H^{\tau'}(Y) \cong K_H^{p^*\tau - \mathbb{P}R}(Y) \xrightarrow{\otimes[R]} K_H^{p^*\tau}(Y) \xrightarrow{p_!} K_H^\tau(X). \tag{5.3}$$

By the usual Mayer-Vietoris argument (for *closed* coverings), it suffices to prove this H-locally on the base X. Our slice assumption reduces us to the case when X is a point x and $H = H_x$.

Having a statement about groups alone, it is more convenient to use the model of twistings as (graded) central extensions by $\mathbb{T}$, and K-classes as twisted virtual representations. Decompose a τ-representation V of H under M^τ and distribute the isotypical components into a τ-twisted, H-equivariant vector bundle over the set Y_x of irreducible τ-representations of M. (H permutes Y_x through its conjugation action on M.) This bundle necessarily has the form $R \otimes W$, where R is the tautological bundle from before, and W is a $\tau' = (\tau - \mathbb{P}R)$-twisted H/M-bundle over Y. In the opposite direction, the direct sum V of components $W \otimes R$, for a $\tau - \mathbb{P}R$-twisted H/M-bundle W over Y_x, carries a τ-twisted action of H. The assignments $V \mapsto W$ and $W \mapsto V$ are mutually inverse, and the second assignment is our composition (5.3). $\square$

5.4 Remark. (i) The local statement in the proof is a weaker form of a result (Thm. 10.7 below) that we shall use in the classification of loop group representations.
(ii) The inverse map to (5.3) lifts a class from $K_H^{\tau+*}(X)$ to $K_H^{p^*\tau+*}(Y)$, tensors with the dual class $[R^\vee] \in K_H^{-\mathbb{P}R}(Y)$, and then extracts the M-invariant part. However, the last step requires a bit of care when using projective bundles, as it involves the Morita isomorphism relating two different projective bundle representatives for the same twisting.

6. Computation when the identity component is a torus

To ensure consistency of notation when the identity component G_1 is a torus T, we write N for G and W for $\pi_0 N$. Denoting, for any $f \in N$, by $N(f)$ the stabiliser in N of the component fT, we can decompose $K_N^\tau(N)$ as a sum over representatives $f \in N$ of the conjugacy classes in W:

$$K_N^\tau(N) \cong \bigoplus\nolimits_f K_{N(f)}^\tau\,(fT)\,. \tag{6.1}$$

(6.2) The identity component. With $H = N$ and $M = X = T$ in construction 5.1, Lemma 5.2 gives $K_N^\tau(T) = K_W^{\tau'}(Y)$. It is easy to describe the bundle $p : Y \to T$. A twisting class $[\tau] \in H_N^3(T)$ restricts to $H_T^3(T)$, hence to $H^1(T) \otimes H_T^2$, and contraction with the first factor gives a map $\kappa^\tau : H_1(T) \to H_T^2$. This gives a translation action of $\Pi := \pi_1 T$ on the set Λ^τ of τ-affine weights of T, and Y is the associated bundle $\mathfrak{t} \times_\Pi \Lambda^\tau$. If κ^τ is injective, as per our regularity condition (2.2), Y is a union of copies of $\mathfrak{t}$, labelled by $\Lambda^\tau / \kappa^\tau(\Pi)$, and integration along $\mathfrak{t}$ gives

$$K_W^{\tau'}(Y) = K_W^{\tau' - \sigma(\mathfrak{t}) - \dim T}\left(\Lambda^\tau / \kappa^\tau(\Pi)\right),$$

where the down-shift $\sigma(\mathfrak{t})$ in the twisting is defined by a W-equivariant Thom class of $\mathfrak{t}$, represented by a choice of spinors $\mathbf{S}(\mathfrak{t})$ with projective W-action.

(6.3) Affine Weyl action. We reformulate this construction by observing that the level $[\tau] \in H_N^3(T)$ has a leading term in $H_W^1(T; H_T^2)$, with respect to the Hochschild-Serre spectral sequence $E_2^{p,q} = H_W^p(T; H_T^q) \Rightarrow H_N^{p+q}(T)$. This term captures the W-action on the covering Y of T, but, more importantly, defines an affine action on Λ^τ of the *extended affine Weyl group* $W \ltimes \Pi$, extending the action of Π. Comparing orbits and stabilisers gives an equivalence of categories of equivariant bundles, and hence an isomorphism

$$K_W^{\tau' - \sigma(\mathfrak{t})}\left(\Lambda^\tau / \kappa^\tau(\Pi)\right) = K_{W \ltimes \Pi}^{\tau' - \sigma(\mathfrak{t})}(\Lambda^\tau)\,.$$

(6.4) A general component. Let now T^f be the T-centraliser of $f \in N$ and $\underline{T}$ its identity component. Then, fT is a homogeneous space, with discrete isotropy, for the combined action of $N(f) / \underline{T}$ by conjugation and of $\underline{\mathfrak{t}} := \mathfrak{t}^f$ by translation. We obtain an $N(f) \ltimes \underline{\mathfrak{t}}$-equivariant isomorphism of the form

$$fT \cong \left[(N(f) / \underline{T}) \ltimes \underline{\mathfrak{t}}\right] / W_{\mathrm{aff}}^e\,, \tag{6.5}$$

where the stabiliser W_{aff}^e of f is expressed, by projection to $N(f)/\underline{T}$, as a group extension

$$1 \to \underline{\Pi} \to W_{\mathrm{aff}}^e \to \widetilde{W}^f \to 1, \tag{6.6}$$

where $\underline{\Pi} := \pi_1 \underline{T}$ and $\widetilde{W}^f := [N(f)/\underline{T}]^f$ is in turn an extension of W^f (the W-centraliser of the component fT) by the finite group $[T/\underline{T}]^f$:

$$1 \to [T/\underline{T}]^f \to \widetilde{W}^f \to W^f \to 1.$$

Exactness on the right follows from the vanishing of $H^1_{\langle f\rangle}(T/\underline{T})$; that, in turn, follows from the absence of f-invariants in $\pi_1(T/\underline{T})$.

With $X = fT$, $H = N(f)$ and $M = \underline{T}$ in (5.1), an $N(f)$-equivariant twisting τ defines a covering space $Y \to fT$, with fibres the sets $\underline{\Lambda}^\tau$ of τ-affine weights of $\underline{T}$. Via (6.5), this cover is associated to an affine action of W^e_{aff} on $\underline{\Lambda}^\tau$, which is classified by the leading component of $[\tau] \in H^3_{N(f)}(fT)$ in

$$H^1_{N(f)/\underline{T}}(fT; H^2_{\underline{T}}) = H^1_{W^e_{\mathrm{aff}}}(H^2_{\underline{T}}). \tag{6.7}$$

6.8 Theorem. (i) *We have a natural isomorphism* $K^\tau_{N(f)}(fT) = K^{\tau'}_{W^e_{\mathrm{aff}}}(\underline{\Lambda}^\tau \times \underline{\mathfrak{t}})$.
(ii) *If τ is regular, this is also* $K^{\tau'-\sigma(\underline{\mathfrak{t}})-\dim\underline{\mathfrak{t}}}_{W^e_{\mathrm{aff}}}(\underline{\Lambda}^\tau)$*, and is free, of finite rank over* $\mathbb{Z}$.

Proof. The first part is Lemma 5.2. Provided that all stabilisers of W^e_{aff} on $\underline{\Lambda}^\tau$ are finite, part (ii) follows from (i) by integration along $\underline{\mathfrak{t}}$, and $\sigma(\underline{\mathfrak{t}})$ is the twisting of the equivariant Thom class.

Now, $\underline{\Pi} \subset W^e_{\mathrm{aff}}$ has finite index, and acts on $\underline{\Lambda}^\tau$ by translation, via the linear map $\kappa^\tau : \underline{\Pi} \to \underline{\Lambda}$, defined by restricting $[\tau]$ to $H^3_{\underline{T}}(\underline{T})$. Topological regularity of τ implies finiteness of the quotient $\underline{\Lambda}^\tau/W^e_{\mathrm{aff}}$ and of all stabilisers. □

6.9 Remark. The action of $N(f) \ltimes \underline{\mathfrak{t}}$ on fT leads to the presentation

$$fT \cong N(f) \ltimes \underline{\mathfrak{t}}/N^e_{\mathrm{aff}},$$

where the stabiliser N^e_{aff} of f is an extension $1 \to \underline{T} \to N^e_{\mathrm{aff}} \to W^e_{\mathrm{aff}} \to 1$. Without Lemma 5.2, the isomorphism (6.5) identifies $K^\tau_{N(f)}(fT)$ with $K^\tau_{N^e_{\mathrm{aff}}}(\underline{\mathfrak{t}})$. The right-hand side has a sensible topological interpretation as the K-theory of the associated quotient stack [FHT1]. It is tempting to integrate along $\underline{\mathfrak{t}}$ and land in the N^e_{aff}-equivariant twisted K-theory of a point. However, no *topological* definition of K-theory that we know allows this operation (cf. Remark 3.5); this could perhaps be done by C^*-algebra methods.

(6.10) Loop group interpretation. The isomorphism (6.5) identifies W^e_{aff} with π_1 of the homotopy quotient of fT by $N(f)$. This quotient turns out to be homotopy equivalent to the classifying space BL_fN. We can reveal this using the gauge action of L_fN on connections on a fixed principal N-bundle over the circle, with topological type $[fT]$. Fixing the fibre over a base-point, the space of holonomies becomes fT, while the residual symmetry group is $N(f)$. Finally, the space of connections is contractible.

Thus, $W^e_{\mathrm{aff}} = \pi_0 L_fN$. Now, each component of L_fN contains loops of minimal length, and so the subgroup $\Gamma_fN \subset L_fN$ of f-twisted *geodesic loops* is an extension of W^e_{aff} by $\underline{T}$, just like N^e_{aff}. In fact, Γ_fN is isomorphic to N^e_{aff}: to equate them, re-interpret the presentation of fT in Remark 6.9 as the quotient of $N(f) \times \underline{\mathfrak{t}}$, the set of flat bundles based at one point and with *constant* connection form, under the gauge action of Γ_fN.

The action of W^e_{aff} on $\underline{\Lambda}^\tau$ and its twisting τ' also have a loop group description. The connection picture gives an equivalence between the quotient stacks $fT/N(f)$ (by conjugation) and $\underline{\mathfrak{t}}/N^e_{\mathrm{aff}}$ (via W^e_{aff}). Our twistings come from $\mathbb{T}$-central extensions $(N^e_{\mathrm{aff}})^\tau$ of N^e_{aff}. The action of W^e_{aff} on $\underline{\Lambda}^\tau$ is then induced by the conjugation action of $(N^e_{\mathrm{aff}})^\tau$ on $\underline{T}$. The subgroup of $(N^e_{\mathrm{aff}})^\tau_\lambda$ stabilising a weight $\lambda \in \underline{\Lambda}^\tau$ is an extension of $(W^e_{\mathrm{aff}})_\lambda$ by $\underline{T}^\tau$. Pushing this out by the affine weight λ gives a $\mathbb{T}$-central extension of $(W^e_{\mathrm{aff}})_\lambda$, and these extensions assemble to the twisting τ'.

(6.11) Induction from conjugacy classes. The following result, together with Theorem 7.8 in the next section, is the basis for our original construction [F2] of twisted K-classes. For each element of the natural basis of Theorem 6.8.ii, it selects a distinguished $N(f)$-conjugacy class in fT, up to an overall ambiguity coming from the Lie algebra of the centre of the group (see Remark 6.13). We shall revisit this when discussing the Dirac families in Chapter V.

6.12 Proposition. *If τ is regular, $K_N^\tau(N)$ is spanned by classes supported on single conjugacy classes.*

Proof. Let us focus on the conjugation action of $N(f)$ on fT. An affine action of W_{aff}^e on $\underline{\mathfrak{t}}$ is inherited from the conjugation$\ltimes$translation action of the ambient group $(N(f)/\underline{T}) \ltimes \underline{\mathfrak{t}}$. There is also a W_{aff}^e-action on the affine copy $\underline{\Lambda}^\tau \otimes \mathbb{R}$ of $\underline{\mathfrak{t}}^*$, defined from the earlier action on $\underline{\Lambda}^\tau$. Such actions are classified respectively by the groups

$$H^1_{W_{\mathrm{aff}}^e}(\underline{\mathfrak{t}}) \cong \mathrm{Hom}_{\widetilde{W}^f}(\underline{\Pi}, \underline{\mathfrak{t}}) \quad \text{and} \quad H^1_{W_{\mathrm{aff}}^e}(\underline{\mathfrak{t}}^*) \cong \mathrm{Hom}_{\widetilde{W}^f}(\underline{\Pi}, \underline{\mathfrak{t}}^*)$$

($\widetilde{W}^f$ acts by conjugation). The first class is given by the natural map $\underline{\Pi} \to \underline{\mathfrak{t}}$; the second, by the map $\kappa^\tau \otimes \mathbb{R}$. Hence, the two actions of W_{aff}^e are isomorphic by some translate $\kappa_\nu^\tau : \underline{\mathfrak{t}} \to \underline{\mathfrak{t}}^*$ of the $\mathbb{R}$-linearised map $\kappa^\tau \otimes \mathbb{R}$.

A class in $K_{W_{\mathrm{aff}}^e}^{\tau' - \sigma(\mathfrak{t})}(\underline{\Lambda}^\tau)$ can now be pushed forward to $K_{W_{\mathrm{aff}}^e}^{\tau'}(\underline{\Lambda}^\tau \times \underline{\mathfrak{t}})$ by using the graph of the inverse map $(\kappa_\nu^\tau)^{-1}$. Under (6.8.i), its image in $K_{N(f)}^\tau(fT)$ is supported on a single conjugacy class, whenever the original lived on a single W_{aff}^e-orbit. □

6.13 Remark. (i) κ_ν^τ descends to a W^f-affine isogeny $\iota^\tau : f\underline{T} \to \underline{\Lambda}^\tau \otimes \mathbb{T}$. The quotient spaces $f\underline{T}/\widetilde{W}^f$ and $fT/N(f)$ are isomorphic: this is because $f\underline{T}$ covers fT/T and $\widetilde{W}^f$ surjects onto W^f (§6.4). The conjugacy classes appearing in Proposition 6.12 lie in the fibre of ι over the base-point $\underline{\Lambda}^\tau$ of the second torus. Specifically, a class in the (twisted) $K_{W_{\mathrm{aff}}^e}(\underline{\Lambda}^\tau)$ supported on a W_{aff}^e-orbit Ω corresponds to one in $K_{N(f)}^\tau(fT)$ with support at the single $\widetilde{W}^f$-orbit $f \cdot \exp\left((\kappa_\nu^\tau)^{-1}\Omega\right)$.
(ii) An ambiguity in the construction results from our freedom in identifying $\underline{\mathfrak{t}}$ and $\underline{\mathfrak{t}}^*$ as W^f-affine spaces, as we are free to translate by the W^f-invariant part of $\underline{\mathfrak{t}}$.

7. General compact groups

For any compact Lie group G, we will describe $K_G^\tau(G)$ in terms of the maximal torus T of G and its normaliser N. We must first recall some facts about disconnected groups; readers focusing on the connected case may skip ahead to §7.7. We keep the notations of §6.

(7.1) Diagram automorphisms. Choose a set of simple root vectors in $\mathfrak{g}_{\mathbb{C}}$, satisfying, along with their conjugates and the simple co-roots, the standard $\mathfrak{sl}_2$ relations.

7.2 Proposition. *Every outer automorphism of $\mathfrak{g}$ has a distinguished implementation, called diagram automorphism, which preserves $\mathfrak{t}$ and its dominant chamber and permutes the simple root vectors.*

Proof. The variety of Cartan sub-algebras in $\mathfrak{g}$ has the rational cohomology of a point, so any automorphism of $\mathfrak{g}$ fixes a Cartan sub-algebra, by the Lefschetz theorem. Composing with a suitable inner automorphism ensures that we preserve $\mathfrak{t}$ and the dominant chamber. Conjugation by T provides the freedom needed to permute the simple root vectors without scaling. □

7.3 Corollary. *As extension of $\pi_0 G$ by the identity component G_1, the group G has a reduction to an extension by the centre of G_1.*

Proof. The subgroup of G-elements whose Ad-action on $\mathfrak{g}$ is a diagram automorphisms meets every component of G, and meets G_1 in its centre. This is our reduction. □

(7.4) Conjugacy classes in G. The push-out of the extension (7.3) to the maximal torus T is called a *quasi-torus* $Q_T \subset G$; it meets every component of G in a translate of T. Q_T depends on T and a choice of dominant chamber. Choose $f \in Q_T$; its Ad-action on the dominant chamber must fix some interior points, so $\underline{\mathfrak{t}} = \mathfrak{t}^f$ contains $\mathfrak{g}$-regular elements. The identity component $\underline{T}$ of the invariant subgroup T^f is then a maximal torus of the centraliser G^f of f.

Call $W = N/T$, $W_1 = (N \cap G_1)/T$ the Weyl groups of G and G_1; we have $W = \pi_0 G \ltimes W_1$, by (7.3). Call $[f]$ the image of f in the quotient fT/T by T-conjugation. Conjugation by $N(f)$, the subgroup of N preserving the component fT, descends to an action of the group $W^f = \pi_0 N(f)$ on fT/T. Let $\underline{W} := W^f \cap W_1$, and $\underline{\widetilde{W}}$ its extension by $[T/\underline{T}]^f$ restricted from the group $\widetilde{W}^f$ of (6.6).

7.5 Lemma. *(i) The space of conjugacy classes fG_1/G_1 is $(fT/T)\,/\underline{W}$.*
(ii) The Weyl group of G_1^f is an extension by $\pi_0 T^f$ of the $\underline{W}$-stabiliser of $[f]$.

