Modular Tensor Categories and invariants of 3-manifolds

Tensor categories

- Hom(V, W) - $\mathbb{C}$ vector spaces ($\mathbb{C}$- linear additive)
- $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $(V, W) \mapsto V \otimes W$

  functor of the tensor product

$$\begin{tikzcd}
 & \mathcal{C}\times\mathcal{C} \arrow[dr, "\otimes"] \arrow[dd, phantom, "\Downarrow a"] & \\
\underline{\mathcal{C}\times\mathcal{C}}\times\mathcal{C} \arrow[ur, "\otimes\times id"] \arrow[dr, "id\times\otimes"'] & & \mathcal{C} \\
 & \mathcal{C}\times\mathcal{C} \arrow[ur, "\otimes"'] &
\end{tikzcd}$$

$\mathrm{Vect}_{\mathbb{C}}$

$\underset{(\qquad\quad)}{(V\otimes W)\otimes U}$

a - isomorphism of functors, it satifies the pentagon axiom and gives a system of functorial isomorphisms

$a = \{a_{V,U,W}\}$

$$a_{V,U,W} : (V \otimes U) \otimes W \to V \otimes (U \otimes W)$$

$$\begin{tikzcd}
((V_1 \otimes V_2) \otimes V_3) \otimes V_4 \arrow[r, "a"] \arrow[d, "a"] & (V_1 \otimes (V_2 \otimes V_3)) \otimes V_4 \arrow[d, "a"] \\
(V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \arrow[dr, "a"] & V_1 \otimes ((V_2 \otimes V_3) \otimes V_4) \arrow[d, "a"] \\
 & V_1 \otimes (V_2 \otimes (V_3 \otimes V_4))
\end{tikzcd}$$

!

guarantees that

$$(A_1 \otimes \cdots \otimes A_n)_B \simeq (A_1 \otimes \cdots \otimes A_n)_{B'}$$

assoc. + 5-gon

by compositions of a's.

- An identity object $\mathbb{1} \in Ob(\mathcal{C})$

$$\mathbb{1}\otimes V \simeq V\otimes \mathbb{1} \simeq V$$

- commutativity constraint

$$\begin{tikzcd}
\mathcal{C}\times\mathcal{C} \arrow[r, bend left, "\otimes"{name=U}] \arrow[r, bend right, "\otimes^{op}"'{name=D}] & \mathcal{C}
\arrow[from=U, to=D, Rightarrow, "c"']
\end{tikzcd}$$

$\sigma : V\otimes W \longrightarrow W\otimes U$

$v\otimes w \mapsto w\otimes v$

$(V,W) \xrightarrow{\otimes} V\otimes W$ $a, \mathbb{1}, \Delta$

$(V,W) \xrightarrow{\otimes^{op}} W\otimes V$ . . . .

compatibility with a for $\otimes$ is guaranteed by two hexagon axioms.

comm. constraint

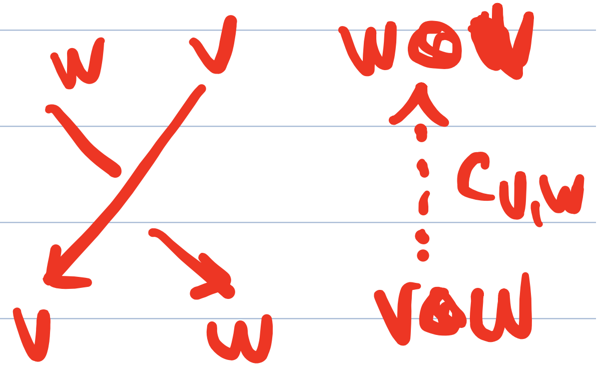
$c = \{c_{V,W}\}$ is a system of functorial isomorphisms

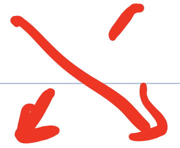
$c_{V,W} : V\otimes W \xrightarrow{\sim} W\otimes V$ hexagon axioms:

$$\begin{tikzcd}
(V\otimes W)\otimes U \arrow[rr, "c\otimes id"] \arrow[dd, dash, "a"] & & (W\otimes V)\otimes U \arrow[d, dash] \\
 & & U\otimes(W\otimes V) \arrow[d, dash] \\
V\otimes(W\otimes U) \arrow[dr, dash, "c"'] & & U\otimes(W\otimes V) \\
 & (W\otimes U)\otimes V \arrow[ur, dashed, dash] &
\end{tikzcd}$$

h.w.

2nd $c \to c^{-1}$



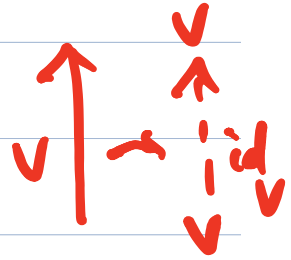


its inv.

