Topological Field Theories in 2 dimensions

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Origins

The notion of a *Topological Field Theory* (TFT) was formalised by Atiyah and Witten ($\sim$ 1990) and modelled on Graeme Segal's notion of 2-dimesional *Conformal Field Theory*.

This was to provide a framework for the new topological invariants of the 1980's (4D Donaldson theory, 3D Chern-Simons theory).

The distinguishing feature of the new invariants is *multiplicativity* under unions, rather than the *additivity* common to algebraic topology (e.g. characteristic numbers). Additivity comes from the Mayer-Vietoris sequence.

Quantum field theory explains this behaviour heuristically: the invariants of a manifold X are integrals, not over X, but over a space of *fields* on X (maps to another fixed space). This space of fields is "multiplicative in pieces of X".

Definition

An n-dimensional topological field theory is a strong symmetric monoidal functor from the category of n-dimensional oriented bordisms to that of complex vector spaces. The monoidal structures are disjoint union and tensor product, respectively.

This means that to each closed oriented $(n-1)$-manifold we assign a vector space, to disjoint unions we assign tensor products, to a bordism we assign linear maps between the boundary spaces, and the gluing of bordisms corresponds to the composition of linear maps.

PICTURE GOES HERE. SOME DAY

There are variations of this definitions; in the case of surfaces ($n = 2$) they are substantial (Cohomological Field Theories; Open-Closed Theories).

Two dimensions

The classification of (compact, connected, oriented) topological surfaces has long been known. The only invariants are the number of components and the Euler characteristic. TFT's in dimension 2 were initially studied as a toy model, not as a source of invariants. Their structure was understood early on.

Theorem (folklore)

A 2-dimensional oriented TFT over $\mathbb{C}$ is equivalent to the datum of a commutative Frobenius algebra A over $\mathbb{C}$.

Recall that an (associative) algebra A is *Frobenius* if it comes equipped with a *trace* $\theta : A \to \mathbb{C}$ for which $a, b \mapsto \theta(a \cdot b)$ gives a perfect (symmetric) paring. (In particular, $\dim A < \infty$.)

Yet, divertingly enough, it is in 2D that the notion of TFT and its variations has seen the most powerful applications!

Gromov-Witten theory

My application is to *Gromov-Witten theory*, which generalises a classical and delicate question in enumerative geometry: counting algebraic curves in a projective manifold, with prescribed degree and intersection conditions.

For example, there is a unique linear map $\mathbb{P}^1 \to \mathbb{P}^n$ sending $0, 1, \infty$ to three general-position linear subspaces of total dimension $n - 1$.

GW theory encodes this by *deforming* the cohomology algebra

$$H^*(\mathbb{P}^n) = \mathbb{C}[\omega]/\langle \omega^{n+1} \rangle$$

into the *quantum cohomology algebra*, parametrised by $q \in \mathbb{C}^*$,

$$QH^*(\mathbb{P}^n) = \mathbb{C}[\omega]/\langle \omega^{n+1} - q \rangle$$

The coefficient 1 of q is the uniqueness, and its exponent 1 is the degree of a straight line. ($q \in \mathbb{C}^* = \exp H^2$.)

Frobenius structure on quantum cohomology

The quantum cohomology of a projective manifold X (like the ordinary cohomology) is a Frobenius algebra: trace = integration. Thus, we get a family of 2D TFT's parametrised by $H^2(X;\mathbb{C}^*)$.

There is a general method to extend the space of parameters to the rest of $H^{\mathrm{ev}}(X;\mathbb{C})$. The properties of the resulting structure were abstracted into the notion of a *Frobenius manifold* (Dubrovin, Givental, Manin). This includes the *grading*; broken by quantum multiplication, it is restored by grading the parameter space (using the *Euler vector field*): one grades H^2 using $c_1(X)$ and the rest of cohomology by the normalised degree $\deg/2 - 1$.

The Frobenius manifold contain (almost) all answers to enumerative questions about rational curves in X. The geometric explanation lies in a theorem that *genus zero cohomological Field theories in two dimensions are equivalent to germs of Frobenius manifolds*.