Proof. Part (i) reformulates Theorem ?.? of [BtD]: indeed, fT/T is the quotient of $f\underline{T}$ under conjugation by $[T/\underline{T}]^f$, whence it follows that $(fT/T)\,/\underline{W} \cong f\underline{T}/\underline{\widetilde{W}}$. That is the description in [BtD].

The normaliser of $\underline{T}$ in G_1^f is $N \cap G_1^f$, by regularity, and exactness of $1 \to T^f \to N \cap G_1^f \to \underline{W}$ implies (ii). $\square$

7.6 Remark. Translation by f identifies fT/T with the co-invariant torus T_f (quotient of T by the sub-torus $\{xfx^{-1}f^{-1} \mid x \in T\}$). The $\underline{W}$-action on fT/T is *affine* with respect to the quotient $\underline{W}$-action on T_f. However, the two $\underline{W}$-actions *agree* when f is a diagram automorphism ε: $\underline{W}$ indeed isomorphic to the Weyl group of $\mathfrak{g}^\varepsilon$ (Appendix A).

(7.7) The Weyl map ω. Decompose $K_G^\tau(G) = \bigoplus_f K_{G(f)}^\tau(fG_1)$ over a collection of representatives $f \in Q_T$ of the conjugacy classes in π_0; $G(f)$ denotes the stabiliser of the component fG_1. The $G(f)$-equivariant map

$$\omega : G(f) \times_{N(f)} fT \to fG_1, \qquad g \times ft \mapsto g \cdot ft \cdot g^{-1}$$

induces two morphisms in twisted K-theory, *restriction* ω^* and *induction* ω_*:

$$K_{N(f)}^\tau(fT) \cong K_{G(f)}^\tau\Big(G(f) \times_{N(f)} fT\Big) \underset{\omega^*}{\overset{\omega_*}{\rightleftarrows}} K_{G(f)}^\tau(fG_1).$$

The names will be justified in §14.

7.8 Theorem. *The composition $\omega_* \circ \omega^*$ is the identity.*

Consequently, $K_{G(f)}^\tau(fG_1)$ is a summand in $K_{N(f)}^\tau(fT)$, split as an $R(G)$-module. We will now idenfity it.

(7.9) Affine-regular weights. Recall that regular conjugacy classes in fG_1 are those with minimal stabiliser dimension. Any fT-representative then has infinitesimal stabiliser $\underline{\mathfrak{t}}$ (because $\underline{\mathfrak{t}}$ does contain $\mathfrak{g}$-regular elements, as noted in §7.4). Call a weight in $\underline{\Lambda}^\tau$ *affine-regular* if it corresponds to a regular conjugacy class in $f\underline{T}$, under the isomorphism $\kappa_\nu^\tau : \underline{\mathfrak{t}} \to \underline{\mathfrak{t}}^*$ from in the proof of Proposition 6.12. While there is an ambiguity in defining κ_ν^τ, it is subsumed by translation by the $\underline{W}$-invariant part of $\underline{\mathfrak{t}}$, which lies in the centre of $\mathfrak{g}$ (see §A.1): so it does not affect regularity. Clearly, affine regularity is preserved by the group W_{aff}^e defined in the previous section.

7.10 Theorem. $K_{G(f)}^\tau(fG_1)$ *is the summand in* $K_{N(f)}^\tau(fT)$ *corresponding to the regular weights:*

$$K_{G(f)}^\tau(fG_1) = K_{W_{\mathrm{aff}}^e}^{\tau' - \sigma(\underline{\mathfrak{t}}) - \dim T^f}\left(\underline{\Lambda}_{reg}^\tau\right).$$

With respect to Proposition 6.12, we are keeping the K-theory classes induced from regular conjugacy classes in G.

7.11 Remark. W_{aff}^{e} is called the *f-twisted, extended affine Weyl group* of G. It contains the f-twisted affine Weyl group $W_{\text{aff}}(\mathfrak{g}, f)$ which is generated by affine reflections in $\mathfrak{t}$. Regular weights are those *not* fixed by any *affine reflection*: see §10.4 and §A.9.

Proof of (7.8). The quotient spaces $fT\,/N(f)$ and $fG_1\,/G(f)$ are isomorphic under ω (Lemma 7.5). We shall show that $\omega_* \circ \omega^*$ is the identity on small neighbourhoods of conjugacy classes: a Mayer-Vietoris argument then implies that the map is a global isomorphism. However, $K^{\tau}_{N(f)}(fT)$ is spanned by classes induced from single orbits (Prop. 6.12). Their ω_*-images are fixed by $\omega_* \circ \omega^*$, so the theorem follows.

We need a local model for the Weyl map. We work near f, which was arbitrary in Q_T. Because $\underline{T}$ contains regular elements, $N^f := N \cap G^f$ is the normaliser of $\underline{T}$ in G^f. Now, the translate $f \cdot \exp(\mathfrak{g}^f)$ is a local slice for G_1-conjugation near f, while $f \cdot \exp(\underline{\mathfrak{t}})$ is one for T-conjugation in Q_T. Therefore, a local, G^f-equivariant model for ω is the Dirac induction map of §4.12,

$$G^f \times_{N^f} \underline{\mathfrak{t}} \to \mathfrak{g}^f, \tag{7.12}$$

and our claim reduces to Proposition 4.13. □

Proof of (7.10). We use the construction (6.12) of K-classes from single conjugacy classes. Let $f \in Q_T$ and observe, from $f\underline{T} \cong f \cdot \underline{\mathfrak{t}}/\underline{\Pi}$ and Lemma 7.5, that the Weyl group of G^f is identified with the stabiliser in W_{aff}^{e} of the associated weight. Singular weights are then fixed by the Weyl reflection in some $\mathfrak{sl}_2$ centralising f, and their K-classes are killed by the local induction (7.12). Near a regular f, on the other hand, the local model for ω is an isomorphism, so regular weights contribute non-zero generators in $K^{\tau}_{G}(G)$. □

IV Loop groups and admissible representations

In this chapter, we summarise some basic facts about loop groups, twisted loop groups and their Lie algebras, as well as the classification of admissible representations in terms of the action on affine regular weights of the extended affine Weyl group. The key result, Thm. 10.2, combines the theorem of the lowest weight with Mackey's irreducibility criterion; while it is certainly known, it does not seem to appear in the literature in this form. (See, however, [TL] for some of the simple groups).

We need to distinguish between representations of the polynomial loop algebras and their Hilbert space completions, and we convene to mark *uncompleted* spaces by a prime.

8. Refresher on affine algebras

(8.1) Affine algebras. We use the notations of §4; in particular, $\mathfrak{g}$ is now simple. The *Fourier polynomial loop algebra* $L'\mathfrak{g}_{\mathbb{C}}$ has the Fourier basis $\xi_a(m) = z^m\xi_a$. Its *basic central extension* $\widetilde{L}'\mathfrak{g} := \mathrm{i}\mathbb{R}K \oplus L'\mathfrak{g}$, with central generator K, is defined by the 2-cocycle sending $\xi \wedge \eta \in \Lambda^2 L'\mathfrak{g}$ to $K \cdot \mathrm{Res}_{z=0}\langle d\xi|\eta\rangle$. The *affine Lie algebra* $\widehat{L}'\mathfrak{g} = \widetilde{L}'\mathfrak{g} \oplus \mathrm{i}\mathbb{R}E$ arises by adjoining a new element $\mathrm{i}E$, where the *energy* E satisfies $[E, K] = 0$ and $[E, \xi(n)] = n\xi(n)$, for any $\xi \in \mathfrak{g}$. Unlike $\widetilde{L}'\mathfrak{g}$, $\widehat{L}'\mathfrak{g}$ carries an ad-invariant bilinear form, extending the basic one on $\mathfrak{g}$:

$$\langle k_1K + \xi_1 + e_1E \,|\, k_2K + \xi_2 + e_2E\rangle \mapsto \frac{1}{2\pi}\int_0^{2\pi} \langle \xi_1(t)|\xi_2(t)\rangle dt + k_1e_2 + k_2e_1. \tag{8.2}$$

(8.3) Lowest-weight modules. A projective representation of $L'\mathfrak{g}$ has *level k* if it extends to a strict representation of $\widetilde{L}'\mathfrak{g}$ in which K acts as the scalar k. This means that we can choose the action $R_a(m)$ of $\xi_a(m)$ so that

$$[R_a(m), R_b(n)] = f_{ab}^c R_c(m+n) + km\delta_{ab}\delta_{m,-n}.$$

Call $\mathfrak{h} := \mathrm{i}\mathbb{R}K \oplus \mathfrak{t} \oplus \mathrm{i}\mathbb{R}E$ a *Cartan sub-algebra* of $\widehat{L}'\mathfrak{g}$, and let $\mathfrak{N} := \bigoplus_{n>0} z^n \mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{n} \subset L'\mathfrak{g}_{\mathbb{C}}$. A *lowest weight vector* in an $\widehat{L}'\mathfrak{g}$-module $\mathbf{H}'$ is an $\mathfrak{h}$-eigenvector killed by the complex conjugate Lie algebra $\overline{\mathfrak{N}}$. Call $\mathbf{H}'$ a *lowest weight module*, with lowest weight $(k, -\lambda, m)$, if it is generated by a lowest weight vector $\mathbf{v}$ of that $(K, \mathfrak{t}, E)$-weight. The factorisation $U(\widehat{L}'\mathfrak{g}) = U(\mathfrak{N}) \otimes U(\mathfrak{h}) \otimes U(\overline{\mathfrak{N}})$ shows that $\mathbf{H}'$ is generated by $\mathfrak{N}$ from $\mathbf{v}$. Defining the *positive alcove* $\mathfrak{a} \subset \mathfrak{t}$ as the subset of dominant elements ξ satisfying $\theta(\xi) \leq 1$, we have the following:

8.4 Proposition. *In a $(k, -\lambda, m)$-lowest-weight module, the weight (k, ω, n) of any other $\mathfrak{h}$-eigenvector satisfies $n - m \in \mathbb{Z}$ and $(\omega + \lambda)(\xi) + n > m$, for all ξ inside $\mathfrak{a}$.*

Proof. All weights of $\mathfrak{N}$ verify these conditions, with $\lambda, m = 0$. □

(8.5) Integrable modules. A lowest-weight module is *integrable* if the action exponentiates to the associated simply connected loop group. We are dealing with infinite-dimensional spaces, so a precise definition is a bit delicate; but there are some easy equivalent Lie algebra criteria: for instance, it suffices that the action should exponentiate on all root $\mathfrak{sl}_2$-subgroups, [K, III], [PS, VII]. Integrable representations are unitarisable, completely reducible, and the irreducible ones are parametrised by their lowest weights $(k, -\lambda, m)$, in which k must be a non-negative integer and λ a dominant T-weight satisfying $\lambda \cdot \theta \leq k$ in the basic inner product. These weights correspond to points of the scaled alcove $k \cdot \mathfrak{a}$.

(8.6) Spinors. The *complex Clifford algebra* $\mathrm{Cliff}(L'\mathfrak{g}^*)$ is generated by the odd elements $\{\psi^a(m)\}$ dual to $\{\xi_a(m)\}$, satisfying

$$\psi^a(m)\psi^b(n) + \psi^b(n)\psi^a(m) = 2\delta_{ab}\delta_{m-n}.$$

Choose an irreducible, $\mathbb{Z}/2$-graded, positive energy module $\mathbf{S}'$ of $\mathrm{Cliff}(L'\mathfrak{g}^*)$. As a vector space, this can be identified with the graded tensor product $\mathbf{S}(0) \otimes \Lambda^\bullet(z\mathfrak{g}_{\mathbb{C}}[z])$, for an irreducible, graded spin module $\mathbf{S}(0)$ of $\mathrm{Cliff}(\mathfrak{g}^*)$. $\mathbf{S}'$ carries a hermitian metric, in which $\psi^a(n)^* = \psi^a(-n)$; so $\psi(\mu)$ is self-adjoint for $\mu \in L'\mathfrak{g}^*$. There are obvious actions of $\mathfrak{g}$ and E on $\mathbf{S}'$, intertwining with $\mathrm{Cliff}(L'\mathfrak{g}^*)$. The lowest E-eigenvalue is 0, achieved on $\mathbf{S}(0) \otimes 1$. Setting

$$K \mapsto h^\vee, \qquad \xi_a(m) \mapsto \sigma_a(m) := -\frac{1}{4} \sum\nolimits_{p+q=m} f_{bc}^a \, \psi^b(p)\psi^c(q) \tag{8.7}$$

extends them to an action of $\widehat{L}'\mathfrak{g}$, with intertwining relation $\left[\sigma_a(m), \psi^b(n)\right] = f_{ca}^b \psi^c(m+n)$. One derives (8.7) by considering the adjoint representation $L'\mathfrak{g}$ to the orthogonal Lie algebra $\mathfrak{so}_{\mathrm{res}}(L\mathfrak{g})$, "restricted" as in [PS] with respect to the splitting $L'\mathfrak{g}_{\mathbb{C}} = z\mathfrak{g}_{\mathbb{C}}[z] \oplus \mathfrak{g}_{\mathbb{C}}[z^{-1}]$. Formula (8.7) is then the quadratic expression of the spin representation of $\mathfrak{so}_{\mathrm{res}}$ in terms of Clifford generators [KS].

The following key result follows from the Kac character formula. It is part of affine algebra lore; but see [FHT2] for a proof.

8.8 Proposition. *As a representation of $\widehat{L}'\mathfrak{g}$, $\mathbf{S}'$ is a sum of copies of the integrable irreducible representation of level $h^\vee$ and lowest weight $(-\rho)$. The lowest weight space, which is isomorphic to the multiplicity space, is also the $\mathfrak{g}$-lowest-weight space in $\mathbf{S}(0)$, and is a graded, irreducible $\mathrm{Cliff}(\mathfrak{t}^*)$-module.*

9. Twisted affine algebras

The loop algebras $L\mathfrak{g}$ have *twisted versions*, arising from the automorphisms of non-trivial principal G-bundles over the circle. They are closely related to the outer automorphisms of $\mathfrak{g}$, and also to the *twisted simple affine algebras* in [K, Tables Aff 2,3]: each entry of the latter is the universal central extension of a twisted loop algebra, plus an outer derivation E.

(9.1) The algebra $L_\varepsilon\mathfrak{g}$ of loops in $\mathfrak{g}$ twisted by an automorphism ε depends, up to isomorphism, only on the conjugacy class of ε in the outer automorphism group of $\mathfrak{g}$. Thanks to Proposition 7.2, we may as well assume that ε is a diagram automorphism. When $\mathfrak{g}$ is simple, this will have order $1, 2$ or 3; in general, we insist that the order r should be finite. This leads to an attractive algebraic model for $L_\varepsilon\mathfrak{g}$ as the invariant part of a copy of $L\mathfrak{g}$, based on the r-fold cover $\sqrt[r]{S^1}$ of the unit circle S^1, under the Galois automorphism which rotates the cover by $2\pi/r$ and applies ε point-wise. To find the geometric meaning of this construction, let G_1 be the simply connected group with Lie algebra $\mathfrak{g}$ and $G = \mathbb{Z}/r \ltimes_\varepsilon G_1$. The quotient of the trivial bundle $\sqrt[r]{S^1} \times G$ under the action of $\mathbb{Z}/r$ which rotates the circle and left-translates the fibres G is a principal G-bundle P over S^1, and its Lie algebra of gauge transformations is $L_\varepsilon\mathfrak{g}$.

(9.2) Standard form. Let $\mathfrak{g}$ be simple, for the rest of this section. The (smooth) twisted affine algebra $\widehat{L}_\varepsilon\mathfrak{g}$ is the invariant part of $\widehat{L}\mathfrak{g}$, in our Galois construction above. Its structure is described in [K, VI, VIII]. Inherited from the ambient $\widehat{L}\mathfrak{g}$ is a linear decomposition $\widehat{L}_\varepsilon\mathfrak{g} = \mathrm{i}\mathbb{R}K \oplus L\mathfrak{g} \oplus \mathrm{i}\mathbb{R}E$. We now *rescale* K and E to $r\times$, resp. $1/r\times$ their original values. Then, E is the natural generator for the rotation of the unit (downstairs) circle, while the bilinear form (8.2) is still ad-invariant. Using the *standard connection* ∇_0 on P, descended from the trivial connection on $\sqrt[r]{S^1}$, the 2-cocycle of the central extension $\widetilde{L}_\varepsilon\mathfrak{g} := \mathrm{i}\mathbb{R}K \oplus L_\varepsilon\mathfrak{g}$ is again expressed as an integral over the unit circle, and the Lie bracket takes the following form:

$$[\xi, \eta]\,(t) = [\xi(t), \eta(t)] + \frac{K}{2\pi\mathrm{i}} \oint \langle \nabla_0\xi | \eta \rangle. \tag{9.3}$$

(9.4) Lowest-weight modules. The rôles of $\mathfrak{t}$, $\mathfrak{h}$ and $\mathfrak{N}$ are taken over by their Galois invariants within the ambient $L'\mathfrak{g}$; we denote them by underlines. The structure of weights and roots parallel the untwisted case; the details are summarised in Appendix A. Note, however, that Corollary 9.9 below will imposes a small distinction for the weight lattice for twisted $\mathrm{SU}(2\ell+1)$; see (A.10).

The underlined Doppelgängers for ρ, θ and $\mathfrak{a}$ require a comment: $\underline{\rho}$ has the obvious meaning, the half-sum of positive roots for $\underline{\mathfrak{g}} := \mathfrak{g}^\varepsilon$, but $\underline{\theta}$, which cuts out the *positive alcove* $\underline{\mathfrak{a}}$ from the dominant chamber of $\underline{\mathfrak{t}}$ by the relation $\underline{\theta}(\xi) \le 1/r$, is *not* the highest root of $\underline{\mathfrak{g}}$, but rather the highest weight of $\mathfrak{g}/\underline{\mathfrak{g}}$. Therewith, the analogue of Proposition 8.4 holds true.