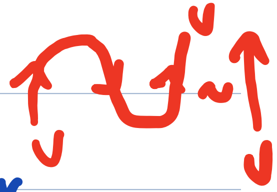
They guarantee that

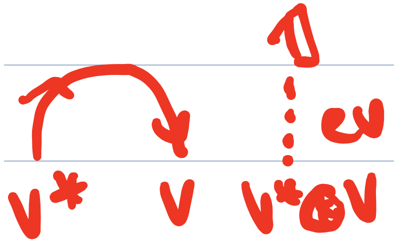
$$(A_{\underline{\sigma_1}}\otimes\cdots\otimes A_{\underline{\sigma_n}})_{\underline{B}} \cong (A_1\otimes\cdots\otimes A_n)_{\underline{B'}}$$

for any $\sigma\in S_n$ and brackets B, B'



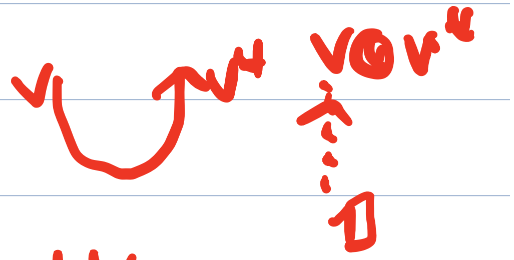
• A right dual to V = (ev_V, η_V, V^*)

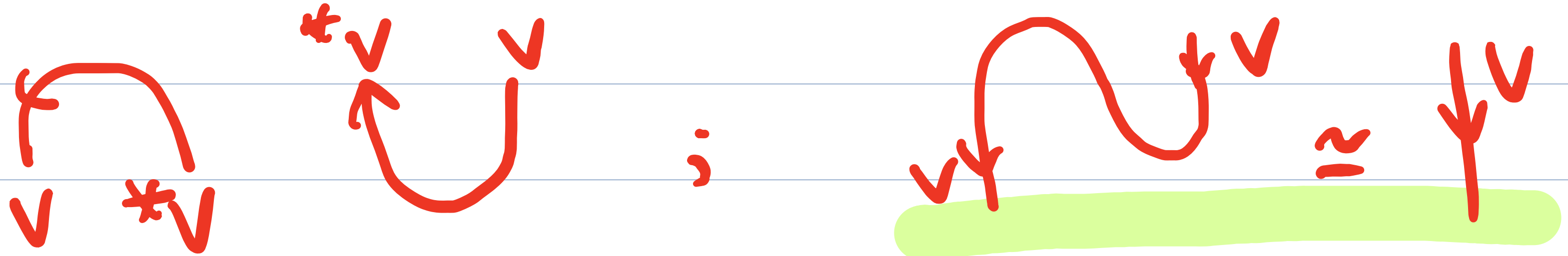




$ev_V : V^* \otimes V \to \mathbb{1}$, $\eta_V : \mathbb{1} \to V \otimes V^*$, axiom

similarly a left dual ${}^*V$





in general ${}^{***}V$, $V^{**\cdots}$ $\left(\begin{array}{l}\text{in } Vect_{\mathbb{C}} \\ V^{**} \cong V\end{array}\right)$

Ribbon categories

A tensor category is ribbon if $\exists\ \theta : id_{\mathcal{C}} \rightleftarrows$

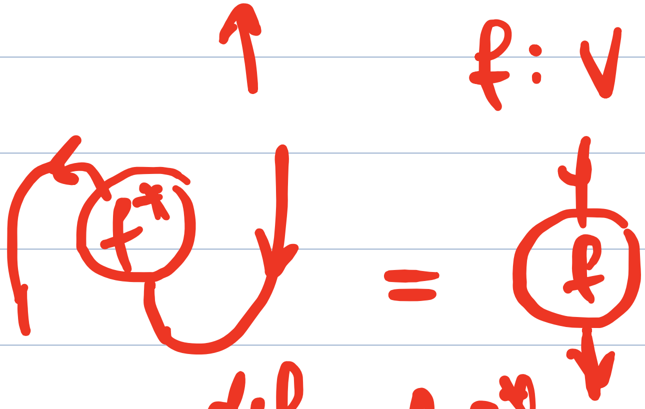
i.e. $\theta = \{\theta_V : V \to V$ functorial$\}$ s.t.

✓ • $\theta_{\mathbb{1}} = id_{\mathbb{1}}$

✓ · $\vartheta_{V\otimes W} = (\vartheta_V \otimes \vartheta_W)\, c^{-1}_{VW}\, c^{-1}_{WV}$

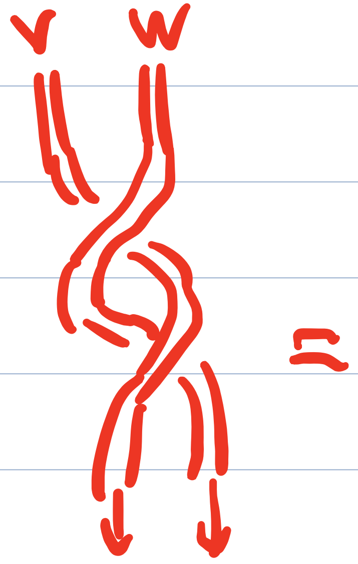
✓ · $\vartheta_V^{*} = \vartheta_{V^*}$

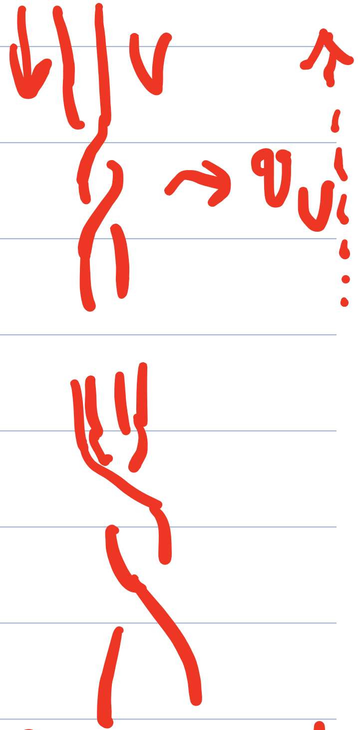
$\uparrow$



$f: V \to W$

def. of $f^*: W^* \to V^*$





full twist of framing

<u>Example:</u> $\mathrm{Vect}_{\mathbb{C}}$, assume for each $V, W \in$ Vect-spaces $\exists\ R^{VW}: V\otimes W \circlearrowleft$ s.t. $R^{VW}_{12} R^{VU}_{13} R^{WU}_{23}$ = opp. product and R is invertible & $(R^{V,W})^{\mathrm{id}\otimes t}$ is also invertible ($M: \mathbb{C}^n \otimes \mathbb{C}^m \circlearrowleft$, M^t, M^{t_1}, M^{t_2}, $M^t = M^{t_1 t_2}$

M is invertible. M^{t_2} also invertible.)