Givental's Reconstruction Conjecture

Gromov-Witten theory extends quantum cohomology to curves of any genus. A fundamental result (Ruan, Tian, Li, McDuff, Salomon) ensures that GW invariants are govered by the structure of a (all-genus) Cohomological Field theory. I'll focus on one important consequence of this structure.

Conjecture (Givental, $\sim$ 1999)

For a compact symplectic manifold X whose quantum cohomology ring is generically semi-simple, all Gromov-Witten invariants are determined from genus zero information.

Remark

- Loosely speaking: counting rational curves determines the answer to enumerative questions for curves of all genera.
- Givental gave a formula for the generating function of GW invariants, in terms of quantised quadratic Hamiltonians.

Classification of semi-simple theories

Theorem (T.)

Givental's conjecture holds. More precisely, the GW (descendent) invariants are determined by a recursive relation from the quantum multiplication at a single (but generic) value of the parameter.

This theorem follows from a structural classification.

Theorem (T.)

A CohFT based on a semi-simple Frobenius algebra A is determined by a power series $R(z) \in \mathrm{End}(A)[\![z]\!]$, $R = \mathrm{Id} \pmod{z}$ subject to Givental's symplectic constraint $R(z)R^*(-z) \equiv \mathrm{Id}$.

Remark

- Givental describes R from the Frobenius manifold. But *one* sufficiently generic (quantum) multiplication suffices.
- The essential input in the classification theorem is the *Mumford conjecture* (Madsen-Weiss, 2002)

When does the theorem apply?

Semi-simplicity of quantum cohomology is a very strong condition.

- Projective spaces, Grassmannians and many toric Fanos work.
- Toric manifolds always have semi-simple deformations: to their torus-equivariant cohomology. This was used by Givental in computing their GW theory (verifying his conjecture there).
- Semi-simplicity is preserved by blowing up points (Bayer 2004); in particular, there exist non-Fano examples.
- 36 of the 59 families of 3D Fanos with no odd cohomology have been checked (Ancona-Maggesi 2002, Ciolli 2004).
- On the negative side, if the *even* part of quantum cohomology is semi-simple, then the manifold has even cohomology only and (in the algebraic case) (p, p)-cohomology only (Bayer and Manin 2004, Manin-Hertling-T 2008). This contradicts claims in the literature about complete intersection Fanos.

Dubrovin's conjecture

A remarkable conjecture has gathered experimental support. Recall that an ordered collection $\{\mathscr{E}_i\}$ of objects in a triangulated $\mathbb{C}$-linear category is *exceptional* if $\mathrm{Ext}^*(\mathscr{E}_i, \mathscr{E}_i) = \mathbb{C}$, in degree 0, while if $j > i$, $\mathrm{Ext}^k(\mathscr{E}_j, \mathscr{E}_i) = 0, \forall k$. The collection is *complete* if it generates the (triangulated) category.

Conjecture (Dubrovin)

A projective manifold has semi-simple quantum cohomology iff its derived category of coherent sheaves contains a complete exceptional collection.

Remark

- Ciolli (2004) checks this for 36 families of 3D Fanos.
- Dubrovin also relates the Ext-Euler characteristics to quantum cohomology data. This would be a consequence of some formulations of Mirror symmetry.

Related results of Kontsevich

In the mid-1990's, Kontsevich initiated a programme, *Homological Mirror Symmetry*, which among others should give a far-reaching adaptation of Givental's reconstruction conjecture. (This preceded Givental's cited work, but only converged with it later.) For a recent update, see Katzarkov-Kontsevich-Pantev (2008).

A key step is to replace the notion of cohomological field theory with that of *chain-level, open-closed field* theory. (See also Costello for an implementation of these ideas.)

This is required by the fact that cohomological field theories seem unclassifiable with our limited understanding of Deligne-Mumford spaces. The semi-simple classification was a (pleasant) surprise.

However, applying this programme to Gromov-Witten theory requires the construction a good *Fukaya category* for a symplectic manifold. This is not yet within reach.