A geometric sense in which $\underline{\mathfrak{a}}$ plays tho rôle of $\mathfrak{a}$ is the following. Let $\mathcal{A}$ denote the space of smooth connections on the bundle P; the quotients $\mathcal{A}\,/L_\varepsilon G_1$ (by gauge transformations) and $\varepsilon G_1\,/G_1$ (by conjugation) are isomorphic by the holonomy map. The classification (A.7) of twisted conjugacy classes gives the following.

9.5 Proposition. *Every smooth connection on P is a smooth gauge transform of $\nabla_0 + \xi dt$, for a unique $\xi \in \underline{\mathfrak{a}}$. That is, $\underline{\mathfrak{a}}$ is a global slice for $L_\varepsilon G$: $\underline{\mathfrak{a}} \cong \mathcal{A}\,/L_\varepsilon G$.* □

In a *level k* representation of $\widehat{L}'_\varepsilon\mathfrak{g}$, K acts as the scalar k. A *lowest weight vector* is an $\mathfrak{h}$-eigenvector killed by $\overline{\underline{\mathfrak{N}}}$, and a *lowest-weight module* is one generated by a lowest weight vector. We call such a module *integrable* if the action of all the root $\mathfrak{sl}_2$ sub-algebras of $\widehat{L}'_\varepsilon\mathfrak{g}$ is so; in that case, the module is unitarisable, and the Lie algebra action exponentiates to one of $\widehat{L}_\varepsilon G$ on the Hilbert space completion. Integrable representations are semi-simple, and the irreducible ones are parametrised by their level k and their lowest weight $(k, -\lambda)$, in which λ is dominant and satisfies $\underline{\theta} \cdot \lambda \le k/r$.

(9.6) The Clifford algebra. A basis for $\widehat{L}'_\varepsilon\mathfrak{g}_\mathbb{C}$ suited to calculations arises from a complex orthonormal ε-eigen-basis $\{\xi_a\}$ of $\mathfrak{g}_\mathbb{C}$, so chosen that the indexing set carries an involution $a \leftrightarrow \bar{a}$ with $\xi_{\bar{a}} = -\xi_a^*$. If $\varepsilon(a) \in \mathbb{Z}/r$ corresponds to the ε-eigenvalue of ξ_a, then $\{\xi_a(m)\}$ forms a basis of $\mathfrak{g}$, as $m + \varepsilon(a)/r$ ranges over $\mathbb{Z}$. Raising and lowering indexes involves a bar; for instance, the relations in the complex Clifford algebra of $L'_\varepsilon\mathfrak{g}^*$ are $\left[\psi^a(m), \psi^b(n)\right] = 2\delta_{a\bar{b}} \cdot \delta_{m-n}$.

A positive energy, graded spin module $\mathbf{S}'$ can be identified, as a vector space, with $\mathbf{S}(0) \otimes \Lambda^\bullet(\underline{\mathfrak{N}})$, for a graded spin module $\mathbf{S}(0)$ of Cliff$(\underline{\mathfrak{g}}^*)$. As in (8.7), the obvious actions of $\underline{\mathfrak{g}}$ and E extend to a lowest-weight representation of $\widehat{L}'_\varepsilon\mathfrak{g}$, with a bar in the raised index a, but, remarkably enough, with the same $h^\vee$, independently of ε. As in Proposition 8.8, the representation can be identified using the Kac character formula; its lowest-weight space is the lowest $\underline{\mathfrak{g}}$-weight space in $\mathbf{S}(0)$, has pure weight $(-\underline{\rho})$ and is a graded irreducible Cliff$(\underline{\mathfrak{t}}^*)$-module.

(9.7) The loop group. The extensions in §8.1 and (9.3) are so normalised as to generate all central extensions of LG by the circle group $\mathbb{T}$. Call $L_\varepsilon G_1$ the twisted loop group of G_1.

9.8 Proposition. *$\widetilde{L}_\varepsilon\mathfrak{g}$ is the Lie algebra of a* basic central extension *$\widetilde{L}_\varepsilon G$ of $L_\varepsilon G$: the central circle is parametrised by $\left\{ z^K \middle| |z| = 1\right\}$, and its Chern class generates $H^2\left(L_\varepsilon G_1; \mathbb{Z}\right) = \mathbb{Z}$.*

Proof. The untwisted case is handled in [PS], so we focus on $r > 1$. Being the space of sections of a G_1-bundle over S^1, $L_\varepsilon G_1$ is connected and simply connected. Further, $\pi_2 L_\varepsilon G_1 = H^1\left(S^1; \pi_3 G_1\right) = \mathbb{Z}$, and Hurewicz gives us $H^2(L_\varepsilon G_1; \mathbb{Z}) = \mathbb{Z}$. Since $\pi_2 LG_1 = H^1(\sqrt[r]{S^1}; \pi_3 G_1)$, the restriction $H^2(LG_1) \to H^2(L_\varepsilon G_1)$ has index r. Our extension of $L_\varepsilon G_1$ will be the r-th root of the restriction of $\widetilde{L}G_1$, the basic extension of the ambient, untwisted loop group. Having fixed the cocycle (9.3), the obstructions to existence and uniqueness of this root are topological, living in H^2 and H^1 of $L_\varepsilon G_1$ with $\mathbb{Z}/r$-coefficients, respectively; and they vanish as seen. Finally, we have a semi-direct decomposition $L_\varepsilon G \cong \mathbb{Z}/r \ltimes L_\varepsilon G_1$, and the ε-action on $L_\varepsilon G_1$ preserves the cocycle (9.3), so it lifts to an automorphism action on the central extension (again, by vanishing of the topological obstructions). We let $\widetilde{L}_\varepsilon G = \mathbb{Z}/r \ltimes_\varepsilon L_\varepsilon G_1$. $\square$

9.9 Corollary. *The basic extension $\widetilde{L}_\varepsilon G$ restricts trivially to the constant subgroup G_1^ε, except when $G_1 =$ SU$(2\ell+1)$ and $r = 2$, in which case $G_1^\varepsilon =$ SO$(2\ell+1)$, and we obtain the $Spin^c$-extension.*

Proof. The flag variety $L_\varepsilon G_1/G_1^\varepsilon$ is simply connected, with no H^3. (This follows, for instance, from its Bruhat stratification by even-dimensional cells.) The Leray sequence for the fibre bundle $L_\varepsilon G_1 \twoheadrightarrow L_\varepsilon G_1/G_1^\varepsilon$ shows that $H^2(L_\varepsilon G_1)$ surjects onto $H^2(G_1^\varepsilon)$. However, G_1^ε is simply connected, save in the cases listed, whence the result. $\square$

10. Representations of $L_f G$

We now classify the admissible representations of the loop groups at levels $\tau - \underline{\sigma}$ for which τ is regular, in terms of the affine Weyl action on regular weights.

(10.1) Notational refresher. Let f be an element of the quasi-torus Q_T and call $L_f G$ is the f-twisted smooth loop group of G (§1.5), τ a regular central extension and $\underline{\sigma}$ the extension defined by the spin module $\mathbf{S}$ of $L_f\mathfrak{g}^*$ (§1.6). Gradings are incorporated into our twistings. The extended affine Weyl group $W_{\text{aff}}^e = \pi_0 L_f N$ acts on $\underline{\Lambda}^\tau$ by conjugating the central extension of $\underline{T}$, and a tautological twisting τ' is defined for this action, wherein each τ-affine weight defines a $\mathbb{T}$-central extension of its stabiliser in W_{aff}^e (§6.10). We now restate Theorem 4 without Clifford algebras; it is the lowest-weight classification of integrable representations of affine Lie algebras, enhanced to track the action of the components of $L_f G$.

10.2 Theorem. (i) *The category of admissible representations of $L_f G$ of level $\tau - \underline{\sigma}$ is equivalent to that of finite-dimensional, W_{aff}^e-equivariant, $\tau' - \sigma(\underline{\mathfrak{t}})$-twisted vector bundles over $\underline{\Lambda}^\tau_{reg}$.*

(ii) *The K-groups of graded admissible representations are naturally isomorphic to the twisted equivariant K-theories* $K^{\tau'-\sigma(\underline{\mathfrak{t}})+*}_{W^e_{\mathrm{aff}}}(\underline{\Lambda}^\tau_{reg})$.

Briefly, the equivalence in part (i) arises as follows. A regular weight μ defines a polarisation of $L_f\mathfrak{g}$, which selects, for each admissible representation $\mathbf{H}$, a lowest-weight space in $\mathbf{H}\otimes\mathbf{S}$ with respect to $L_fG \ltimes \mathrm{Cliff}(L_f\mathfrak{g}^*)$. The $(-\mu)$-eigen-component under $\underline{T}$ of this lowest weight-space is a $\mathrm{Cliff}(\underline{\mathfrak{t}})$-module, and factoring out the spinors on $\mathfrak{t}$ gives the fibre of our vector bundle at $\mu \in \underline{\Lambda}^\tau$. The reader may wish to consult the simple Example 10.9 at the end of this section, where $G = N$.

The inverse equivalence arises "morally" from Dirac induction. Each $\mu \in \underline{\Lambda}^\tau_{reg}$ determines a regular co-adjoint orbit $\mathfrak{O}_\mu \subset L_f(\mathfrak{g}^*)^\tau$, over which a twisted representation of the W^e_{aff}-stabiliser defines a $(\tau-\sigma(\underline{\mathfrak{t}}))$-twisted, L_fG-equivariant vector bundle. The Dirac index of this bundle along $\mathfrak{O}_\mu$, coupled to the *highest-weight* spinors, is the desired representation of L_fG. Its level $(\tau-\underline{\sigma})$ arises from the shift by the level $\sigma(\underline{\mathfrak{t}})-\underline{\sigma}$ of the highest-weight spinors on $L_f\mathfrak{g}/\underline{\mathfrak{t}}$.

Dirac induction in infinite dimensions is only a heuristic notion, but can be realised in this case by the *Borel-Weil construction*, as a space of holomorphic sections [PS]. We will review that in §16, where it is needed, but we will make no use of it this section.

Proving (10.2) requires some preparation. Split $\mathfrak{g}$ into its centre $\mathfrak{z}$ and derived sub-algebra $\mathfrak{g}'$.

10.3 Proposition. *$L_f\mathfrak{g}'$ splits canonically into a sum of simple, possibly twisted loop algebras. Central extensions of $L_f\mathfrak{g}$ are sums of extensions of $L_f\mathfrak{z}$ and of the simple summands.*

Proof. In the decomposition of $\mathfrak{g}'$ into simple ideals, f-conjugation permutes isomorphic factors. To a cycle C of length $\ell(C)$ in this permutation, we assign one copy of the underlying simple summand $\mathfrak{g}(C)$ and the automorphism $\varepsilon(C) := \mathrm{Ad}(f)^\ell$. This is a diagram automorphism of $\mathfrak{g}(C)$, whose fixed-point sub-algebra is isomorphic to that of $\mathrm{Ad}(f)$ on the summand $\mathfrak{g}(C)^{\oplus\ell}$ in $\mathfrak{g}'$. Then, $L_f\mathfrak{g}'$ is isomorphic to the sum of loop algebras $L_{\varepsilon(C)}\mathfrak{g}(C)$, with the loops parametrised by the $\ell(C)$-fold cover of the unit circle. The splitting arises from the eigenspace decomposition of $\mathrm{Ad}(f)$ on $\mathfrak{g}(C)^{\oplus\ell}$. As the summands are simple ideals, uniqueness is clear. Finally, the splitting of the extension follows from the absence of one-dimensional characters of the simple summands. ☐

(10.4) More on W^e_{aff}. The proposition splits $\underline{\mathfrak{t}}$ into $\underline{\mathfrak{z}} := \mathfrak{z}^f$ and the sum of the Cartan sub-algebras $\underline{\mathfrak{t}}(C)$. Call $\tau\cdot\underline{\mathfrak{a}} \in \underline{\mathfrak{t}}$ the product of $\underline{\mathfrak{z}}$ and the positive alcoves (§8.3, §9.2) in the $\underline{\mathfrak{t}}(C)$, scaled by the simple components of the level $[\tau]$, and let $\tau\cdot\underline{\mathfrak{a}}^*$ be its counterpart in $\mathfrak{t}^*$ in the basic inner product on $\mathfrak{g}'$. Reflection about the walls of $\tau\cdot\underline{\mathfrak{a}}^*$ generate a normal subgroup $W_{\mathrm{aff}}(\mathfrak{g},f) \subset W^e_{\mathrm{aff}}$, under whose action the transforms of the alcove are distinct and tessellate $\underline{\mathfrak{t}}^*$ (A.5). The two groups agree when G is simply connected, but in general we have an exact sequence

$$1 \to W_{\mathrm{aff}}(\mathfrak{g},f) \to W^e_{\mathrm{aff}} \to \pi := \pi_0 L_fG \to 1; \tag{10.5}$$

note that the sequence is *split* by the inclusion of π in W^e_{aff} as the stabiliser of $\tau\cdot\mathfrak{a}$.

Regular are those weights not lying on any alcove wall. The alcoves correspond to positive root systems on $L_f\mathfrak{g}$ which are conjugate to the standard one (§9.4), the simple roots being the outward normals to the walls. The positive root spaces span a polarisation of $L_f\mathfrak{g}'$; the various polarisations, plus the original one on $L_f\mathfrak{z}$, are conjugate under $\Gamma_f N \subset L_fG$, so they define the same class of admissible representations.

(10.6) Mackey decomposition in K-theory. Let H be a group, M a normal subgroup, v a central extension of H. Conjugation leads to an action of H/M on isomorphism classes of v-representations of M. Let Y be a family of isomorphism classes, satisfying the conditions

(i) Y is stable under H/M;
(ii) Every point in Y has finite stabiliser in H/M;

(iii) The M-automorphisms of any representation in Y are scalars.

There is a tautological projective vector bundle $\mathbb{P}R$ over Y, whose fibre $\mathbb{P}R_y$ at $y \in Y$ is the projective space on a representation of isomorphism type y. Its uniqueness up to canonical isomorphism, and hence H-equivariance, follow from condition (iii). The bundle defines a $\mathbb{T}$-central extension of the action groupoid of H^v on Y. This central extension is split over M^v, so dividing out by the latter gives a central extension, or twisting, v' for the H/M-action on Y.

Call an H-representation *Y-admissible* if its restriction to M is a finite-multiplicity sum of terms of type in Y, with only finitely many H/M-orbit types. For instance, this includes all induced representations $\mathrm{Ind}_M^K(R_y)$. The same construction as in Lemma 5.2 establishes the following:

10.7 Proposition. *The category of Y-admissible representations of H is equivalent to that of v'-twisted, H/M-equivariant vector bundles over Y, supported on finitely many orbits.* □

In this equivalence, a M-representation $\mathbf{H}$ is sent to the bundle whose fibre at y is $\mathrm{Hom}^M(R_y, \mathbf{H})$. Conversely, to a bundle over Y we associate its space of sections. The relation to Construction 5.1 can be made explicit by choosing a representation $\mathbf{H}$ of H^v containing all elements of Y. The projective bundle $\mathbb{P}\mathrm{Hom}^M(R; \mathbf{H})$ over Y gives a model for the twisting v' of the H/M-action.

Proof of (10.2). The unitary lowest-weight representations of the Lie algebra correspond to the admissible ones of the simply connected cover of the identity component $(L_f G)_1$. For the simple summands, integrable representations are classified by lowest-weights [K]. Analytic regularity of τ on the centre $L_f\mathfrak{z} \cong \underline{\mathfrak{z}} \oplus L_f\mathfrak{z}/\underline{\mathfrak{z}}$ means that the second summand has a unique irreducible lowest-weight representation. Unitary irreducibles of $\underline{\mathfrak{z}}$ are labelled by the points of the τ-affine dual space. Descent of representations to $(L_f G)_1$ is controlled by an integrality constraint imposed by $\underline{T}$: parametrising the admissible irreducibles of $(L_f G)_1$ by their lowest weights $(-\lambda)$, the shifted weights $(\lambda + \underline{\rho})$ range over $\underline{\Lambda}^\tau_{reg+} := \underline{\Lambda}^\tau_{reg} \cap \tau \cdot \underline{\mathfrak{a}}^*$.

As $W^e_{\mathrm{aff}}(\mathfrak{g}, f)$ acts freely on $\underline{\Lambda}^\tau_{reg}$, and the orbits are in bijection with the points in $\underline{\Lambda}^\tau_{reg+}$, we get an identification

$$K^{\tau' - \sigma(\underline{\mathfrak{t}})}_{W^e_{\mathrm{aff}}}(\underline{\Lambda}^\tau_{reg}) = K^{\tau' - \sigma(\underline{\mathfrak{t}})}_{\pi}(\underline{\Lambda}^\tau_{reg+}). \tag{10.8}$$

We apply Proposition 10.7 to $H = L_f G$, $M = (L_f G)_1$, $v = \tau - \underline{\sigma}$, $Y = \underline{\Lambda}^\tau_{reg+}$. The actions of π described in §10.1 and §10.7 do match, because the (sign-reversed) lowest weight $(\underline{\sigma}, \underline{\rho})$ of $\mathbf{S}$ is π-invariant. To conclude the proof, it remains to identify the π-twistings v' and $\tau' - \sigma(\underline{\mathfrak{t}})$.