Then $(\mathrm{Vect}_{\mathbb{C}}, R)$ is a <u>tensor category</u>

with $c_{VW} = P^{VW} R^{VW}$, $P^{VW}(v \otimes w) = w \otimes v$

Ribbon structure $\Longleftarrow$ assume certain $\sqrt{\cdots}$ can be consistently taken.
(R. 1989) (Algebra & Analysis)

<u>Examples</u>: H - Hopf algebra. <u>H-mod</u> is always a tensor categ. (no commu. constr.)

if $R \in H \otimes H$ s.t. $\Delta^{op}(a) = R\,\Delta(a)\,R^{-1}$

$(\Delta \otimes id)(R) = R_{13}R_{23}$, $(id \otimes \Delta)R = R_{13}R_{12}$

$\Rightarrow$ <u>H-mod</u> is a tensor cat. $R^{VW} = (\pi^V \otimes \pi^W)(R)$.

? $(H, R \in H \otimes H)$

Drinfeld double: H - any (f.d.) Hopf algebra, $D(H) = H \otimes H^{\text{cop.}}$

s.t.

$$\begin{tikzcd} R^D = \sum_i e_i \otimes e^i \arrow[r, hook] & H \arrow[rrrr, bend right] & \otimes & H^{\text{cop}} \arrow[r, hook] \arrow[rrr, bend right] & D & \otimes & D \end{tikzcd}$$

satisfies (above)

"$a = id$"

$U_q(\mathfrak{g})$, . . .

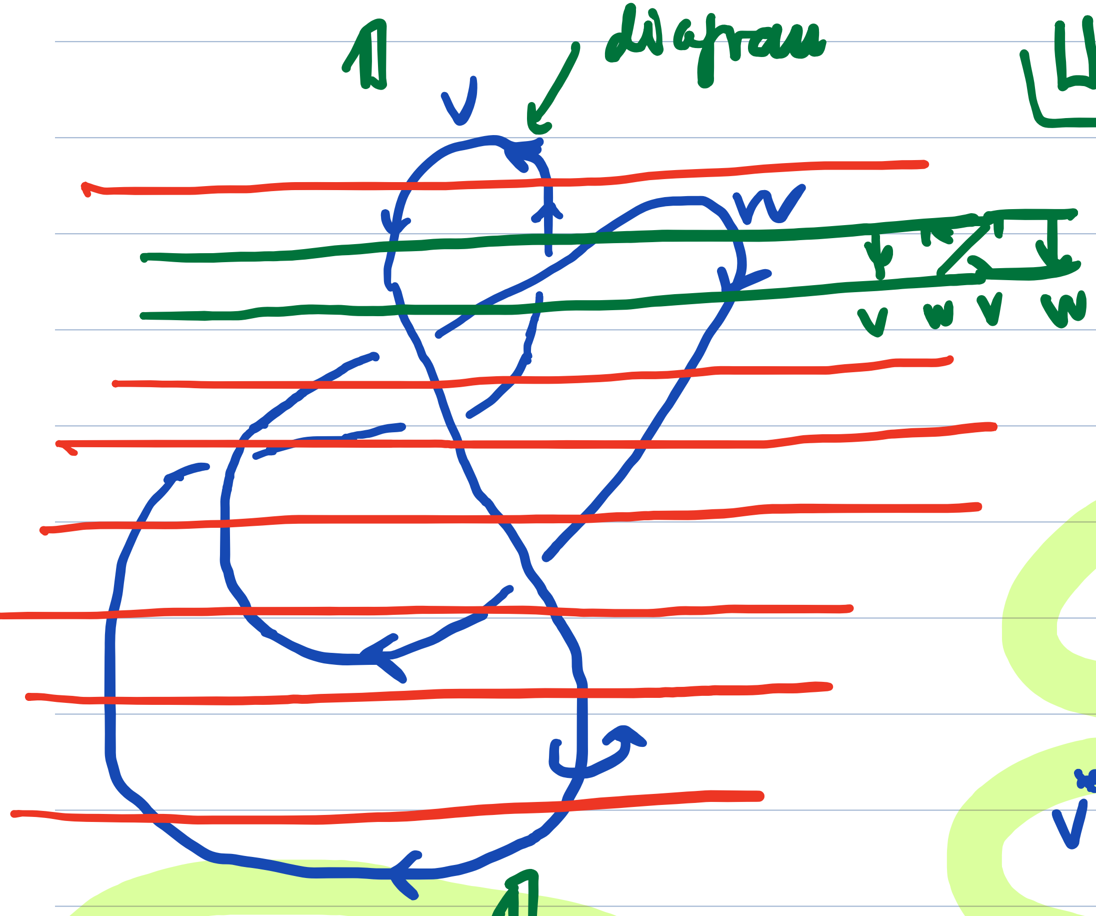
(quasiHopf alg. $a \neq id$)