Cohomological Field Theory (CohFT)

- Defined by Kontsevich and Manin for application to GW theory, but related notions (Segal's Topological Conformal Field theory) had a parallel development.
- Is a version of TFT for *families of surfaces*, taking values in the cohomology of the parameter space instead of numbers.
- A family of closed surfaces over B gives a class in $H^*(B;\mathbb{C})$.
- A family of surfaces with m input and n output boundaries gives a class in $H^*\big(B;\mathrm{Hom}(A^{\otimes m};A^{\otimes n})\big)$.
- "Gluing $=$ composition" applies in families.
- Nodal degenerations (Lefschetz fibrations) are allowed.
- All this is functorial in the base B.
- It suffices to specify the classes for the universal Lefschetz fibrations, over the *Deligne-Mumford spaces* of *stable curves*.
- Have skipped some details: stability, flat identity ...

Details I: What makes the classification work?

The *Euler class* of a Frobenius algebra A is the product of the co-product of 1: $1 \mapsto A \otimes A \mapsto A$. Pictorially, this is represented by a torus with one outgoing boundary.
For the cohomology ring of a manifold, this is the usual Euler class. However, the quantum Euler class *can* be *inveritble*: this happens iff the quantum multiplication is semi-simple.
Hence, in the semi-simple case, one can increase the genus of surfaces without loss of information in the CohFT.
The *Mumford conjecture* (Madsen-Weiss) describes the complex cohomology of the open moduli space M_g^n of smooth curves in the $g \to \infty$ limit as a free $\mathbb{C}$-algebra in the *tautological* classes κ_j, ψ_i.
From here, we can classify the M_g^n part of semi-simple CohFT's.
Finally, in large g the boundary divisors of $\overline{M}_g^n$ have Euler classes which are not zero-divisors. This controls the problem of extending cohomology classes to the boundary.

Details II: Moral Meaning of $R(z)$

The Frobenius algebra A is associated to the circle in a 2D field theory. Heuristically, it should be viewed as the cohomology of a space Y with circle action. (In all known applications, A is the Hochschild cohomology of a category, so its chain-level model has an algebraic circle action.)

The series R should give a *splitting* of S^1-equivariant cohomology:

$$H^*_{S^1}(Y) \cong H^*(Y) \otimes \mathbb{C}[\![z]\!],$$

where z is the generator of $H^*_{S^1}(\text{point})$.

The *existence* of such a splitting is *necessary* for the extension of a cohomological field theory from the open moduli space M_g^n over its Deligne-Mumford boundary. (Fairly easy.) The core of the classification is that a choice of splitting *determines* this extension.

Details III: Construction of the theory

A cohomological field theory based on A assigns to each allowed pair (n, g) a class in $H^*(\overline{M}_g^n; A^{\otimes n})$. These classes make A into an algebra over the *modular operad* $H^*(\overline{M}_g^n; \mathbb{C})$. Restriction to boundary divisors is subject to *gluing rules*.

Theories can be constructed using the Morita-Mumford-Miller (tautological) classes. Start with $\exp(\sum a_j \kappa_j) \in H^*(\overline{M}_g^n; A)$, co-multiplied out to $A^{\otimes n}$. (The $a_j \in A$ are determined from R.) Twist each output by $R(\psi_i)$, with the ψ-class at the respective marked point. Finally, add recursively, for all boundary strata, terms of the form $(\mathrm{Id} - R(\psi_+)^* R(\psi_-))/(\psi_+ + \psi_-)$, contracted with the classes already constructed on the boundary stratum, and pushed forward by the Thom class.

This construction can be captured by a certain action of matrices $R(z)$ on the cohomology of Deligne-Mumford spaces.

Open Questions

- *Degeneration.* Semi-simple theories come in families with non-semi-simple degenerations: classical cohomology for GW theory, the Jacobian ring for the Landau-Ginzburg B-model (potential with an isolated critical point). The Givental data for semi-simple theories degenerates at such a classical point. Nonetheless, some theories are continuous.

  Problem: Understand this phenomenon.

- *Formality.* GW theory can be defined at *chain level*. A cohomological classification leaves open the possibility of higher operations (Massey products).

  Question: Can this happen in semi-simple field theories?

  The expected answer is no, because of formality of Deligne-Mumford spaces.

Open Questions

- *Twisted Frobenius manifolds.* Gromov-Witten K-theory and the twisted Gromov-Witten invariants (Coates, Givental) as well as other example do not fit the standard definition of Frobenius manifolds. Variations of the notion have been studied by Dubrovin and Manin.
  Problem: Describe Givental's higher genus reconstruction in this more general setting.
- Thank you for your patience!