The subgroup of N^e_{aff} lying over π preserves the lowest-weight space in any v-representation $\mathbf{H}$ of $L_f G$, and so the projective action of π on the resulting lowest-weight bundle over Y represents v'. Similarly, a model for τ' arises from the action of π on the lowest-weight space in $\mathbf{H} \otimes \mathbf{S}$, distributed over the (sign-reversed) eigenvalues in $\underline{\Lambda}^\tau_{reg+}$. The second bundles differs from the first by a factor of $\mathbf{S}(\underline{\mathfrak{t}})$, and this represents the twisting $\sigma(\underline{\mathfrak{t}})$. □

(10.9) Example: $G = N$. Let $V := L_f\mathfrak{t} \ominus \underline{\mathfrak{t}}$ and $L_f N \cong N^e_{\mathrm{aff}} \ltimes \exp(V)$, as in §2.3. Regularity of τ confines us to sums of Heisenberg extensions of V and topologically regular extensions $\Gamma_f N^\tau$. The lowest-weight module $\mathbf{F}$ of $\exp(V)$ carries a (projective) intertwining action of $\pi_0 N^e_{\mathrm{aff}}$. An admissible representation $\mathbf{H}$ of $L_f N$ factors then as $\mathbf{F} \otimes \mathrm{Hom}^V(\mathbf{F}; \mathbf{H})$, where the second factor is (the ℓ^2 completion of) a *weight module* of N^e_{aff}, which means that it is $\underline{T}$-semi-simple, of finite type. Our classification now becomes the following, more precise

10.10 Proposition. *Global sections give an equivalence from the category of W^e_{aff}-equivariant, τ'-twisted vector bundles on $\underline{\Lambda}^\tau$ with that of weight τ-modules of N^e_{aff}.*

It is understood here that $\underline{T}^\tau$ acts with weight λ on the fibre at $\lambda \in \underline{\Lambda}^\tau$. The proposition follows directly from Prop. 10.7. Weight modules split into irreducibles, which are induced from stabilisers of single weights.

V From representations to K-theory

To an admissible representation $\mathbf{H}$ of LG at fixed level $\tau - \sigma$, we assign a family of Fredholm operators parametrised by an affine copy of $L\mathfrak{g}^*$, equivariant for the *affine action* of the loop group LG at the shifted level τ, defined in (§12). The underlying space of the family is $\mathbf{H} \otimes \mathbf{S}$, and the operator family is the analogue of the one in §4, but is based on the *Dirac-Ramond operator*. We recall this operator in §11, and reproduce the calculation [L, T] of its Laplacian, which we extend to twisted algebras. Our family defines an LG-equivariant twisted K-theory class over $L\mathfrak{g}^*$, which we identify, when $\mathbf{H}$ is irreducible, with the Thom push-forward of the natural line bundle on a single, integral co-adjoint orbit. The passage from representation to orbit and line bundle is an inverse of Kirillov's quantisation of co-adjoint orbits. Finally, the affine copy of $L\mathfrak{g}^*$ carrying our family can be identified with the space of $\mathfrak{g}$-connections over the circle with the gauge action, leading to an interpretation of our family as a cocycle for $K_G^\tau(G)$.

11. The affine Dirac operator and its square

Let $\mathfrak{g}$ be simple and let $\mathbf{H}'$ be a lowest weight module for $\widehat{L}'\mathfrak{g}$, with lowest weight $(k, -\lambda, 0)$. Consider the following formally skew-adjoint operator on $\mathbf{H}' \otimes \mathbf{S}'$:

$$\not{D} = \not{D}_0 := R_a(m) \otimes \psi^a(-m) + \frac{1}{3} \cdot \sigma_a(m)\psi^a(-m). \tag{11.1}$$

This is known to physicists as the Dirac-Ramond operator [M]; in the mathematical literature, it may have been first considered by Taubes [T], and was more recently studied in detail by Landweber [L], based on Kostant's compact group analogue. Denote by $T_a(m)$ the total action $R_a(m) + \sigma_a(m)$ of $\xi_a(m)$ on $\mathbf{H}' \otimes \mathbf{S}'$, and let $k^\vee := k + h^\vee$.

11.2 Proposition. $\left[\not{D}, \psi^b(n)\right] = 2T_b(n),\ \left[\not{D}, T_b(n)\right] = -nk^\vee \cdot \psi^b(n).$

We postpone the proof for a moment and explore the consequences. Clearly, the commutation action of the Dirac Laplacian $\not{D}^2$ on the $T_\bullet$ and the ψ agrees with that of $-2k^\vee E$. Normalise the total energy operator E on $\mathbf{H}' \otimes \mathbf{S}'$ to make it vanish on its lowest eigenspace $\mathbf{H}(0) \otimes \mathbf{S}(0)$. This last space is $\not{D}$-invariant, and the only terms in (11.1) to survive on it are those with $m = 0$. These sum to the Dirac operator for $\mathfrak{g}$, acting on its representation $\mathbf{H}(0)$. As we saw in §4, the latter squares to $-(\lambda + \rho)^2$. Since $\mathbf{H}' \otimes \mathbf{S}'$ is generated by the actions of the $T_\bullet$ and the ψ on $\mathbf{H}(0) \otimes \mathbf{S}(0)$, the following formula for the Dirac Laplacian results:

$$\not{D}^2 = -2k^\vee E - (\lambda + \rho)^2. \tag{11.3}$$

In particular, $\not{D}$ is invertible, with discrete, finite multiplicity spectrum.

11.4 Remark. Because the σ are expressible in terms of the ψ, the Dirac operator (11.1) is expressible in terms of the operators $T_\bullet$ and ψ alone. Define the *level $k^\vee$ universal enveloping algebra* of $L'\mathfrak{g}$, $U_{k^\vee}(L'\mathfrak{g}) := U(\widetilde{L}'\mathfrak{g})/(K - k^\vee)$. Then, $\not{D}$ is an odd element in a certain completion of the "semi-direct tensor product" of Cliff$(L'\mathfrak{g}^*)$ by $U_{k^\vee}(L'\mathfrak{g})$, acting on Cliff via ad. (The most natural completion is that containing infinite sums of normal-ordered monomials, of bounded degree and energy; this acts on all lowest weight modules of $L'\mathfrak{g} \ltimes \psi(L'\mathfrak{g}^*)$.) The first equation in (11.2) determines $\not{D}$ uniquely, because no odd elements of the completed algebra commute with all the ψ. However, a definite lifting $T_\bullet$ of $L\mathfrak{g}$ into $U_{k^\vee}(L\mathfrak{g})$ has been chosen at this point. This choice will show up more clearly in the next section, where we consider the family of $\not{D}$'s parametrised by all possible linear splittings of the central extension $\widetilde{L}\mathfrak{g}$ (cf. also §13.3).

Proof of (11.2). The first identity follows by adding the two lines below (in which summation over $m \in \mathbb{Z}$ is implied, in addition to the Einstein convention):

$$\begin{aligned}\left[R_a(m) \otimes \psi^a(-m), \psi^b(-n)\right] &= 2R_b(-n), \\ \left[\sigma_a(m)\psi^a(-m), \psi^b(-n)\right] &= 2\sigma_b(-n) + f^b_{ac}\psi^c(m-n)\psi^a(-m) = 6\sigma_b(-n).\end{aligned}$$

The second identity in (11.2) follows from the first. Indeed:

$$\begin{aligned}\left[[\not{\mathbb{D}}, T_b(n)], \psi^c(p)\right] &= [\not{\mathbb{D}}, [T_b(n), \psi^c(p)]] - [T_b(n), [\not{\mathbb{D}}, \psi^c(p)]] \\ &= f^c_{db}\left[\not{\mathbb{D}}, \psi^d(p+n)\right] - 2\,[T_b(n), T_c(p)] \\ &= 2f^c_{db}T_d(p+n) - 2f^d_{bc}T_d(p+n) - 2nk^\vee \cdot \delta_{bc}\delta_{n+p} \\ &= -2nk^\vee \cdot \delta_{bc}\delta_{n+p} \\ &= -nk^\vee\left[\psi^b(n), \psi^c(p)\right],\end{aligned}$$

whence we conclude that the odd operator $[\not{\mathbb{D}}, T_b(n)] + nk^\vee\psi^b(n)$ commutes with all the ψ; hence it is zero, as explained in Remark (11.4). □

(11.5) The twisted-affine case. With the same notation and the same definition (11.1) of $\not{\mathbb{D}}$, we have

$$\left[\not{\mathbb{D}}, \psi^b(n)\right] = 2T_{\bar{b}}(n), \quad [\not{\mathbb{D}}, T_b(n)] = -nk^\vee \cdot \psi^{\bar{b}}(n); \tag{11.6}$$

and we obtain, as before, the formula for the Dirac Laplacian:

$$\not{\mathbb{D}}_0^2 = -2k^\vee E - (\lambda + \underline{\rho})^2. \tag{11.7}$$

(11.8) The affine Dirac operator. The relation between the finite and affine Dirac Laplacians, (4.6) and (11.3), becomes more transparent if we use spinors on the full Kac-Moody algebra. Let $\widehat{L}\mathfrak{g}^* = \mathrm{i}\mathbb{R}K^* \oplus L\mathfrak{g}^* \oplus \mathrm{i}\mathbb{R}\delta$, where K^* (which is denoted Λ in [K]) is dual to K and δ to E. Identifying it with $\widehat{L}\mathfrak{g}$ by the bilinear form (8.2), the co-adjoint action of $\widehat{\xi} = (k, \xi, e)$ on $\widehat{L}\mathfrak{g}^*$ becomes

$$\begin{aligned}K^* &\mapsto \mathrm{i}\, d\xi/dt = -[E, \xi], \quad \delta \mapsto 0, \\ \mu \in L\mathfrak{g}^* &\mapsto \mathrm{ad}^\vee_{\xi(t)}\mu(t) + e \cdot \mu'(t) + \mathrm{i}\delta \cdot \oint \mu\xi' dt.\end{aligned} \tag{11.9}$$

The Spin module for Cliff$(\widehat{L}\mathfrak{g}^*)$ is $\widehat{\mathbf{S}} = \mathbf{S} \oplus \psi^{K^*} \cdot \mathbf{S}$. The corresponding Dirac operator,

$$\widehat{\not{\mathbb{D}}} := \not{\mathbb{D}} + E\psi^\delta + K\psi^{K^*},$$

commutes with the (new) total action $T_\bullet$ of $\widehat{L}\mathfrak{g}$ and satisfies the simpler formula $\widehat{\not{\mathbb{D}}}^2 = -(\lambda + \rho)^2$, whose verification we leave to the reader.

12. The Dirac family on a simple affine algebra

Assume now the representation $\mathbf{H}'$ of $\widehat{L}'\mathfrak{g}$ to be integrable; it is then unitarisable, and its Hilbert space completion $\mathbf{H}$ carries an action of the smooth loop group LG. Furthermore, we must have $k^\vee > 0$.

(12.1) The level hyperplanes. The co-adjoint action (11.9) preserves the fixed-level hyperplanes $\mathrm{i}k^\vee K^* + \widetilde{L}\mathfrak{g}^* \subset \widehat{L}\mathfrak{g}^*$. Ignoring δ leads to the *affine action at level* $k^\vee$ on $L\mathfrak{g}^*$. The correspondence

$$\mathrm{i}k^\vee K^* + \mu \quad \leftrightarrow \quad d/dt + \mu/k^\vee$$

identifies this action with the gauge action on the space $\mathcal{A}$ of $\mathfrak{g}$-valued connections on the circle.

12.2 Proposition. *The assignment* $\mu \mapsto \not{D}_\mu := \not{D} + \mathrm{i}\psi(\mu)$*, from* $L\mathfrak{g}^*$ *to* $\mathrm{End}(\mathbf{H}' \otimes \mathbf{S}')$*, intertwines the affine action of* $\widehat{L}\mathfrak{g}$ *at level* $k^\vee$ *with the commutator action.*

Proof. $\left[T(\xi), \not{D}_\mu\right] = k^\vee\psi\left([E,\xi]\right) + \mathrm{i}\left[\sigma(\xi), \psi(\mu)\right] = \mathrm{i}\psi\left(-k^\vee d\xi/dt + \mathrm{ad}_\xi^\vee(\mu)\right)$, as desired. □

(12.3) The Laplacian. Formulae (11.2) and (11.3) give

$$\begin{aligned} \not{D}_\mu^2 &= \not{D}^2 + \mathrm{i}\left[\not{D}, \psi(\mu)\right] - \psi(\mu)^2 \\ &= -2k^\vee E - (\lambda+\rho)^2 + 2\mathrm{i}\langle T|\mu\rangle - \mu^2 \\ &= -2\left(k^\vee E - \mathrm{i}\langle T\,|\mu\rangle + \langle \lambda+\rho\,|\mu\rangle\right) - (\lambda+\rho-\mu)^2. \end{aligned} \tag{12.4}$$

When $\mu \in \mathfrak{t}^*$, we can view this formula as a generalisation of (11.3), as follows. The first term in (12.4) is $-2k^\vee E_\mu$, with a *modified energy operator*

$$E_\mu = E - \mathrm{i}\langle T\,|\mu/k^\vee\rangle + \langle\lambda+\rho\,|\mu/k^\vee\rangle.$$

This is associated to the connection $d/dt + \mu/k^\vee$ in the same way that E is associated to the trivial connection: they intertwine correctly with the action of $L\mathfrak{g}$. Furthermore, E_μ is additively normalised so as to vanish on the $-(\lambda+\rho)$-weight space within $\mathbf{H}(0)\otimes\mathbf{S}(0)$. As we are about to see, when $\mu/k^\vee \in \mathfrak{a}^*$, that weight space is the lowest eigenspace for the Dirac Laplacian on $\mathbf{H}\otimes\mathbf{S}$.

(12.5) The Dirac kernels. To study a general $\not{D}_\mu$, we conjugate by a suitable loop group element to bring μ into $k^\vee\mathfrak{a}^*$. As $\not{D}_\mu$ now commutes with $\mathfrak{t}$ and E, we can evaluate (12.4) on a weight space of type (ω, n), where $T(\mu) = \mathrm{i}\,\langle\omega|\mu\rangle$, and obtain

$$\not{D}_\mu^2 = -2\left(k^\vee n + \langle\omega+\lambda+\rho|\mu\rangle\right) - (\lambda+\rho-\mu)^2 \tag{12.6}$$

Now, a weight of $\mathbf{H}\otimes\mathbf{S}$ splits as $(\omega,n) = (\omega_1,n_2) + (\omega_2,n_2)$, into weights of $\mathbf{H}$ and $\mathbf{S}$. Proposition (8.4) asserts that $(\omega_i+\lambda)\cdot\mu + k^\vee n_i \geq 0$,with equality only if $\mu/k^\vee$ is on the boundary of $\mathfrak{a}^*$, or else if $\omega = -(\lambda+\rho)$ and $n=0$. But then, (12.6) can only vanish if, additionally, $\mu = \lambda+\rho$. Since that lies in the interior of $k^\vee\mathfrak{a}^*$, we obtain the following.

12.7 Theorem. *The kernel of* $\not{D}_\mu$ *is nil, unless* μ *is in the affine co-adjoint orbit of* $(\lambda+\rho)$ *at level* $k^\vee$*. If so, the transformation mapping* μ *to* $\lambda+\rho$ *identifies* $\ker\not{D}_\mu$ *with the* $-(\lambda+\rho)$*-weight space in* $\mathbf{H}(0)\otimes\mathbf{S}(0)$*.* □

The last space is the product of the lowest-weight space $\mathbb{C}\mathbf{v}$ of $\mathbf{H}(0)$ with that of $\mathbf{S}(0)$; this last weight space is a graded, irreducible $\mathrm{Cliff}(\mathfrak{t})$-module. As in finite dimensions, the more canonical statement is that the kernels of the $\not{D}_\mu$ on the "critical" co-adjoint orbit $\mathfrak{O}$ of $\lambda+\rho$ in $\mathrm{i}k^\vee K^* + L\mathfrak{g}^*$ assemble to a vector bundle isomorphic to $\mathbf{S}(\mathcal{N})(-\lambda-\rho)$, the normal spinor bundle twisted by the natural line bundle on $\mathfrak{O}$. This vector bundle has a natural continuation to a neighbourhood of $\mathfrak{O}$ as the lowest eigen-bundle of $\not{D}_\mu$. We can describe the action of $\not{D}_\mu$ there, when μ moves a bit off $\mathfrak{O}$.

12.8 Theorem. *Let* $\mu \in \mathfrak{O}$*,* $\nu \in \mathcal{N}_\mu$ *a normal vector to* $\mathfrak{O}$ *at* μ *in* $\mathcal{A}$*. The Dirac operator* $\not{D}_{\mu+\nu}$ *preserves* $\ker(\not{D}_\mu)$ *and acts on it as Clifford multiplication by* $\mathrm{i}\psi(\nu)$*.* □

(12.9) Twisted K-theory class. Proposition 12.2 shows that our constructions are preserved by the action of LG, so the Fredholm bundle $(\mathbf{H}\otimes\mathbf{S}, \not{D}_\mu)$ over $L\mathfrak{g}^*$ defines a twisted, LG-equivariant K-theory class supported on $\mathfrak{O}$. Formula (12.6) bounds the complementary spectrum of $\not{D}_\mu$ away from zero, so the embedding of the lowest eigenbundle induces an equivalence of twisted, LG-equivariant K-theory classes in some neighbourhood of $\mathfrak{O}$. Proposition 12.8 identifies the K-class with the Thom push-forward of the line bundle $\mathcal{O}(-\lambda-\rho)$, from $\mathfrak{O}$ to $L\mathfrak{g}^*$. Finally, identifying the level $k^\vee$ hyperplane in $\widetilde{L}\mathfrak{g}$ with $\mathcal{A}$ as in §12.1 and using the holonomy map from $\mathcal{A}$ to G interprets our Dirac family as a class in $K^\tau_G(G)$, in degree $\dim\mathfrak{g} \pmod 2$.

(12.10) Twisted affine algebras. The results extend verbatim to twisted affine algebras, if we use the presentation $L_\varepsilon\mathfrak{g}$ of §9. Let $\mathcal{A}_\varepsilon$ be the space of smooth connections on the G-bundle of type ε and recall the distinguished connection ∇_0 of §9.2.