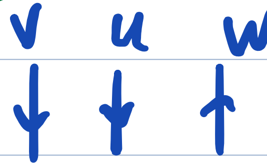
Invariants of framed links from ribbon categories

links ⊂ $\mathbb{R}^3$ ⟶ $\mathbb{R}^2 \subset \mathbb{R}^3$

$\sqcup(S^1)$

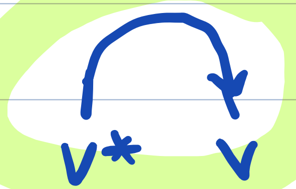
diagrams



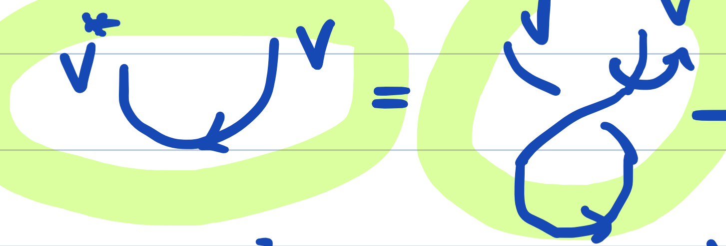


$\to \mathrm{id}_{V\otimes U\otimes W^*}$

assume $a_{V,U,W} = \mathrm{id}_{V\otimes U\otimes W}$



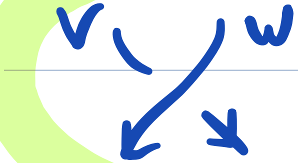
$\to ev_V : V^*\otimes V \to \mathbb{1}$



$\to$

$\to \mathbb{1} \xrightarrow{2} V\otimes V^* \xrightarrow{c} V^*\otimes V \xrightarrow{\mathrm{id}\otimes\nu}$

$\to V^*\otimes V$



$\to c_{V,W}$

Redemeister thm.