12.11 Proposition. (i) *The identification of the affine hyperplane* $\mathrm{i}k^\vee K^* + L_\varepsilon\mathfrak{g}^* \subset$ *with* $\mathcal{A}_\varepsilon$ *sending* μ *to* $\nabla_0 + \mu/k^\vee$ *is equivariant for the action of* $L_\varepsilon\mathfrak{g}$.

(ii) *The assignment* $\mu \mapsto \not{D} + \mathrm{i}\psi(\mu)$ *intertwines the affine co-adjoint and commutator actions.*

(iii) *Formula (12.4) for* $\not{D}^2_\mu$*, and its consequences (12.7) and (12.8), carry over, with* ρ *replaced by* $\underline{\rho}$. ☐

13. Arbitrary compact groups

We now extend the construction of the Dirac family, and the resulting map from representations to twisted K-classes, to the space $\mathcal{A}_P$ of connections on a principal bundle P over the circle, with arbitrary compact structure group G. The Lie algebra $L_P\mathfrak{g}$ of the loop group L_PG of gauge transformations splits into a sum of abelian and simple loop algebras, and the central extension preserves the splitting (Prop. 10.3). To assemble the families for the individual summands, we must still discuss the abelian case and settle their equivariance under the non-trivial components of L_PG.

(13.1) The Abelian case. Assume, as in Def. 2.4, that the the central extension $\widetilde{L}\mathfrak{z}$ takes the form $[\xi,\eta] = b(S\xi,\eta)\cdot K$, for the L^2 pairing in an inner product on the abelian Lie algebra $\mathfrak{z}$. Letting $\widehat{L}\mathfrak{z} := \mathrm{i}\mathbb{R}K \oplus L\mathfrak{z} \oplus \mathrm{i}\mathbb{R}S$, the discussion of the Dirac family in §11 and §12 carries over, with $\mathfrak{a} = \mathfrak{z}$, $L\mathfrak{z}$ acting trivially on the spin module $\mathbf{S}$, and ρ and $h^\vee$ null. For instance, $\not{D} := R_a \otimes \psi^a$, summing over a basis of $L\mathfrak{z}$, and relations affdiracrels and (11.6) are clear in $U_k(L\mathfrak{z}) \otimes \mathrm{Cliff}$ (Remark 11.4).

An admissible irreducible representation of $\widehat{L}\mathfrak{z}$ has the form $\mathbf{F}\otimes\mathbb{C}_{-\lambda}$, for the Fock representation $\mathbf{F}$ of $\widehat{L}\mathfrak{z}/\mathfrak{z}$ and a τ-affine weight λ of $\mathfrak{z}$, and we obtain

$$\not{D}^2 = -2S - \lambda^2, \qquad \not{D}^2_\mu = -2S - (\lambda-\mu)^2.$$

The kernel is identified as before: it is supported on the affine subspace $\mathrm{i}K^* + \lambda + L\mathfrak{z}^* \ominus \mathfrak{z}^*$ of $\widetilde{L}\mathfrak{z}^*$. This is a single co-adjoint orbit of the identity component of LZ, and the family represents the Thom push-forward of the LZ-equivariant line bundle $\mathcal{O}(-\lambda)$, from that orbit to the ambient space.

(13.2) Spectral flow over Z. The positive polarisation $\mathfrak{U} \subset L\mathfrak{z}_\mathbb{C} \ominus \mathfrak{z}_\mathbb{C}$ of §2.8 leads to vector space identifications $\mathbf{S}' \cong \mathbf{S}(0)\otimes\Lambda^\bullet(\mathfrak{U})$ and $\mathbf{F}' \cong \mathrm{Sym}(\mathfrak{U})$. Decomposing $\not{D}_\mu = \not{D}^{\mathfrak{z}}_\mu + \not{D}^{L\mathfrak{z}/\mathfrak{z}}$ into zero-modes and $\mathfrak{U}$-modes, we recognise in the first term is the Dirac family of §3, lifted to $\mathfrak{z}^*$ and restricted to the single summand $\mathbb{C}_{-\lambda} \subset \mathbf{F}_{[-\lambda]}$; whereas $\not{D}^{L\mathfrak{z}/\mathfrak{z}} = \partial + \partial^*$, for the Koszul differential

$$\partial : \mathrm{Sym}^p(\mathfrak{U})\otimes\Lambda^q(\mathfrak{U}) \to \mathrm{Sym}^{p+1}(\mathfrak{U})\otimes\Lambda^{q-1}(\mathfrak{U}).$$

Thus, $\not{D}_\mu$ is quasi-isomorphic to the finite-dimensional family $(\mathbb{C}_{-\lambda}, \not{D}^{\mathfrak{z}}_\mu)$ over $\mathfrak{z}^*$. The induced LZ-module will have the form $\mathbf{F}'\otimes\mathbf{F}_{[-\lambda]}$, and dropping the factor $\Lambda^\bullet(\mathfrak{U})\otimes\mathbf{F}'$, which is equivalent to $\mathbb{C}$, recovers our spectral flow family of §3.3.

(13.3) Characterisation of $\not{D}_\mu$. Proposition 12.2 ensures the equivariance of our Dirac family under the connected part of the loop group. When G is not simply connected, we must extend this to the other components; in particular, this is needed for tori. This extension is accomplished by an intrinsic characterisation of $\not{D}_\mu$. We first restate the Dirac commutator relations without coordinates:

$$\left[\not{D}_\mu, \psi(\nu)\right] = 2\langle T \,|\, \nu\rangle + 2\mathrm{i}\langle \mu \,|\, \nu\rangle \qquad \left[\not{D}_\mu, T(\xi)\right] = \psi\left(\mathrm{ad}^\vee_\xi(k^\vee K^* - \mathrm{i}\mu)\right),$$

where the bracket in the first equation is contraction in the bilinear form (8.2). Observe now that the right-hand side of first formula expresses the total action of ν on $\mathbf{H} \otimes \mathbf{S}$, *in the lifting of $L_P\mathfrak{g}$ to $\widetilde{L}_P\mathfrak{g}$ defined by the line* through $(\mathrm{i}k^\vee K^* + \mu)$ in $\widetilde{L}_P\mathfrak{g}^*$. In the second formula, we have used the co-adjoint action of §11.8. As explained in Remark 11.4, the first relation uniquely determines $\not{D}_\mu$, and we conclude

13.4 Proposition. *The assignment $\mu \mapsto \not{D}_\mu$ is equivariant under all compatible automorphisms of $L_P\mathfrak{g}$,* **H** *and* **S** *which preserve the bilinear form on $L_P\mathfrak{g}$.* □

(13.5) Coupling to representations. The Dirac family $\not{D}_\mu$ lives naturally on an affine copy of $L_P\mathfrak{g}^*$, namely the hyperplane over $\mathrm{i} \in \mathrm{i}\mathbb{R}$ in the projection $(L_P\mathfrak{g}^*)^\tau \to \mathrm{i}\mathbb{R}$, dual to the central extension (2.6). We transport it to $\mathcal{A}_P$ by identifying the two as L_PG-affine spaces. For the simple factors, this identification is described in §12; but on the abelian part, there is an ambiguity: we can translate by the Lie algebra of the centre of L_PG. Under the holonomy map from connections to conjugacy classes, this ambiguity matches the one encountered in (6.13.ii), where we identified $\tau \cdot \mathfrak{a}^*$ with the space of holonomies. Note, however, that the regularity and singularity of the affine weights matches the one of the underlying (twisted) conjugacy classes in G, irrespective of the chosen identification.

Coupling $\not{D}_\mu$ to graded, admissible representations results in twisted K-classes on $\mathcal{A}_P$, equivariant under L_PG. This is also an Ad-equivariant twisted K-classes over G, supported on the components which carry the holonomies of P.

13.6 Proposition. *The isomorphism of Theorem 3 is induced by the Dirac family map, from admissible representations to K-classes.*

Proof. This follows by comparing the Dirac kernels to the classification of irreducibles by their lowest-weight spaces in §10, and again with the basis of $K^\tau_G(G)$ described in Proposition 6.12. □

VI Variations and Complements

In this chapter we exploit the correspondence between representations and K-classes to produce analogues of known representation-theoretic constructions in purely topological terms.

14. Semi-infinite cohomology

In this section, we give alternative formulae (14.3), (14.10) for the Dirac operator $\not{D}$. With the Lie algebra cohomology results of Bott [B] and Kostant [K1] and with Garland's loop group analogues [G], the new formulae explain the magical appearance of the kernel on the correct orbit. The relative Dirac operators of [K2] and [L] allow us to interpret the morphisms ω^* and ω_* of §7 in terms of well-known constructions for affine algebras, namely *semi-infinite cohomology* and *semi-infinite induction* [FF].

We work here with polynomial loop algebras and lowest-weight modules; for simplicity, we omit f-twist, underlines and the primes from the notation. We shall also use $\mathrm{ad}^\vee$ to denote the co-adjoint action of a Lie algebra on its dual, reserving the "$*$" for hermitian adjoints.

(14.1) Lie algebra cohomology. The triangular decomposition $L\mathfrak{g}_{\mathbb{C}} = \mathfrak{N} \oplus \mathfrak{t}_{\mathbb{C}} \oplus \overline{\mathfrak{N}}$ factors the spin module as $\mathbf{S} = \mathbf{S}(\mathfrak{t}^*) \otimes \Lambda^\bullet \overline{\mathfrak{N}}^*$. The action of $\overline{\mathfrak{N}}$ on a lowest-weight module $\mathbf{H}$ leads to a Chevalley differential on the Lie algebra cohomology complex,

$$\begin{aligned} \bar\partial &: \mathbf{H} \otimes \Lambda^k \overline{\mathfrak{N}}^* \to \mathbf{H} \otimes \Lambda^{k+1} \overline{\mathfrak{N}}^*, \\ \bar\partial &= R_{-\alpha} \otimes \psi^\alpha + \frac{1}{2}\psi^\alpha \cdot \mathrm{ad}^\vee_{-\alpha} \end{aligned} \tag{14.2}$$

where we have used a root basis of $\overline{\mathfrak{N}}$ and its dual basis ψ^α of Clifford generators. Let $\bar\partial^*$ be the hermitian adjoint of $\bar\partial$, and denote by $\not{D}^{\mathfrak{t}}_{-\rho}$ the $\mathfrak{t}$-Dirac operator with coefficients in the representation $\mathbf{H} \otimes \Lambda^\bullet \overline{\mathfrak{N}}^* \otimes \mathbb{C}_{-\rho}$ of T.

14.3 Proposition. *$\not{D} = \bar\partial + \bar\partial^* + \not{D}^{\mathfrak{t}}_{-\rho}$; moreover, $\not{D}^{\mathfrak{t}}_{-\rho}$ commutes with $\bar\partial + \bar\partial^*$.*

Proof. Commutation is obvious. It is also clear that the R-terms on the two sides agree; so, it remains to compare the Dirac $(\sigma\psi)/3$-term in (11.1) with $\psi \cdot \mathrm{ad}^\vee/2 + (\psi \cdot \mathrm{ad}^\vee)^*/2$, plus the ad-term in $\not{D}^{\mathfrak{t}}$. Now, all three terms have cubic expressions in the Clifford generators, and we will check their agreement. We have

$$\begin{aligned} \frac{1}{2}\psi^\alpha \cdot \mathrm{ad}^\vee_{-\alpha} &= \frac{1}{4}\sum\nolimits_{\substack{\alpha,\beta>0 \\ \gamma<0}} f_{\alpha\beta\gamma}\psi^\alpha\psi^\beta\psi^\gamma, \\ \frac{1}{2}\left(\psi^\alpha \cdot \mathrm{ad}^\vee_{-\alpha}\right)^* &= \frac{1}{4}\sum\nolimits_{\substack{\alpha,\beta>0 \\ \gamma<0}} \bar f_{\alpha\beta\gamma}\psi^{-\gamma}\psi^{-\beta}\psi^{-\alpha}. \end{aligned}$$

Disregarding the order of the generators, their difference contains precisely the terms in $\sigma\psi/3$ involving two positive roots and a negative one, respectively two negative roots and a positive one; whereas the ad-term in $\not{D}^{\mathfrak{t}}$ similarly collects the $\sigma\psi/3$-terms involving exactly one $\mathfrak{t}^*$-element. Clearly, this accounts for all terms in $\sigma\psi/3$. We have thus shown that the symbols of these operators agree in (a completion of) $\Lambda^3(L\mathfrak{g}^*)$.

The difference between the two must then be a linear ψ-term. However, both operators commute with the maximal torus T and with the energy E; so the difference is $\psi(\mu)$, for some $\mu \in \mathfrak{t}^*_{\mathbb{C}}$. A quick computation gives, for $\nu \in \mathfrak{t}^*$,

$$\begin{aligned} \left[\bar\partial, \psi(\nu)\right] &= \left[\bar\partial^*, \psi(\nu)\right] = 0, \\ \left[\not{D}^{\mathfrak{t}}_{-\rho}, \psi(\nu)\right] &= 2T(\nu) = [\not{D}, \psi(\nu)]\,; \end{aligned}$$

so $[\psi(\mu), \psi(\nu)] = 0$ for all ν, and it follows that $\mu = 0$, as desired. □

(14.4) The Dirac kernels. Proposition 14.3 gives a new explanation for the location of $\ker \not{D}_\mu$. If $\mathbf{H}$ is irreducible with lowest weight $(-\lambda)$, we have, on $\mathbf{H} \otimes \Lambda^q \overline{\mathfrak{N}}^* \otimes \mathbf{S}(\mathfrak{t}^*)$,

$$\ker\left(\bar\partial + \bar\partial^*\right) \cong H^q\left(\overline{\mathfrak{N}}; \mathbf{H}\right) \otimes \mathbf{S}(\mathfrak{t}^*) = \bigoplus_{\ell(w)=q} \mathbb{C}_{w(-\lambda-\rho)+\rho} \otimes \mathbf{S}(\mathfrak{t}^*), \tag{14.5}$$

embedded in the Lie algebra complex as harmonic co-cycles; the sum ranges over the elements of length q in the affine Weyl group of $\mathfrak{g}$. If $\mu \in \mathfrak{t}^*$, then $\not{D}^{\mathfrak{t}}_{\mu-\rho} = \not{D}^{\mathfrak{t}}_{-\rho} + i\psi(\mu)$ commutes with $\left(\bar\partial + \bar\partial^*\right)$, so $\ker \not{D}_\mu$ is also the kernel of $\not{D}^{\mathfrak{t}}_{\mu-\rho}$ on the space in (14.5). Clearly, the latter is non-zero precisely when μ is one of the $w(\lambda+\rho)$; otherwise, it follows that the highest eigenvalue of $\not{D}^2_\mu$ is the negative squared distance to the nearest such point, in agreement with §12.3.

(14.6) Semi-infinite cohomology. A similar construction applies to a decomposition of a rather different kind. Splitting $L\mathfrak{g}_{\mathbb{C}} = L\mathfrak{n} \oplus L\mathfrak{t}_{\mathbb{C}} \oplus L\bar{\mathfrak{n}}$ gives a factorisation

$$\mathbf{S}(L\mathfrak{g}^*) = \mathbf{S}(L\mathfrak{t}^*) \otimes \Lambda^{\infty/2+\bullet}(L\mathfrak{n}^*),$$

where the "semi-infinite" right-most factor is the exterior algebra on the non-negative Fourier modes in $\mathfrak{n}^*$ and the duals of the negative ones, the latter carrying degree (-1) [FGZ, FF]. The analogue of formula 14.2 defines a differential ∂ for semi-infinite Lie algebra cohomology, acting on $\mathbf{H} \otimes \Lambda^{\infty/2}(L\mathfrak{n}^*)$. With the same $\mathbf{H}$, the semi-infinite cohomology can be expressed as a sum of positive energy Fock spaces $\mathbf{F} \otimes \mathbb{C}_\mu$ for $L\mathfrak{t}\,/\,\mathfrak{t}$, on which T acts with weight μ:

$$H^{\infty/2+q}(L\mathfrak{n}; \mathbf{H}) = \bigoplus_{\ell(w)=q} \mathbf{F} \otimes \mathbb{C}_{w(-\lambda-\rho)+\rho}. \tag{14.7}$$

Because the splitting of $L\mathfrak{g}_{\mathbb{C}}$ was LT-equivariant, LT act on $\Lambda^{\infty/2}(L\mathfrak{n}^*)$; it commutes with ∂, so acts on the cohomology; but the non-trivial components shift the degree. Passing to Euler characteristics, we can collect terms into the irreducible representations $\mathbf{F} \otimes \mathbf{F}_{[\mu]}$ of LT described in §3, and we obtain a sum over the finite Weyl group

$$\textstyle\sum_q (-1)^q H^{\infty/2+q}(L\mathfrak{n}; \mathbf{H}) = \sum_{w \in W} \varepsilon(w) \cdot \mathbf{F} \otimes \mathbf{F}_{[w(-\lambda-\rho)+\rho]}. \tag{14.8}$$

In the f-twisted case, the Weyl group is replaced by the extension $\widetilde{W}^f$ of (6.6), and the $\mathbf{F}_{[\mu]}$ are the irreducible τ-modules of $\underline{\Pi} \times \underline{T}$.

(14.9) Relative Dirac operator. Define $\not{D}^{L\mathfrak{g}/L\mathfrak{t}} := \partial + \partial^*$; its index is given by (14.8).

14.10 Proposition. $\not{D} = \not{D}^{L\mathfrak{t}} + \not{D}^{L\mathfrak{g}/L\mathfrak{t}}$*, and all three operators commute.* □

The proof is very similar to the one of Prop. 14.3; see [L] for more help. Similarly, we have $\not{D}_\mu^{L\mathfrak{g}} = \not{D}_\mu^{L\mathfrak{t}} + \not{D}^{L\mathfrak{g}/L\mathfrak{t}}$, and the three operators commute when $\mu \in L\mathfrak{t}^*$. As in §14.4, it follows that the restriction to $L\mathfrak{t}^*$ of our Dirac family on $\mathbf{H} \otimes \mathbf{S}$ is stably equivalent to $\not{D}_\mu^{L\mathfrak{t}}$, acting on the alternating sum of spaces in (14.8). Comparing this with the construction (6.12) of K-classes from conjugacy classes and with the local model of the Weyl map (7.7), we obtain the following

14.11 Theorem. *Under the Dirac family construction, the map from $R^{\tau-\sigma}(LG)$ to $R^\tau(LT)$ defined by the semi-infinite $L\mathfrak{n}$-Euler characteristic corresponds to the Weyl restriction $\omega^* : K_G^\tau(G) \to K_T^\tau(T)$.* □

14.12 Remark. (i) In the twisted case, this applies to the restriction $\omega^* : K^\tau_{G(f)}(fG_1) \to K_T^\tau(fT)$.
(ii) We have used LT for simplicity, but the result applies to LN, which preserves the relative Dirac $\not{D}^{L\mathfrak{g}/L\mathfrak{t}}$ (though not the semi-infinite differential ∂). We can then detect the restriction to $K^\tau_{N(f)}(fT)$.

15. Loop rotation, energy and the Kac numerator

In this section, we study a rotation-equivariant version of $K_G^\tau(G)$ and relate it to the positive energy representations of the loop groups.

(15.1) Conditions for rotation-equivariance. The admissible loop group representations of greatest interest admit a circle action intertwining with the loop rotations (§1.9). This will be the case iff the following two conditions are met:

(i) The loop rotation action lifts to the central extension LG^τ,
(ii) The polarisation used in defining admissibility is rotation-invariant (§2.9).

A lifting in (i) defines a semi-direct product $\mathbb{T} \ltimes LG^\tau$. Subject to condition (ii), the Borel-Weil construction of admissible representations [PS] shows that they all carry actions of the identity component of this product, and the $\mathbb{T}$-action is determined up to an overall shift on each irreducible. The rotation action can be extended to the entire loop group as in the discussion of §10, leading to the same classification of irreducibles, but with the extra choice of normalisation for the circle action.

With respect to condition (i), it is convenient to allow *fractional* circle actions: that is, we allow the circle of loop rotations to be replaced by some finite cover. A lifting of the rotation action to LG^τ refines the level $[\tau]$ to a class in $H^3\left(B(\mathbb{T} \ltimes LG)\right)$. The obstruction to such a refinement is the differential $\delta_2 : H^3_G(G_1) \to H^2(B\mathbb{T}) \otimes H^2_G(G_1)$ in the Leray sequence for the projection to $B\mathbb{T}$. All torsion obstruction vanish when $\mathbb{T}$ is replaced by a suitable finite cover. Rationally, $H^*_G(G_1)$ is the invariant part of $H^*_T(T)$ under the Weyl group W of G, and for the torus we have the following.

15.2 Lemma. *A class in $H^3(T \times BT)$ lifts to a rotation-equivariant one iff its component in $H^1(T) \otimes H^2_T$ is symmetric.*

Proof. The differential δ_2 vanishes on the $H^*(T)$ factor, and is determined its effect on H^2_T: this is mapped isomorphically onto $H^2(B\mathbb{T}) \otimes H^1(T)$. On $H^3(T \times BT)$, this becomes the anti-symmetrisation map $H^1(T) \otimes H^2_T \to H^2(T) = \Lambda^2 H^1(T)$. □

15.3 Remark. For semi-simple G, symmetry is ensured by Weyl invariance.

Adding loop rotations to the landscape leads to the quotient stack of the space $\mathcal{A}$ of smooth connections by the action of $\mathbb{T} \ltimes LG$. This is a smooth stack, with compact quotient and proper stabiliser, but unlike the quotient stack G_G of G by its own Ad-action, it cannot be presented as a quotient of a manifold by a compact group. The K-theory of such stacks was discussed in [FHT1]. Let $\widehat{\Lambda}^\tau = \Lambda^\tau \oplus \mathbb{Z}\delta$ be the level τ slice of the affine weight lattice (A.9).

15.4 Proposition. *We have isomorphisms $R^{\tau-\sigma}(\mathbb{T} \ltimes LG) \cong K^{\tau'-\sigma(\mathfrak{t})}_{W^e_{\mathrm{aff}}}(\widehat{\Lambda}^\tau) \cong K^{\tau+\dim\mathfrak{g}}_{\mathbb{T}}(G_G)$, obtained by tracking the loop rotation in Thm. 7.10 and in the Dirac family.*

The middle group is a free $R_{\mathbb{T}}$-module, with the generator acting on $\widehat{\Lambda}^\tau$ by δ-translation. Killing the augmentation ideal forgets the circle action in the outer groups and δ in the middle group, and recovers the isomorphisms in Theorems 3 and 4.

Proof. The argument is a repetition of (5.2), (6.8) and (7.10), with the extra $\mathbb{T}$-action. The main difference is that we are now dealing with the K-theories of some smooth, proper stacks, which are no longer global quotients, but only locally so. However, the proofs of (5.2) and (7.10) proceed via the same local step, which continues to apply, globalised using the Mayer-Vietoris principle. □

(15.5) Positive energy. The natural choices for the Fredholm operator S defining the Lie algebra cocycle in (2.4) are multiples of the derivative $-\mathrm{i}d/dt$; the polarisation $\mathfrak{P}$ is then the semi-positive Fourier part of $L\mathfrak{g}_{\mathbb{C}}$. With those choices, lowest-weight modules of $L\mathfrak{g}$ carry a bounded-below energy operator E, unique up to additive normalisation, generating the intertwining loop rotation action. If the restriction to $H^1(T) \otimes H^2_T$ of $[\tau]$ is symmetric, loop rotations lift fractionally to LG^τ; and, if that same bilinear form is positive, E is bounded below on admissible τ-representations of the group.

This generalises easily to the *twisted* loop groups L_PG of gauge transformations of a principal bundle P over S^1. The diffeomorphisms of the bundle P which cover the loop rotation form an extension of the rotation group $\mathbb{T}$ by L_PG; this group replaces $\mathbb{T} \ltimes LG$ from the trivial bundle case. (The extension is not too serious, any connection on P whose holonomy has finite order gives a fractional splitting.) The topological constraint for rotation-equivariance of a extension τ is now the symmetry of the map κ^τ in §6.4.

(15.6) The Kac numerator. For the remainder of this section, we make the simplifying assumption that G is connected, with $\pi_1 G$ free. Positive energy representations of LG are then determined by their restriction to the subgroup $\mathbb{T} \times G$ of circle rotations and constant loops; moreover, loop rotations extend to a trace-class action of the semi-group $\{q \in \mathbb{C}^\times | |q| < 1\}$. If $\mathbf{H}$ is irreducible with lowest-weight $(-\lambda)$, the value of its character at $q \in \mathbb{C}$ and $g \in G$ is given by the *Kac formula* [K]

$$\mathrm{Tr}\left(q^E g|\mathbf{H}\right) = \frac{\sum_\mu \varepsilon(\mu) \cdot q^{\|\mu\|^2/2} \cdot \mathrm{Tr}\left(g|V_{\rho-\mu}\right)}{\Delta(g;q)}, \tag{15.7}$$

where μ ranges over the dominant regular affine Weyl transforms of $(\lambda+\rho)$ at level $[\tau]$, $\varepsilon(\mu)$ is the signature of the transforming affine Weyl element, V_μ the G-representation with lowest weight μ, $\|\mu\|^2 := \langle (\kappa^\tau)^{-1}(\mu)|\mu\rangle$ defined by the level $[\tau]$, and the *Kac denominator* for $(L\mathfrak{g}, \mathfrak{g})$

$$\Delta(g;q) = \prod_{n>0} \det\left(1 - q^n \cdot \mathrm{ad}(g)\right)$$

independent of λ and τ, representing the (super)character of the spinors on $L\mathfrak{g}/\mathfrak{g}$. We shall now see how (15.7) is detected by our $K_{\mathbb{T}}$-group.

Including the identity $e \in G$ defines a Gysin map

$$\mathrm{Ind} : R^{\tau-\sigma(\mathfrak{g})}(\mathbb{T} \times G) \to K_{\mathbb{T}}^{\tau+\dim G}(G_G),$$

with τ on the left denoting the restricted twisting and $\sigma(\mathfrak{g})$ the Thom twist of the adjoint representation. Dualising over $R_{\mathbb{T}}$, while using the bases of irreducible representations to identify $K_{\mathbb{T}}^\tau(G_G)$ with its $R_{\mathbb{T}}$-dual, leads to an $R_{\mathbb{T}}$-module map

$$\mathrm{Ind}^* : K_{\mathbb{T}}^{\tau+\dim G}(G_G) \to \mathrm{Hom}_{\mathbb{Z}}\left(R^{\tau-\sigma(\mathfrak{g})}(G); R(\mathbb{T})\right);$$

the right-hand side is the $R(\mathbb{T})$-module of formal sums of (twisted) G-irreducibles with Laurent polynomial coefficients. The choice of basis gives an indeterminacy by an overall power of q for each irreducible, which must be adjusted to give an exact match in the following theorem. Let $[\mathbf{H}]$ be the $K_{\mathbb{T}}^\tau(G_G)$-class corresponding to $\mathbf{H}$.

15.8 Theorem. $\mathrm{Ind}^*[\mathbf{H}]$ *is the Kac numerator in* (15.7)*.*

Proof. The theorem is a consequence of two facts. First is the relation

$$q^{\|\lambda+\rho\|^2/2} \cdot \mathrm{Ind}\left(V_{-\lambda}\right) = \varepsilon(\mu) \cdot q^{\|\mu\|^2/2} \cdot \mathrm{Ind}\left(V_{\rho-\mu}\right), \tag{15.9}$$

holding for any μ in the affine Weyl orbit of $(\lambda+\rho)$. Second is the fact that, with our simplifying assumption that G is connected with free π_1, the twisted K-class $\mathrm{Ind}(V_{-\lambda})$ corresponds to an irreducible representation of LG^τ. (There are no affine Weyl stabilisers of regular weights).

We can check (15.9) by restriction to the maximal torus T. The Weyl denominator is the Euler class of the inclusion $T \subset G$; multiplying by it while using the Weyl character formula converts the Kac numerator for $(L\mathfrak{g}, \mathfrak{g})$ to that of $(L\mathfrak{t}, \mathfrak{t})$, and we are reduced to verifying the theorem for the torus (with $\varepsilon(\mu) = 1$ and without ρ-shifts, as the affine Weyl group is now the lattice $\pi_1 T$).

The stack $\mathcal{A}/\mathbb{T} \ltimes LT$ of T-valued connections on the circle, modulo gauge transformations and circle rotations, is equivalent to the classifying stack of a bundle of groups over the quotient space T, with fibre $\mathbb{T} \times T$. The bundle of groups is described by its holonomy around loops $\gamma \in \pi_1 T$, given by the automorphism

$$\mathbb{T} \times T \ni (q,t) \mapsto (q, tq^\gamma).$$

The holonomy on the associated bundle $\mathcal{R}$ of representation rings is $q^m t^{\mathrm{i}\lambda} \mapsto q^{m+\langle\lambda|\gamma\rangle} t^{\mathrm{i}\lambda}$, for any $m \in \mathbb{Z}$ and integral weight $\lambda : \pi_1 T \to \mathbb{Z}$.

The twisting τ defines a bundle $\mathcal{R}^\tau$ of free rank one modules over $\mathcal{R}$. (With respect to Construction (5.1), $\mathcal{R}^\tau$ is the free $\mathbb{Z}$-module over the fibres of Y.) The holonomy describing $\mathcal{R}^\tau$ must vary from that of $\mathcal{R}$ by multiplication by a unit $q^{\phi(\lambda,\gamma)} \cdot t^{\mathrm{i}\kappa^\tau(\gamma)}$. (The correct exponent $\kappa^\tau(\gamma)$ of t is detected by restricting to the known case $q = 1$.) We claim that the only option, up to automorphism, is $\phi(\lambda,\gamma) = \langle \kappa^\tau(\gamma) | \gamma \rangle / 2$, resulting in the holonomy

$$q^{m+\|\lambda\|^2/2} t^{\mathrm{i}\lambda} \mapsto q^{m+\|\lambda+\kappa^\tau(\gamma)\|^2/2} \cdot t^{\mathrm{i}(\lambda+\kappa^\tau(\gamma))}.$$

Travelling now around γ shows that inductions from the characters $q^{\|\lambda\|^2/2} t^{\mathrm{i}\lambda}$ and from $q^{\|\lambda+\kappa^\tau(\gamma)\|^2/2} \cdot t^{\mathrm{i}(\lambda+\kappa^\tau(\gamma))}$ of $\mathbb{T} \times T$ lead to the same twisted K-class, proving (15.9) and hence our theorem.

To check the claim, note the two relations

$$\begin{aligned} \phi(\lambda+\mu,\gamma) &= \phi(\mu,\gamma), \\ \phi(\lambda,\gamma+\gamma') &= \phi(\lambda,\gamma) + \phi(\lambda,\gamma') + \langle \kappa^\tau(\gamma) | \gamma' \rangle \end{aligned}$$

the first, by computing the holonomy of $t^{\mathrm{i}(\lambda+\mu)} = t^{\mathrm{i}\lambda} t^{\mathrm{i}\mu}$ in two different ways (using the module structure of $\mathcal{R}^\tau$) and the second, from the homomorphism condition. These imply that $\phi(\lambda,\gamma) = \langle \kappa^\tau(\gamma) | \gamma \rangle / 2$, modulo a linear γ-term; but the latter can be absorbed by a shift $t^{\mathrm{i}\lambda} \mapsto t^{\mathrm{i}(\lambda+\nu)}$ in T-characters, representing an automorphism of $\mathcal{R}^\tau$. $\square$

15.10 Remark. This discussion can be generalised to twisted loop groups and their disconnected versions. However to determine a representation uniquely, we must restrict it to a larger subgroup of the loop group, one which meets at least every *torsion* component in a translate of the maximal torus). We then expect to find an extension of the Kac character, which is due to Wendt [W].

16. Fusion with G-representations

For positive energy representations, the *fusion product* of conformal field theory defines an operation $* : R(G) \otimes R^\tau(LG) \to R^\tau(LG)$. We will now recall its construction and prove its agreement with the topologically defined $R(G)$-action on $K_G^\tau(G)$ by tensor product. For notational clarity, we only write out the argument for the untwisted loop groups, the twisted result following by judicious insertion of underlines and f-subscripts.

(16.1) Example: G_1 is a torus. Recall from §2.3 that, when $G = N$, $LN \cong \Gamma N \ltimes \exp(L\mathfrak{t} \ominus \mathfrak{t})$, where $\Gamma N = N^e_{\mathrm{aff}}$ is the subgroup of geodesic loops. Evaluating geodesic loops at a point x in the circle gives a homomorphism $E_x : LN \to N$. If V is a finite-dimensional N-representation, the pull-back $E_x^* V$ is an admissible LN-representation, and fusing with V is simply tensoring with $E_x^* V$.

Note that E_x is not the "evaluation at x" homomorphism on the whole of LN; indeed, the latter would *not* lead to admissible representations. Because of this, for non-abelian G_1, we need the more complicated definition that follows, essentially moving the base-point x inside the disk.

(16.2) Segal's holomorphic induction. Let $\mathbf{H}$ be a positive energy admissible τ-representation of LG, and V a G-module whose ρ-shifted highest weights lie in the alcove $\tau \cdot \mathfrak{a}^*$ (§10.4). Such G-modules will be called *small*. Let also A be a complex annulus, with an interior base-point x. The obvious group $\mathcal{O}(A; G_\mathbb{C})$ of holomorphic maps with smooth boundary values acts on $\mathbf{H}$, by restriction to the inner boundary, on V by evaluation at x, and maps into a copy of $LG_\mathbb{C}$ by restriction to the outer boundary. G. Segal defines the fusion of $\mathbf{H}$ with V along A as the *holomorphic induction*

$$\mathbf{H} * V_x := \mathrm{Ind}_{\mathcal{O}(A;G_\mathbb{C})}^{LG_\mathbb{C}} \left(\mathbf{H} \otimes V_x \right), \tag{16.3}$$

by which we mean the space of right $\mathcal{O}(A; G_\mathbb{C})$-invariant holomorphic maps from $LG_\mathbb{C}$ to $\mathbf{H} \otimes V$. Conjecturally, this is a completion of an admissible representation.

The rigorous implementations of this construction that we know are algebraic. The direct product $\widehat{\mathbf{H}}$ of energy eigenspaces in $\mathbf{H}$ is a representation of the *Laurent polynomial loop group* $L'G_{\mathbb{C}} := G_{\mathbb{C}}[z, z^{-1}]$. After evaluation at $z = x$, $L'G_{\mathbb{C}}$ also acts on V. The completion of $L'G_{\mathbb{C}}$ at $z = \infty$ is the group of formal Laurent loops $G_{\mathbb{C}}((w))$ ($w = z^{-1}$). Its algebraic, positive energy $\tau - \sigma$-modules are completely reducible, and the irreducibles are precisely the direct sums $\mathbf{H}'$ of energy eigenspaces in irreducible admissible representations $\mathbf{H}$ of LG.[5] Constructing the induced representation now from *algebraic* functions, the following important lemma permits the subsequent definition.

16.4 Lemma. $\mathrm{Ind}_{L'G_{\mathbb{C}}}^{G_{\mathbb{C}}((w))}\left(\widehat{\mathbf{H}} \otimes V\right)$ *is a finitely reducible, positive-energy representation of* $G_{\mathbb{C}}((w))$.