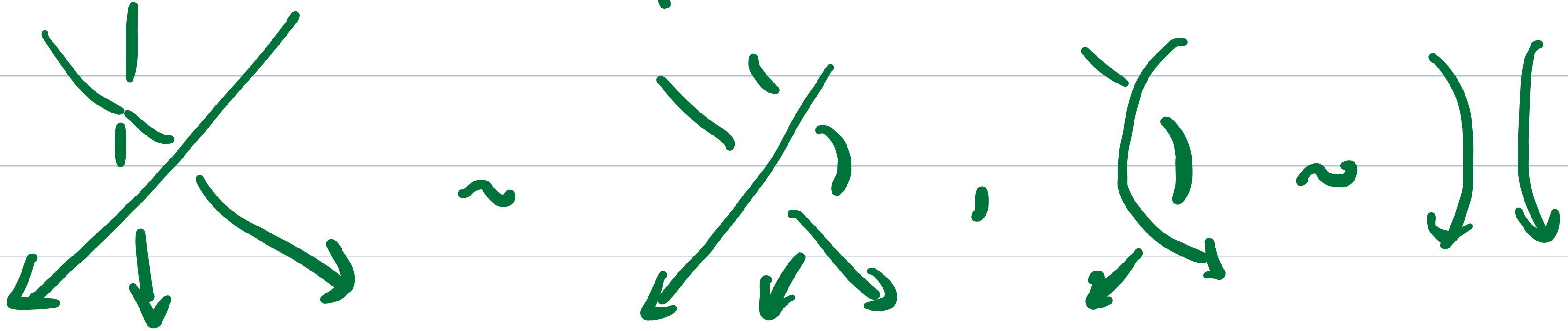
framed

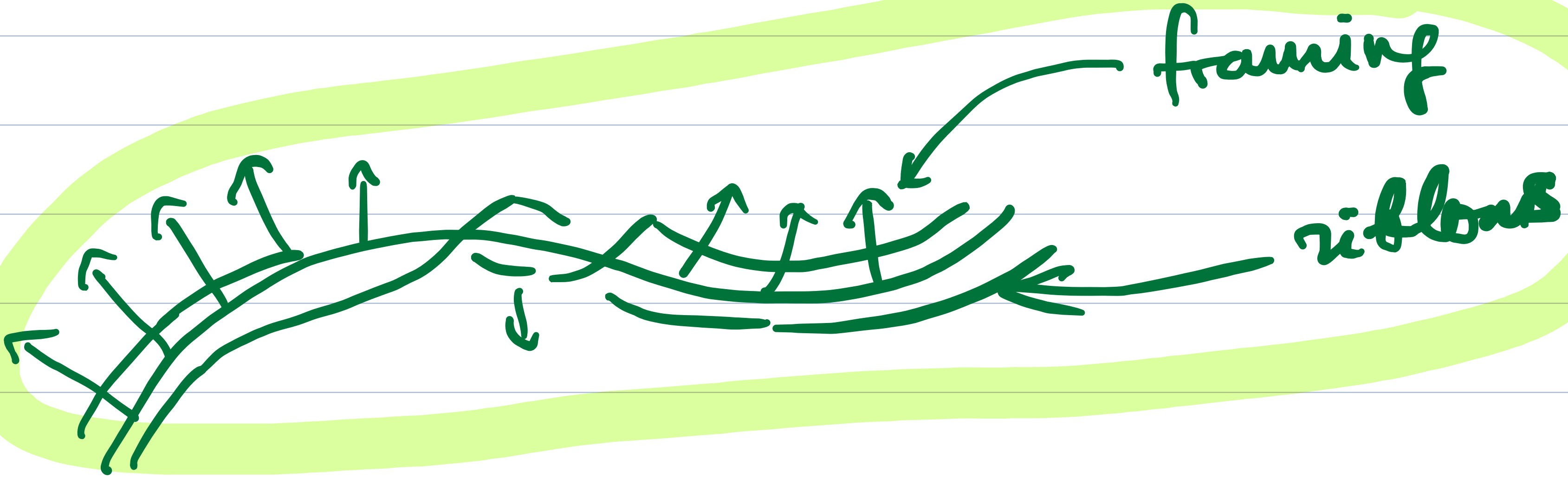
$$D_L \longrightarrow D'_L$$

↑

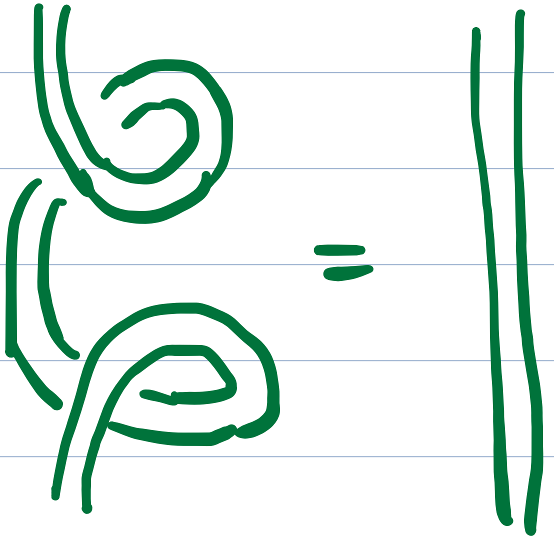
a sequence of Red. moves

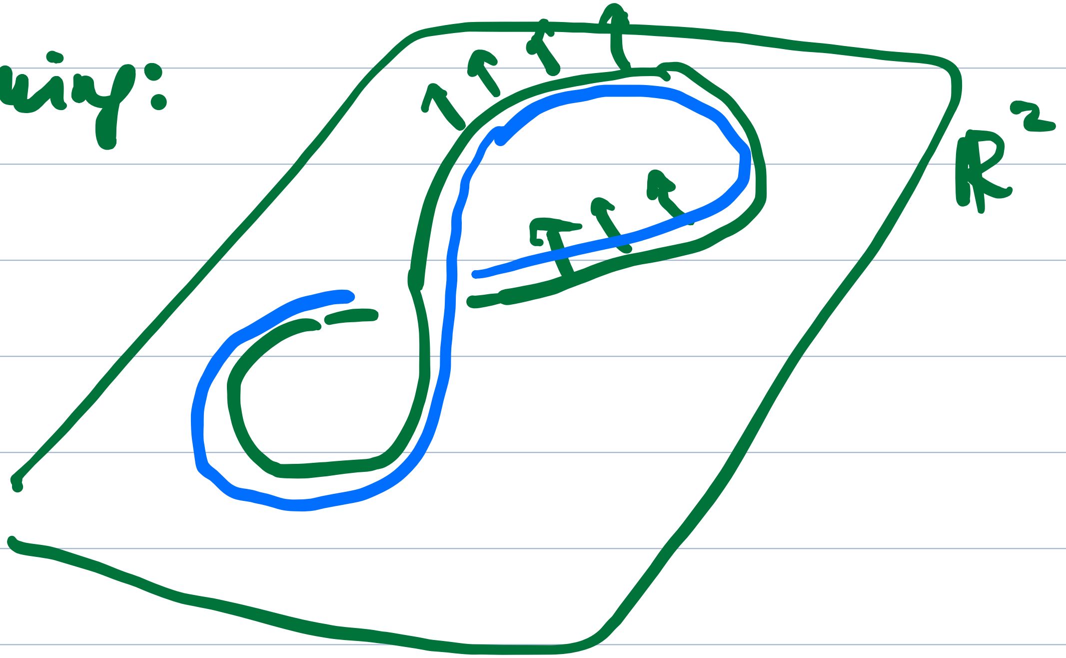
moves for diagrams with Black b. framing





Blackboard framing:





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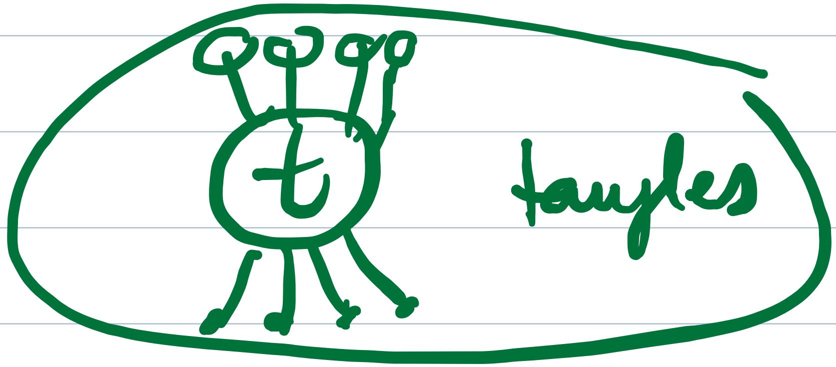


→ (Turaev, book)

(Turaev, R. 1999)

(inv. of tangles)

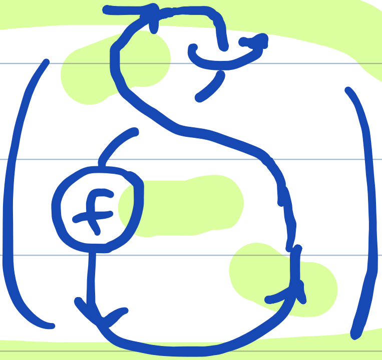
Reidemeister invariant

The composition of such elem. morphisms

Gives $\operatorname{inv}(L^{V_1,\dots,V_\ell}) : \underline{\mathbb{1} \longrightarrow \mathbb{1}}$, i.e. $\operatorname{inv}(L) \in \underset{\operatorname{End}(\mathbb{1})}{\overset{\shortparallel}{\mathbb{C}}}$

$$\operatorname{Tr}(f) : \mathbb{1} \xrightarrow{2} V \otimes V^* \xrightarrow{f \otimes \mathrm{id}} V \otimes V^* \xrightarrow{c} V^* \otimes V \xrightarrow{\mathrm{id} \otimes \nu} V^* \otimes V \to \mathbb{1}$$

$\operatorname{Tr}(f) = \operatorname{inv}\Big($



$\Big)$,

invariant of links colored by objects of $\mathcal{C}$.