16.5 Definition. The fusion product $\mathbf{H} * V_x$ is $\mathrm{Ind}_{L'G_{\mathbb{C}}}^{G_{\mathbb{C}}((w))}\left(\widehat{\mathbf{H}} \otimes V\right)$.

Using brackets to denote the associated K-classes, the fusion is identified by the following

16.6 Theorem. *In* $K_G^\tau(G)$ *with its topological* $R(G)$*-action,* $[\mathbf{H} * V_x] = [\mathbf{H}] \otimes [V]$.

The proof of this theorem requires some preliminary constructions.

(16.7) Borel-Weil construction. We need to review the construction of $\mathbf{H}$ by algebraic induction from a Borel-like subgroup, the *Iwahori subgroup*, but minding the group $\pi_0 LG$ of components. To see the problem, recall that every representation of a *connected* compact Lie group is holomorphically induced from a Borel subgroup B. However, this fails for disconnected groups, where induction from the *quasi-Borel subgroup* $Q_T \cdot B$ is required instead. (Q_T is the quasi-torus of §7.) This is neatly accomplished by using an old idea of Beilinson and Bernstein.

The *quasi-iwahori subgroup* $Q_I \subset L'G_{\mathbb{C}}$ is the normaliser of $\mathfrak{N}$; it meets every component of $L'G_{\mathbb{C}}$ in a translate of the standard Iwahori subgroup. We can factor $Q_I = Q_L \ltimes \exp(\mathfrak{N})$, with a subgroup $Q_L \subset (N^e_{\mathrm{aff}})_{\mathbb{C}}$ which plays the rôle of a complexified quasi-torus for the loop group. In fact, $Q_L = Q_I \cap (N^e_{\mathrm{aff}})_{\mathbb{C}}$. There is a Cartesian square

$$\begin{array}{ccc} Q_L & \longrightarrow & (N^e_{\mathrm{aff}})_{\mathbb{C}} \\ \downarrow & & \downarrow \\ \pi_0 LG & \longrightarrow & W^e_{\mathrm{aff}}, \end{array}$$

where the bottom horizontal arrow is the splitting of (10.5) defined by the positive alcove.

Over the full flag variety $X' := L'G_{\mathbb{C}}/Q_I$, there is an algebraic vector bundle $\mathcal{U}$, whose fibre at a coset γQ_I is the space $\mathbf{H}'/\mathfrak{N}^\gamma \mathbf{H}'$ of co-invariants in $\mathbf{H}'$, with respect to the conjugated nilpotent $\mathfrak{N}^\gamma := \gamma \mathfrak{N} \gamma^{-1}$. (This fibre is isomorphic to the lowest-weight space for the opposite polarisation.) Then, $\widehat{\mathbf{H}}$ is the space of algebraic sections of $\mathcal{U}$ over X'. A result of Kumar [Ku] ensures the vanishing of higher cohomologies of this bundle.

16.8 Remark. (i) Q_I acts (projectively) on the space $U := \mathbf{H}'/\mathfrak{N}\mathbf{H}'$, which defines a projective $L'G_{\mathbb{C}}$-vector bundle over X'; "unprojectivising" this bundle at level $\tau - \sigma$ results in $\mathcal{U}$.
(ii) The same prescription defines $\mathcal{U}$ over the "thicker" flag variety $X := G_{\mathbb{C}}((w))/Q_I$, and its sections there lead to the "thin" version $\mathbf{H}'$ of the same representation.

(16.9) Derived induction. The fibre of $\mathcal{U}$ at 1 is a representation of Q_I which factors through Q_L, and whose highest weights are in Λ^τ_{reg+}, as discussed in §10. We now study the "derived induction" $\mathbf{R}$Ind from Q_L-modules to LG-modules, by which we mean the Euler characteristic over X of a vector bundle associated to a general $(\tau - \sigma)$-module of Q_L. By §10 again,

$$^{\tau-\sigma}R(Q_L) \cong {}^{\tau'-\sigma(\mathfrak{t})}K_\pi(\Lambda^\tau),$$

[5] Experts will know that, when G is not semi-simple, these algebraic loop groups are highly non-reduced group (ind)-schemes, and their formal part must be included in the discussion.

with the action and twistings defined there, and we claim that **R**Ind is the result of the direct image map, followed by restriction to the regular part:

$$ {}^{\tau'-\sigma(\mathfrak{t})}K_{\pi}(\Lambda^{\tau}) \to {}^{\tau'-\sigma(\mathfrak{t})}K_{W^{e}_{\mathrm{aff}}}(\Lambda^{\tau}) \to {}^{\tau'-\sigma(\mathfrak{t})}K_{W^{e}_{\mathrm{aff}}}(\Lambda^{\tau}_{reg}). \tag{16.10} $$

From §10 and the vanishing of higher cohomology, this is known for weights in Λ^{τ}_{reg+}. Because $W^{e}_{\mathrm{aff}} \cong \pi \ltimes W^{e}_{\mathrm{aff}}(\mathfrak{g})$ and $\tau \cdot \mathfrak{a}^{*}$ is a fundamental domain for $W^{e}_{\mathrm{aff}}(\mathfrak{g})$, it suffices to show that **R**Ind is anti-symmetric under this last group and that weights on the walls of $\tau \cdot \mathfrak{a}^{*}$ induce 0. Both statements follow from Bott's reflection argument [B] applied to the simple affine reflections.

Proof of (16.4). Q_I acts on on V by evaluation at $z = x$; calling $\mathcal{V}_x$ the associated vector bundle over X, transitivity of induction shows that

$$ \mathrm{Ind}_{L'G_{\mathbb{C}}}^{G_{\mathbb{C}}((w))}\left(\widehat{\mathbf{H}} \otimes V\right) \cong \Gamma\left(X; \mathcal{U} \otimes \mathcal{V}_x\right). $$

and the Lemma now follows from Theorem 4 of [T1]. □

Proof of (16.6). Theorem 4 of [T1] also ensures the vanishing of higher cohomologies when V is small. We will identify $\mathbf{H} * V_x$ by deforming $\mathcal{V}_x$. Scaling $x \mapsto 0$ deforms the action of Q_I on V_x into the representation V_0, pulled back from the quotient map $Q_I \to Q_L$. More precisely, any point-wise evaluation $LG \to G$ embeds Q_L into N, and V_0 is obtained from V under $Q_L \to N \subset G$. The Euler characteristic of the bundle $\mathcal{U} \otimes \mathcal{V}_x$ is unchanged under deformation, because of the rigidity of admissible representations of $G((w))$ (and the techniques of [T1], which reduce this to a "finite type" problem). We conclude that

$$ \mathbf{H} * V_x \cong \mathbf{R}\mathrm{Ind}(\mathcal{U} \otimes V_0). $$

To prove the theorem, we must show that $\mathbf{R}\mathrm{Ind} : R(Q_L) \to K^{\tau}_{G}(G)$ is an $R(G)$-module map, under the inclusion $Q_L \subset G$. Factoring **R**Ind as in (16.10), this property is clear for the second step, restriction to Λ^{reg}, since that is nothing but the map ω^{*} of §7.10. A different description makes the same obvious for the first step, the direct image. Indeed,

$$ {}^{\tau'-\sigma(\mathfrak{t})}K_{W^{e}_{\mathrm{aff}}}(\Lambda^{\tau}) \cong K^{\tau}_{N^{e}_{\mathrm{aff}}}(\mathfrak{t}), \qquad {}^{\tau'-\sigma(\underline{\mathfrak{t}})}K_{\pi}(\Lambda^{\tau}) \cong K^{\tau}_{Q_L}(\mathfrak{t}), $$

as in Remark 6.9. The direct image map becomes now induction along the inclusion $Q_L \subset N^{e}_{\mathrm{aff}}$, and this is clearly a module homomorphism under the super-ring $R(N)$ (as N^{e}_{aff} maps to N by evaluation at any fixed point in the loop). □

17. Topological Peter-Weyl theorem

We now describe a topological version of the Peter-Weyl theorem for loop groups; beyond its entertainment value, the result can be used to confirm that the bilinear form in the Frobenius algebra $K^{\tau}_{G}(G)$ of [FHT1] agrees with the natural duality pairing in the Verlinde ring, as we claimed in [FHT3, §8]. The TFT interpretation is only available for twistings that are transgressed from BG in a suitable sense [FHT4], but our description of the duality pairing applies to any regular twisting.

(17.1) Compact groups. One version of the Peter-Weyl theorem for a compact Lie group G asserts that the two-sided regular representation — the space of continuous functions on G, under its left and right translation actions — is a topological completion of the direct sum $\bigoplus V \otimes V^{*}$, ranging over the irreducible finite-dimensional modules V. (The direct sum describes the polynomial functions.) A variation of this, for a central extension G^{τ} by $\mathbb{T}$, describes the space of sections of the associated line bundle over G as the corresponding sum over irreducible τ-representations.

Qua $G \times G$-module, the regular representation of G is induced from the trivial G-module, under the diagonal inclusion $G \subset G \times G$. For *finite* G, the result can be expressed in terms of equivariant K-theory: it asserts that the trivial representation $[1] \in R(G)$ maps, under diagonal inclusion $G \subset G \times G$, to the class $\sum[V \otimes V^*] \in R(G \times G)$. To see this more clearly, identify $R(G)$ with $K_{G\times G}(G)$, with the left×right action, and push forward to a point with $G \times G$ action. In the presence of a twisting τ for $R(G)$, we map $[1] \in R(G)$ instead to $R^{\tau\times(-\tau)}(G \times G)$. In constructing this last push-forward, we have used the natural trivialisation of the sum of a central extension τ of G with its opposite, so that the required twisting on $K_{G\times G}(G)$ is canonically zero and the "trivial" class [1] is well-defined.

17.2 Remark. When τ is graded, our formulation of Peter-Weyl conceals a finer point. The module $R^\tau(G)$ of graded representations has now an odd component $R^{\tau+1}(G)$, defined from the *super-symmetric representations* [FHT3, §4]. These are graded G-modules with a commuting action of the rank one Clifford algebra Cliff(1). The contribution of such a super-symmetric representation V to the Peter-Weyl sum is the (graded) tensor product $V \otimes_{\mathrm{Cliff}(1)} V^*$ over Cliff(1), and not over $\mathbb{C}$. However, this is exactly what we need to match the cup-product

$$R^{\tau+1}(G) \otimes R^{-\tau+1}(G) \to R^{\tau\times(-\tau)+0}(G \times G);$$

indeed, the (graded) tensor product $V \otimes_{\mathbb{C}} V^*$ has a commuting Cliff(2) action, and defines an element of K^2, which is indeed where the cup-product initially lands [LM]. Tensoring over Cliff(1) instead of $\mathbb{C}$ is the Morita identification of complex Cliff(2)-modules with vector spaces, which implements the Bott isomorphism to K^0.

(17.3) Loop groups. Before discussing the loop group analogue of this, let us recall the algebraic Peter-Weyl theorem for loop groups; this is a special case of the Borel-Weil theorem of [T1]. As in the preceding section, denote by $G_{\mathbb{C}}((z))$ and $G_{\mathbb{C}}((w))$ be the two Laurent completions of the loop group LG at the points 0 and ∞ on the Riemann sphere. The Laurent polynomial loop group $L'G_{\mathbb{C}} = G[z, z^{-1}]$ embeds in both (with $w = z^{-1}$). The quotient variety $Y := G((w)) \times_{L'G_{\mathbb{C}}} G((z))$ for the diagonal action is a homogeneous space for the product of the two loop groups, which should be thought regarded as a generalised flag variety. For any twisting τ, the product $\mathcal{O}(\tau - \sigma) \boxtimes \mathcal{O}(\sigma - \tau)$ of the opposite line bundles on the two factors carries an action of $L'G_{\mathbb{C}}$, so it descends to an (algebraic) line bundle on Y. A special case of the Borel-Weil-Bott theorem of [T1] asserts that, as a representation of $G((w)) \times G((z))$,

$$\Gamma\left(Y; \mathcal{O}(\tau - \sigma) \boxtimes \mathcal{O}(\sigma - \tau)\right) \cong \bigoplus\nolimits_{\mathbf{H}} \mathbf{H}' \otimes \overline{\mathbf{H}}',$$

with the sum ranging over the lowest-weight representations $\mathbf{H}$ for $G_{\mathbb{C}}((w))$ at level $\tau - \sigma$.

(17.4) Topological interpretation. The topological construction in §17.1 breaks down for infinite compact groups, but remarkably, it does carry over to loop groups. To start with, the diagonal self-embedding of G leads to a Gysin map

$$\iota_* : K_G(G) \to K_{G\times G}(G \times G),$$

with the Ad-action in both cases. When G is *connected*, this is a topological model for the classifying map of the diagonal $LG \to LG \times LG$. For general G, the restriction of ι_* to $K_G(G_1)$ corresponds to the diagonal of LG, whereas the restriction to $K_{G(f)}(fG_1)$, in the notation of §7.7, captures the diagonal embedding for the twisted loop group $L_f G$. Finally, for any τ, we get a map

$$\iota_*^\tau : K_G(G) \to K_{G\times G}^{\tau\times(-\tau)}(G \times G), \tag{17.5}$$

cancelling the pulled-back twisting by the earlier observation: the sum of extensions $\tau + (-\tau)$ is trivial on the diagonal LG.

To describe ι_*, we replace $K_G(G)$ with the isomorphic group $K^*_{G\times G}(G\times G)$, the action being now

$$(g_1,g_2).(x,y) = (g_1 x g_1^{-1}, g_1 y g_2^{-1}).$$

The isomorphism with $K^*_G(G)$ arises by restriction to the diagonal G's. The map $G\times G \to G\times G$ inducing ι_* sends (x,y) to $(x, y^{-1}xy)$. Note that the relative tangent bundle of this map is (stably) equivariantly trivial, and there is a preferred relative orientation, if we use the same dual pair of Spin modules on each pair of $\mathfrak{g}$'s, so there is no ambiguity coming from orientations.

17.6 Theorem (Peter-Weyl for Loop Groups). *When τ is regular, we have*

$$\iota^\tau_*(1) = \textstyle\sum_{\mathbf{H}}[\mathbf{H}\otimes\mathbf{H}^*],$$

summing over the irreducible admissible representations $\mathbf{H}$ of LG, with the correspondence of defined in Theorem 3. The analogue holds for each twisted loop group of G.

Without using Theorem 3, we can assert that $\iota_*(1)$ has a diagonal decomposition in the basis of $K^\tau_G(G)$ produced from regular affine Weyl orbits and irreducible representations of the centralisers (Theorem 7.10), and the complex-conjugate basis for $K^{-\tau}_G(G)$. The two formulations are of course related by Theorem 10.2. Let us state this more precisely: consider the "anti-diagonal" class $[\Delta^-]$ on $\underline{\Lambda}^\tau \times \underline{\Lambda}^{-\tau}$, which is identically 1 on pairs $(\lambda, -\lambda)$ and null elsewhere. It is equivariant for the diagonal W^e_{aff}-action. Also let $\tau'' = \tau' - \sigma(\underline{\mathfrak{t}})$.

17.7 Lemma. *The sum in the right-hand side of (17.6) corresponds to the direct image of $[\Delta^-]$ under the direct image map*

$$K_{W^e_{\mathrm{aff}}}\left(\underline{\Lambda}^\tau_{reg}\times\underline{\Lambda}^{-\tau}_{reg}\right) \longrightarrow K^{\tau''\times(-\tau'')}_{W^e_{\mathrm{aff}}\times W^e_{\mathrm{aff}}}\left(\underline{\Lambda}^\tau_{reg}\times\underline{\Lambda}^{-\tau}_{reg}\right).$$

Proof. Replacing both sides with the sets of orbits, represented by weights $\mu\in\underline{\Lambda}^\tau_+$ and stabilisers $\pi_\mu\subset W^e_{\mathrm{aff}}$, we get the direct sum over μ of the diagonal push-forwards

$$R(\pi_\mu)\to R^{\tau''}(\pi_\mu)\otimes R^{-\tau''}(\pi_\mu),$$

and apply the topological Peter-Weyl theorem to each π_μ. □

For a torus T, the representation categories of LT^τ and $\Gamma^\tau = (\Pi\times T)^\tau$ are equivalent, and ι^τ_* captures the Peter-Weyl theorem for Γ^τ: diagonal induction of the trivial representation to $\Gamma^\tau\times\Gamma^{-\tau}$ leads to the sum in (17.6). This result generalises to every group N^e_{aff} of (f-twisted) geodesic loops in N, and is the basis for the general proof. To convert it into a topological statement, we will factor both the algebraic and the topological induction (direct image) maps into two steps, with the second step being described by Lemma 17.7. Agreement of the other, first step is then verified by a Dirac family construction generalising slightly the spectral flow family in §3. As the general case may be obscured by the notational clutter imposed by the groups of components, we handle the torus first.

(17.8) Example: $G=T$. Let $\ell=\dim T$ and factor ι^τ_* into the direct images

$$K^0_T(T)\xrightarrow{B\mathrm{diag}_*} K^{\tau\times(-\tau)-\ell}_{T\times T}(T)\xrightarrow{\mathrm{diag}_*} K^{\tau\times(-\tau)+0}_{T\times T}(T\times T), \tag{17.9}$$

along the obvious diagonal morphisms. Describing diag_* is easy. Double use the Key Lemma 5.2, with the *same* group T^2, followed by direct images (along $\mathfrak{t}$ and $\mathfrak{t}^2$) lead to isomorphisms

$$\begin{aligned} K^{\tau\times(-\tau)-\ell}_{T\times T}(T) &\cong K^0(\Lambda^\tau\times_\Pi\Lambda^{-\tau}),\\ K^{\tau\times(-\tau)+0}_{T\times T}(T\times T) &\cong K^0(\Lambda^\tau/\Pi\times\Lambda^{-\tau}/\Pi). \end{aligned}\tag{17.10}$$

Moreover, diag_* becomes the direct image between the groups on the right, and this is the map appearing in Lemma 17.7.

In view of Lemma 17.7, we must check that $B\mathrm{diag}_*[1]$ in the middle group of (17.9) is the anti-diagonal class $[\Delta^-]$. We have a commutative square

$$\begin{array}{ccc} K^0_T(T) & \xrightarrow{B\mathrm{diag}_*} & {}^{\tau\times(-\tau)}K^{-\ell}_{T\times T}(T) \\ \uparrow & \square & \uparrow \\ K^0(T) & \xrightarrow{p_*} & K^{\tau-\ell}_T(T) \end{array}$$

with the vertical arrows being the pull-backs, along the projection of BT to a point and the map $BT^2 \to BT$ induced by group multiplication. Our anti-diagonal class, in the upper right, is the pull-back of the sum of the irreducible classes in $K^{\tau-\ell}_T(T)$. But we identified this in §3.6 with $p_*[1]$, as desired.

Proof of (17.6). Fix a twisting element f in the quasi-torus; we prove the theorem for L_fG. We use the notation of §6 and §7, except that we write G for $G(f)$, N for $N(f)$, W for W^f for simplicity.

Step 1. In view of the following commutative square, in which $\omega_*(1) = 1$,

$$\begin{array}{ccc} K_N(fT) & \xrightarrow{\iota_*} & K^{\tau\times(-\tau)}_{N\times N}(fT\times fT) \\ \big\downarrow \omega_* & & \big\downarrow \omega_* \\ K_G(fG_1) & \xrightarrow{\iota_*} & K^{\tau\times(-\tau)}_{G\times G}(fG_1\times fG_1) \end{array}$$

it suffices to prove the theorem for the upper ι_*: that is, we may assume $G = N$.

Step 2. Let $\delta(N)$ be the left equaliser of the two projections $N^2 \rightrightarrows N/\underline{T}$. Its Ad-action on fT^2 preserves the diagonal copy of fT. With $\underline{\ell} = \dim \underline{T}$, we can factor ι^τ_* as

$$K^0_N(fT) \xrightarrow{B\mathrm{diag}_*} K^{\tau\times(-\tau)-\underline{\ell}}_{\delta(N)}(fT) \xrightarrow{\mathrm{diag}_*} K^{\tau\times(-\tau)+0}_{N^2}(fT\times fT). \tag{17.11}$$

Moreover, we have the "key Lemma" isomorphisms for $M = \underline{T}^2$ in $\delta(N)$ and N^2,

$$\begin{aligned} K^{\tau\times(-\tau)-\underline{\ell}}_{\delta(N)}(fT) &\cong K^{\tau''\times(-\tau'')+0}_{W^e_{\mathrm{aff}}}\left(\underline{\Lambda}^\tau \times \underline{\Lambda}^{-\tau}\right), \\ K^{\tau\times(-\tau)+0}_{N^2}(fT\times fT) &\cong K^{\tau''\times(-\tau'')+0}_{{W^e_{\mathrm{aff}}}^2}\left(\underline{\Lambda}^\tau \times \underline{\Lambda}^{-\tau}\right). \end{aligned} \tag{17.12}$$

and, as in (6.2), diag_* is the push-forward from upper to lower K-groups. We are reduced to showing that $B\mathrm{diag}_*[1] \in K^{\tau\times(-\tau)-\underline{\ell}}_{\delta(N)}(fT)$ is the anti-diagonal class in the upper right group.

Step 3. Call $\delta(N^e_{\mathrm{aff}})$ the left equaliser of the projections $N^e_{\mathrm{aff}} \times N^e_{\mathrm{aff}} \rightrightarrows W^e_{\mathrm{aff}}$. The presentation (6.5) of fT as a homogeneous space for $N \ltimes \underline{\mathfrak{t}}$ leads to the isomorphisms

$$\begin{aligned} K_N(fT) &\cong K^{\tau\times(-\tau)}_{N^e_{\mathrm{aff}}}(\underline{\mathfrak{t}}), \\ K^{\tau\times(-\tau)}_{\delta(N)}(fT) &\cong K^{\tau\times(-\tau)}_{\delta(N^e_{\mathrm{aff}})}(\underline{\mathfrak{t}}), \end{aligned} \tag{17.13}$$

as flagged in Remark 6.9. The twisting $\tau\times(-\tau)$ is null on the diagonal copy of N^e_{aff} in $\delta(N^e_{\mathrm{aff}})$, but trivialising it in relation to the other twistings is the key step in finding $B\mathrm{diag}_*$.

Step 4. Call $\mathcal{O}(\tau)$ the line bundle over $\underline{T} \cong \delta(N^e_{\text{aff}})/N^e_{\text{aff}}$ descended from the line bundle of the extension $\tau \times (-\tau)$ of $\delta(N^e_{\text{aff}})$. This $\mathcal{O}(\tau)$ carries a projective action of $\delta(N^e_{\text{aff}})$, by left translations, and its space of sections over $\underline{T}$ is, by definition, the representation $\text{Ind}[1]$ induced from $\mathbb{C}$ under the embedding $N^e_{\text{aff}} \subset \delta(N^e_{\text{aff}})^{\tau\times(-\tau)}$. This is the sought-after class $[\Delta^-]$ in (17.12).

Step 5. Finally, we show that, under the standard trivialisation of the extension $\tau \times (-\tau)$ over N^e_{aff}, the direct image of $[1]$ along the topological induction

$$K^{\tau\times(-\tau)+0}_{N^e_{\text{aff}}}(\underline{\mathfrak{t}}) \xrightarrow{B\text{diag}_*} K^{\tau\times(-\tau)-\ell}_{\delta(N^e_{\text{aff}})}(\underline{\mathfrak{t}})$$

is represented by the Dirac family on $\underline{\mathfrak{t}}$ coupled to $\text{Ind}[1]$. This implies its agreement with $[\Delta^-]$. The argument repeats the discussion in §3.6, after observing that $[1]$ corresponds to the class of $\mathcal{O}(\tau)$ in the chain of isomorphisms

$$[1] \in K_{N^e_{\text{aff}}}(\underline{\mathfrak{t}}) \cong K^{\tau\times(-\tau)}_{N^e_{\text{aff}}}(\underline{\mathfrak{t}}) \cong K^{\tau\times(-\tau)}_{\delta(N^e_{\text{aff}})}(\underline{T} \times \underline{\mathfrak{t}}).$$

□

Appendix

A. Affine roots and weights in the twisted case

We recall here the properties of diagram automorphisms, which lead to a concrete description of the twisted affine algebras in terms of simple, finite-dimensional ones. The connection between the two questions is due to Kac, to which we refer for a complete discussion [K, §7.9 and §7.10]; but we reformulate the basic facts more conveniently for us.

(A.1) When $\mathfrak{g}$ is simple, the order of a diagram automorphism ε is $r = 1, 2$ or 3, with the last value only possible for $\mathfrak{so}(8)$. Assume that $\varepsilon \neq 1$; $\mathfrak{g}$ must then be simply laced. We summarise the relevant results from [K].

- The invariant sub-algebra $\underline{\mathfrak{g}} := \mathfrak{g}^\varepsilon$ is simple, with Cartan sub-algebra $\underline{\mathfrak{t}} := \mathfrak{t}^\varepsilon$ and Weyl group $\underline{W} := W^\varepsilon$.
- The simple roots are the restrictions to $\underline{\mathfrak{t}}$ of those of $\mathfrak{g}$ (with multiplicities removed).
- The ratio of long to short root square-lengths in $\underline{\mathfrak{g}}$ is r, save for $\mathfrak{g} = \mathfrak{su}(3)$, when $\underline{\mathfrak{g}} = \mathfrak{su}(2)$.
- The ε-eigenspaces are irreducible $\underline{\mathfrak{g}}$-modules. The two $\varepsilon \neq 1$-eigenspaces are isomorphic when $\mathfrak{g} = \mathfrak{so}(8)$ and $r = 3$.

(A.2) The weight $\underline{\theta}$. Denote by $\underline{\theta}$ the highest weight of $\mathfrak{g}/\underline{\mathfrak{g}}$, and let $a_0 = 2$ when $\mathfrak{g} = \mathfrak{su}(2\ell+1)$ and $r = 2$; else, let $a_0 = 1$. Then, $\underline{\theta}/a_0$ is the short dominant root of $\underline{\mathfrak{g}}$. (When $\mathfrak{g} = \mathfrak{su}(2\ell+1)$, $\underline{\mathfrak{g}} = \mathfrak{so}(2\ell+1)$ and $\mathfrak{g}/\underline{\mathfrak{g}}$ is $\text{Sym}^2\mathbb{R}^{2\ell+1}/\mathbb{R}$, whose highest-weight is twice the short root.) The basic inner product on $\mathfrak{g}$ restricts to a_0 times the one on $\underline{\mathfrak{g}}$; so $\underline{\theta}^2 = 2a_0/r$.

A.3 Remark. With reference to [K, VI], we have $\underline{\theta} = \sum a_i\alpha_i - a_s\alpha_s$, where $s = 0$, except when $\mathfrak{g} = \mathfrak{su}(2\ell+1)$, in which case $s = 2\ell$: if so, our $\underline{\theta}$ differs from θ in *loco citato*.

(A.4) Twisted affine Weyl group. Denote by $\underline{\mathfrak{a}}$ the simplex of dominant elements $\xi \in \underline{\mathfrak{t}}$ satisfying $\underline{\theta}(\xi) \leq 1/r$. The *$\varepsilon$-twisted affine Weyl group* $W_{\text{aff}}(\mathfrak{g}, \varepsilon)$ is generated by the reflections about the walls of $\underline{\mathfrak{a}}$. Let $\underline{R}' \subset \underline{\mathfrak{t}}$ correspond to the root lattice $\underline{R}$ in $\underline{\mathfrak{t}}^*$ under the $\mathfrak{g}$-basic inner product.

A.5 Proposition ([K, Props. 6.5 and 6.6]). *$W_{\mathrm{aff}}(\mathfrak{g}, \varepsilon)$ is the semi-direct product $\underline{W} \ltimes \underline{R}'$. Its action on $\underline{\mathfrak{t}}$ has $\underline{\mathfrak{a}}$ as fundamental domain. The $W_{\mathrm{aff}}(\mathfrak{g}, \varepsilon)$-stabiliser of any point in $\underline{\mathfrak{a}}$ is generated by the reflections about the walls containing it.*

Proof. This follows from the analogous result for the untwisted affine algebra based on the Langlands dual to $\underline{\mathfrak{g}}$, in which $\underline{\mathfrak{a}}$ is the fundamental alcove and $\underline{R}'$ the co-root lattice. □

(A.6) Twisted conjugacy classes. When G is simply connected, the points of $\mathfrak{a}$ parametrise the conjugacy classes in G. The alcove $\underline{\mathfrak{a}}$ fulfils the same role for the ε-twisted conjugation $g : h \mapsto g \cdot h \cdot \varepsilon(g)^{-1}$.

A.7 Proposition. *If G is simply connected, every ε-twisted conjugacy class in has a representative* $\exp(2\pi\xi)$, *for a unique $\xi \in \underline{\mathfrak{a}}$. The twisted centraliser of* $\exp(2\pi\xi)$ *in G is connected, and its Weyl group is isomorphic to the stabiliser of ξ in $W_{\mathrm{aff}}(\mathfrak{g}, \varepsilon)$.*

Proof. For the first part, we must show, given (A.5) and (7.5), that the integer lattice of T_ε in $\mathfrak{t}_\varepsilon \cong \underline{\mathfrak{t}}$ is $\underline{R}'$, and that the ε-twisted action of $\underline{W}$ on T_ε is the obvious one. Now, the first lattice is the image, in the quotient $\mathfrak{t}_\varepsilon$ of $\mathfrak{t}$, of the integer (co-root) lattice $R^\vee \subset \mathfrak{t}$ of T. As $\mathfrak{g}$ is simply laced, $R^\vee$ is identified with the root lattice R of $\mathfrak{g}$ in $\mathfrak{t}^*$ by the basic inner product, so the integer lattice of T_ε is also the image of R in $\underline{\mathfrak{t}}^*$. But, by (A.1), this agrees with the root lattice $\underline{R}$ of $\underline{\mathfrak{g}}$. Concerning $\underline{W}$, since that is the Weyl group of G^ε, we can find ε-invariant representatives for its elements, and their ε-action coincides with the usual one.

Connectedness of twisted centralisers, for simply connected G, is due to Borel [Bo]. Moreover, because maximal tori are maximal abelian subgroups, T^ε is connected as well; and (7.5) identifies the Weyl groups of centralisers as desired. □

A.8 Remark. Connectedness of T^ε can also be seen directly, as follows. Clearly, the ε-fixed point set $\exp(\mathfrak{a}^\varepsilon)$ in the simplex $\exp(\mathfrak{a})$, is connected. By regularity of $\underline{\mathfrak{t}}$, every component of T^ε contains a regular element. This must be conjugate to some $a \in \exp(\mathfrak{a})$, hence of the form $w(a)$, with $w \in W$ and $a \in \exp(\mathfrak{a})$. Invariance under ε implies $w(a) = \varepsilon(w(a)) = \varepsilon(w)(\varepsilon(a))$. As a and $\varepsilon(a)$ are both in $\mathfrak{a}$ and regular, it follows that $w = \varepsilon(w)$ and $a = \varepsilon(a)$, so $w(a)$ is in the $\underline{W}$-image of $\exp(\underline{\mathfrak{t}})$, hence in $\underline{T}$.

(A.9) Affine roots and weights. The sub-algebra $\underline{\mathfrak{h}} = i\mathbb{R}K \oplus \underline{\mathfrak{t}} \oplus i\mathbb{R}E$ plays the role of a *Cartan subalgebra* of $\widehat{L}'_\varepsilon\mathfrak{g}$. The *affine roots*, living in $\underline{\mathfrak{h}}^*$, are the $\mathfrak{h}$-eigenvalues of the adjoint action on $\widehat{L}'_\varepsilon\mathfrak{g}$. Define the elements δ and K^* of $\underline{\mathfrak{h}}^*$ by $\delta(E) = 1/r$, $K^*(K) = 1$, $\delta(K) = \delta(\underline{\mathfrak{t}}) = K^*(\underline{\mathfrak{t}}) = K^*(E) = 0$. The *simple affine roots* are the simple roots of $\underline{\mathfrak{g}}$, plus $\delta - \underline{\theta}$; their $\mathbb{Z}$-span is the *affine root lattice* R_{aff}. The *positive roots* are sums of simple roots. The *standard nilpotent sub-algebra* $\mathfrak{N}$ is the sum of the positive root spaces, and a triangular decomposition $\widehat{L}'_\varepsilon\mathfrak{g}_{\mathbb{C}} = \overline{\mathfrak{N}} \oplus \underline{\mathfrak{h}}_{\mathbb{C}} \oplus \mathfrak{N}$ is inherited from $\widehat{L}'\mathfrak{g}_{\mathbb{C}}$.

The *simple co-roots* are those of $\underline{\mathfrak{g}}$, plus $(K - \beta^\vee)/a_0$, where $\beta^\vee$, the long dominant co-root of $\underline{\mathfrak{g}}$, satisfies $\lambda(\beta^\vee) = \langle\lambda|\underline{\theta}\rangle/r$.[6] The restriction $\widetilde{\underline{T}}$ of the basic central extension (9.8) to $\underline{T}$ is the quotient of $i\mathbb{R}K \oplus \underline{\mathfrak{t}}$ by the affine co-root lattice $R^\vee_{\mathrm{aff}}$.

The *weight lattice* $\widetilde{\Lambda}$ of $\widetilde{L}_\varepsilon G$, in $\underline{\mathfrak{h}}^*$, is the integral dual of $R^\vee_{\mathrm{aff}}$, and comprises the characters of $\widetilde{\underline{T}}$. Calling $\underline{\Lambda}$ the (simply connected) weight lattice of $\underline{\mathfrak{g}}$, we have

$$\widetilde{\Lambda} = \begin{cases} \mathbb{Z}K^* \oplus \underline{\Lambda}, & \text{if } \underline{\mathfrak{g}} \neq \mathfrak{su}(2\ell+1), \\ 2\mathbb{Z}K^* \oplus \underline{\Lambda}^+ \cup (2\mathbb{Z}+1)K^* \oplus \underline{\Lambda}^- & \text{if } \underline{\mathfrak{g}} = \mathfrak{su}(2\ell+1) \end{cases} \tag{A.10}$$

the superscript indicating the value of the character on the central element of $\mathrm{Spin}(2\ell+1)$. The *affine weight lattice* $\widehat{\Lambda}$ includes, in addition, the multiples of δ, giving the energy eigenvalue.

[6] Recall from A.2 that $\underline{\theta}/a_0$ is a short root, and $\underline{\theta}^2 = 2a_0/r$.

The *dominant weights* pair non-negatively with the simple co-roots; this means that (k, λ, x) is dominant iff λ is $\underline{\mathfrak{g}}$-dominant and $\lambda \cdot \underline{\theta} \leq k/r$. The affine Weyl group $W_{\mathrm{aff}}(\mathfrak{g}, \varepsilon)$ preserves the constant level hyperplanes, and its lattice part $\underline{R}$ (A.5) acts by k-fold translation at level k. Every positive-level weight has a unique dominant affine Weyl transform. Regular weights are those not fixed by any reflection in $W_{\mathrm{aff}}(\mathfrak{g}, \varepsilon)$. The important identity $\langle \underline{\rho} | \underline{\theta} \rangle + \underline{\theta}^2/2 = h^\vee/r$ [K, VI] implies that an *integral* weight (k, λ, x) is dominant iff the shifted weight $(k + h^\vee, \lambda + \underline{\rho}, x)$ is dominant regular.

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