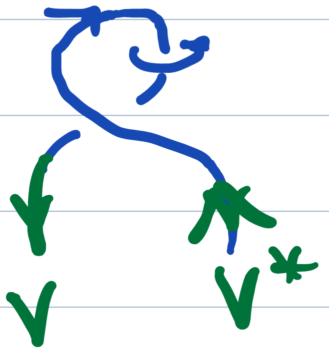
Similarly, invariants of tangles.

$$V^{**} \xrightarrow{\sim} V, \qquad \nu_V : V \circlearrowleft$$

ν can be used to construct $\mu_V : V^{**} \xrightarrow{\sim} V$

$$\mu_{V \otimes W} = \mu_V \otimes \mu_W, \qquad \dim_{\mathcal{C}}(V) = \operatorname{Tr}_V(\mathrm{id}_V)$$

$$\operatorname{Tr}_{V \otimes W}(f \otimes g) = \operatorname{Tr}_V(f)\operatorname{Tr}_W(g)$$



$V \otimes V^* \to \mathbb{1}$

for the right dual:

is the definition of μ

Drinfeld 1989 (A&A) (on central elem.)

R_1 —tr— (explain how to $V \overset{\sim}{\to} V^{**}$)

Modular tensor categories

all our categories are Abelian / $\mathbb{C}$.

Assume that $\mathcal{C}$ - semisimple tensor category with finitely many simple objects V_i / $\mathbb{C}$

- $\mathrm{Hom}(V_i, V_j) = \delta_{ij} \cdot$ 1-dim,

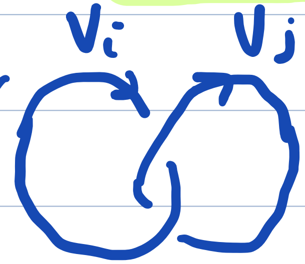
  and any $V \simeq \bigoplus_i V_i^{\oplus n_i}$ (finite sum)

~~finiteness~~ pumbed manifold (Gukov ,...)

(Kruskal, ...) 2020?

Definition A ribbon tensor category is modular (MTC) if

$S_{ij} = \mathrm{Tr}_{V_i \otimes V_j}(c_{V_i,V_j} c_{V_j,V_i}) = \mathrm{inv}\Big($

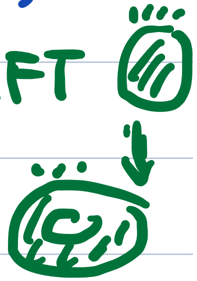


$\Big)$

is nondegenerate

$\boxed{S, T_{ij} = \delta_{ij} \theta_i} \leftarrow SL_2(\mathbb{Z})$ on $\mathbb{C}^{\#(\text{simple objects})}$

(Moore Seiberg) (1990?) CFT



An example: $q = e^{\frac{2\pi i}{K}}$. The category $\mathcal{C}_q$

Simple objects $V_0 = \mathbb{1}, V_1, \cdot\cdot, V_{K-1}$ $\quad 0, 1, \cdot\cdot, K-1 \quad \mathbb{Z}_K$,

- $V_i \otimes V_j \overset{\text{def}}{=} V_{i+j \bmod K}$

- braiding

$$\begin{tikzcd} V_i \otimes V_j \arrow[r, "c"] \arrow[d, equal, "\mathrm{def}"] & V_j \otimes V_i \arrow[d, equal] \\ V_{i+j(k)} \arrow[r] & V_{i+j(k)} \end{tikzcd}, \quad \underset{\underset{c_{ij}}{\shortparallel}}{e}^{\frac{\overset{\sqrt{-1}}{i}\pi}{k}\, \underset{\text{not}}{ij}}$$

- $V_i^* = V_{-i\,(k)}, \; V_i^{**} = V_i$

- $S_{ij} = \exp\left(\frac{2\pi i}{k} ij\right), \quad (S^{-1})_{ij} = \frac{1}{k} e^{-\frac{2\pi i}{k} ij}$

discrete Fourier transform.

Kirby calculus

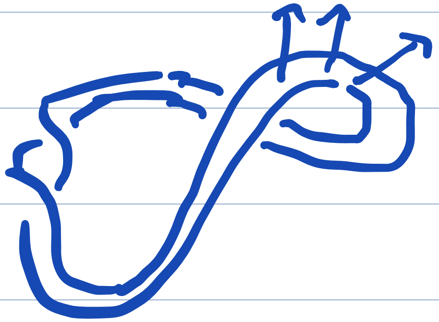
Let $L \subset S^3$, $L = \bigsqcup_{i=1}^{\ell} L_i$

- Remove a tubular nbd along each component of L. The result is a 3d manifold with the boundary $T_1 \sqcup \cdots \sqcup T_\ell$
- Glue solid tori back to T_i's twisting by a diffeomorphism each component.

This is a Dehn surjery.

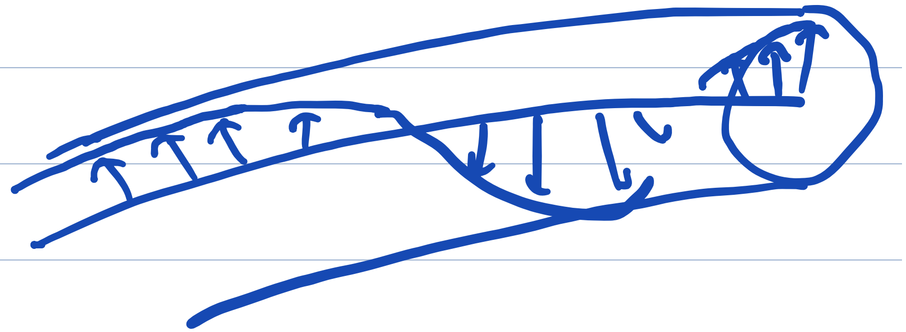
In particular:

- choose a faming on L



- Framing defines a curve on each T_i

It defines the gluing diffeomorphism.



Thm (Lickorish-Wallace, 1960's) Any closed oriented 3d manifold can be obtained by a surjery along a framed link $L \subset S^3$.

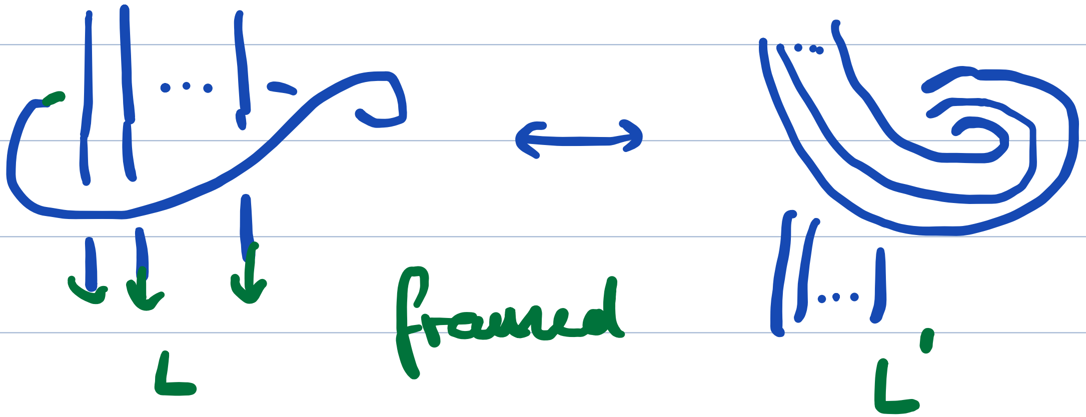
It follows from V.A. Rochlin (1951) thm that each closed oriented 3-manifold bounds a compact oriented 4-Ball.

Kirby calculus.

Thm (Kirby 1978 (Fenn, Rourke 1979)) Let M_L

and $M_{L'}$ be two manifolds obtained by a surjery along L and L' on S^3 respectively.

$M_L \simeq M_{L'}$ if and only if L and L' are related by a sequence of moves



(K)

__Corollary__. If I_L is an invariant of links in S^3 that satisfied (K), then it is an invariant of 3-manifolds (links are not oriented).

Remark This is a nonlocal construction of 3-manifolds.

Local: A triangulation (Turaev-Viro, 199?)

M - closed. $TV(M) = |RT(M)|^2$.

Invariants of closed 3-manifolds

We want to construct invariants of 3-manifolds using Kirby calculus.

closed mflds

Thm (R.-Turaev, MSRI-preprint 1989, $\underline{\underline{1991}}$) ← TQFT

$$\tau(L^{v_1 \cdots v_\ell} \subset M_{L'}) = D^{-|L'|} \left(\frac{p_+}{p_-}\right)^{\sigma(L')}$$

$L_{total} = L \sqcup L'$ ← surgery

$\sigma(L')$: signature of the link matrix of L'

$$\cdot \sum_{\substack{i_1\cdots i_{\ell'} \\ \text{all simple}}} d_{i_1}\cdots d_{i_{\ell'}} \; \underbrace{\mathrm{inv}\left(L^{V_1\cdots V_\ell} \sqcup L'^{V_{i_1}\cdots V_{i_{\ell'}}} \right)}_{\text{invariant of}_{\ell'} \text{ colored link in } S^3}$$

$|L'| = \#$ conn. comp. of $L' = \ell'$; L, L' are framed

$\sigma(L') = \text{sign } lk(L')$

D - funct dim of ℓ.

$$\boxed{d_i = \mathrm{inv}\left(\overset{V_i}{\bigcirc}\right)}, \quad D = \sqrt{\sum_i d_i^2},$$

$$\boxed{p_\pm = \sum_i d_i v_i^{\pm 1}},$$

"$\bigoplus_i V_i$" space where all irreps appear once.

Gelfand "model spaces" $C(G/B)$.

$d_i = \dim_\ell(V_i)$

$D^2 = \sum_i d_i^2$

$D^2 = \dim(k[G])$

G- finite group

$\ell =$ <u>G-mod</u>

invariant with respect to K-moves.

$\underline{\underline{\ell_q}}$

$V_0, \dots, V_{K-1}$, $\quad q^{ij}$, $\quad q = e^{\frac{\pi\sqrt{-1}}{K}}$, $\quad \boxed{U(1)}$

$v_i = q^{i^2}$, $\quad p_\pm = \sum_{i=0}^{K-1} q^{\pm i^2}$, $\quad D = K$,

$$\text{inv} = K^{|L'|} \left(\frac{p_+}{p_-}\right)^{\sigma(L')} \sum_{i_1 \cdots i_{l'}} q^{(i_1, \dots, i_{l'})(\hat{L}')\begin{pmatrix} i_1 \\ \vdots \\ i_{l'} \end{pmatrix}} =$$

$\boxed{\text{no } L}$

$K \to \infty$, "semiclassical limit".

Witten, 1989, $\quad \int e^{ikCS(A)} \mathcal{D}A = \sum_\alpha \tau(A_\alpha) e^{\frac{1}{2} ikCS(A_\alpha)}$

SU(2)

$V_\alpha[A_\alpha]$ = Flat.con/Gauge $\rightarrow$ $(1 + \dots F\alpha d)$

...

How to see this from the comb. side?

L_q-inv ⇝ U(1) Chern-Simons.

path int- pert. side.

Mattes, Polyak. R. ?
199...

Axelrod-Singer!

comb. constr. $U_q(sl_2)$
of inv. of links
with flat conn.
in the complement.

perturb. finite type inv.
of 3-manifolds
$(L \subset M, [A_0])$

$q = e^{\frac{i\pi}{K}}$

$U_q(sl_2)$

$M = S^3 \supset L$ this
is the exp. of
as $K \to \infty$

Specializations to a root of 1.

Relations (Lusztig, 1989) in $U_v(sl_2)$

$$E^{(n)}E^{(m)} = \begin{bmatrix} n+m \\ m \end{bmatrix} E^{(n+m)},$$

$\left\{ E^{(n)}, F^{(n)} - \text{generators} \right.$

$$F^{(n)}F^{(m)} = \begin{bmatrix} n+m \\ m \end{bmatrix} F^{(n+m)},$$

$$E^{(p)}E^{(r)} = \sum_{0 \leq t \leq \min(p,r)} F^{(r-t)} \begin{bmatrix} K;\, 2t-p-r \\ t \end{bmatrix} E^{(p-t)}, \quad p, r \in \mathbb{Z}_{\geqslant 0}$$

Corollary: For $t \in \mathbb{Z}_{\geqslant 0}$, $c \in \mathbb{Z}$

$$\begin{bmatrix} K;c \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{Kv^{c-s+1} - K^{-1}v^{-c+s-1}}{v^s - v^{-s}} \in U_A(sl_2)$$

Claim: $U_A(sl_2)$ is a Hopf subalgebra.

- $R^{V,W} : V \otimes W \to V \otimes W$,
- $R^{V,W} \cdot (\pi^V \otimes \pi^W)(\Delta(a)) = (\pi^V \otimes \pi^W)(\Delta^{op}(a)) \cdot R^{V,W}$
- $$R^{V\otimes W, U} = R_{23}^{W,U} \, R_{13}^{V,U}$$

$$R^{V, W\otimes U} = R_{1,2}^{V,W} \, R_{13}^{V,U}$$

(ii) Lusztig's integral form (with divided powers)

$\mathcal{A} = \mathbb{Z}[q, q^{-1}]$

<u>Definition</u>: $U_{\mathcal{A}}(sl_2) \subset U_v(sl_2)$ is the unital $\mathcal{A}$-subalgebra generated by

$$E^{(n)} = \frac{E^n}{[n]!}, \quad F^{(n)} = \frac{F^n}{[n]!}, \quad n \in \mathbb{Z}_{\geq 1}$$

Relations (Lusztig, 1989) in $U_v(sl_2)$

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$\left\{ E^{(n)}, F^{(m)} - \text{generators} \right.$

$$F^{(n)}F^{(m)} = \begin{bmatrix} n+m \\ m \end{bmatrix} F^{(n+m)},$$

$$E^{(p)}E^{(r)} = \sum_{0 \leq t \leq \min(p,r)} F^{(r-t)} \begin{bmatrix} K;\, 2t-p-r \\ t \end{bmatrix} E^{(p-t)}, \quad p, r \in \mathbb{Z}_{\geq 0}$$

Corollary: For $t \in \mathbb{Z}_{\geq 0}$, $c \in \mathbb{Z}$

$$\begin{bmatrix} K;c \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{Kv^{c-s+1} - K^{-1}v^{-c+s-1}}{v^s - v^{-s}} \in U_A(sl_2)$$

Claim: $U_A(sl_2)$ is a Hopf subalgebra.

(b) $U_q(sl_2)$ is generated by $E, E^{(l)}, F, F^{(l)}$ K, K^{-1}.

(c) $K^{2l} = 1$, K^l is central

(d) $u_q(sl_2) \subset U_q(sl_2)$, $K^{\pm 1}, E, F$ is a Hopf subalgebra

$\underline{u_q(sl_2)-\text{mod}} \supset$ Proj modules. tensor ideal

Thm. $\underline{U_q(sl_2)-\text{mod}}$ / Projectives

is a modular tensor category

Conjecture: corresponding invariants of 3-manifolds when $q = e^{\frac{i\pi}{K}} \xrightarrow[K \to \infty]{}$ semiclassical CS series

- confirmed in cases with only isolated flat connection
- otherwise it is a problem on the semiclassical side.

Fact: Chen, Yang 2015 when $q = e^{\frac{i\pi}{K}m}$ $m \neq 1$ this is not true

instead

$$\tau_M \to e^{K\,\mathrm{vol}(M)} \cdots \quad , \quad K \to \infty$$