TOPOLOGICAL FOUR-MANIFOLDS WITH FINITE FUNDAMENTAL GROUP

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Contents

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Introduction

Let me begin with a quotation from Michael Freedman's celebrated paper *The Disc Theorem for Four-Dimensional Manifolds.*

Metatheorem . *Using the topological surgery and s-cobordism theorems for finite groups, it will be possible to obtain a lot of information on the topological classification of finite groups acting on compact* 1*-connected* 4*-manifolds.*

The purpose of this thesis is to verify Freedman's metatheorem as far as possible in the case of free group actions. More precisely, we want to present methods for a classification of closed topological 4-manifolds with finite fundamental group. For trivial fundamental group, Freedman proved that intersection form and Kirby-Siebenmann invariant classify such manifolds up to homeomorphism and that every unimodular symmetric bilinear form is realized as the intersection form of a 1-connected closed 4-manifold. Besides the topological surgery and s-cobordism theorems, there are four main ingredients in his proof which we have to understand in order to see whether they generalize to other finite groups. These (at that time well-known) results are the following.

1. The homotopy classification of 1-connected 4-dimensional Poincaré complexes.
2. The computation of the surgery groups $L_5^s(1) = 0$ and $L_4^s(1) \cong \mathbb{Z}$.
3. The realization of the torsion classes in
$$\mathcal{N}_4^{TOP}(X) \cong [X, G/TOP] \cong H^2(X;\mathbb{Z}/2) \times H^4(X;\mathbb{Z}) \cong H_2(X;\mathbb{Z}/2) \times \mathbb{Z}.$$
by homotopy self-equivalences of X.
4. The identification of the Kirby-Siebenmann invariant as the map
$$\mathfrak{ks} : \mathcal{N}_4^{TOP}(X) \cong [X, G/TOP] \cong H^2(X;\mathbb{Z}/2) \times H^4(X;\mathbb{Z}) \longrightarrow \mathbb{Z}/2$$
given by $(x, y) \longmapsto \langle x^2 + r_2(y), [X]\rangle$ for a 4-manifold X.

Let me discuss these results in reverse order for a general fundamental group.
ad(4): This is a result of [Kirby-Siebenmann, p.329] and holds for arbitrary fundamental groups.
ad(3): This result was proven in [Wall 1, Ch.16] by the following construction. Given an element $\alpha \in \pi_2 X$, Wall defines a homotopy self-equivalence h_α of X by first pinching off the top cell of X to obtain a map
$$X \longrightarrow X \vee S^4$$
and then mapping X identically and S^4 via $\alpha \circ \eta^2$ to X, where $\eta : S^3 \longrightarrow S^2$ is the Hopf map. Wall then claims that in the surgery sequence h_α maps to the Hurewicz image of α in
$$H_2(X;\mathbb{Z}/2) \subseteq \mathcal{N}_4^{TOP}(X).$$
The exact sequence
$$\pi_2 X \to H_2(X;\mathbb{Z}/2) \to H_2(\pi_1 X;\mathbb{Z}/2) \to 0$$
then shows that in the non simply-connected case, elements in $H_2(\pi_1 X;\mathbb{Z}/2)$ might still cause problems. In fact, even for a cyclic group of order 2, it has only recently been proved by [Hambleton-Kreck 4] (in a completely indirect way) that the nontrivial element in $H_2(\mathbb{Z}/2;\mathbb{Z}/2)$ is also realized by a homotopy self-equivalence of a manifold X. We note that Wall's original argument about the image of the map h_α contains an error which was corrected in a recent paper by [Cochran-Habegger] in the 1-connected case.

ad(2): It is well-known that the surgery obstruction groups are in general very complicated. Although for finite groups there is an induction theory for L-groups, complete computations are carried out only for special finite fundamental groups.

ad(1): This is the main obstacle for extending Freedman's classification result. In the simply-connected case, it is a result of [Milnor] that the intersection form of a Poincaré complex determines the homotopy type and that every unimodular symmetric bilinear form on a finitely generated $\mathbb{Z}$-lattice is realizable. To generalize this result, one has to consider the π_1-equivariant intersection form S_X on the universal covering of X. But again even for $\pi_1 = \mathbb{Z}/2$, this form does not detect the homotopy type of X, the missing invariant in this case being the k-invariant $k_X \in H^3(\pi_1 X; \pi_2 X)$, see [Hambleton-Kreck 1, Rem.4.5]. In this paper, the authors consider the *quadratic* 2-*type*

$$(\pi_1 X, \pi_2 X, k_X, S_X)$$

and show that this algebraic object detects the homotopy type of an oriented Poincaré complex X if $\pi_1 X$ is a finite group with 4-periodic cohomology. (This result was later extended by [Bauer] to finite groups with 4-periodic 2-Sylow subgroups.) For an arbitrary finite group π_1, Hambleton and Kreck show that there is a $\mathbb{Z}/|\pi_1|$-valued primary obstruction and a secondary obstruction with values in a certain finite abelian group $G(\pi_1 X, \pi_2 X)$ such that the vanishing of both obstructions implies that the quadratic 2-type determines the homotopy type of an oriented 4-dimensional Poincaré complex X with fundamental group π_1. In the paper already mentioned, Bauer improves this result by showing that the primary obstruction is in fact $\mathbb{Z}/2$-valued and that the secondary obstruction group is annihilated by 4.

In Part(I) of this thesis we will extend these results in two directions. First we will show that they hold with the appropriate modifications also for nonorientable Poincaré complexes. The arguments will be straightforward generalizations of the oriented case, the interesting difference being that the secondary obstruction group $G(\pi_1 X, \pi_2 X, w_1 X)$ is usually nontrivial in the nonorientable case (and can be arbitrary large), whereas in the oriented case this obstruction group vanishes for all fundamental groups studied so far. Only a very recent calculation by Dr.M.Hennes seems to show that it is nontrivial for the group

$$\pi_1 = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2.$$

In Section 3.3 we apply the nonorientable theory to show that there exists a 4-dimensional Poincaré complex Y with the quadratic 2-type of $\mathbb{RP}^4 \# \mathbb{CP}^2$ but not homotopy equivalent to the same. I conjectured at a very early stage that this space Y is not homotopy equivalent to a manifold but methods for a proof were not then clear. In Section 10.2 we successfully apply a modified surgery approach developed in [Kreck 1] to prove this conjecture.

Our *Main Theorem* (see Section 2.2) on the homotopy classification of Poincaré complexes shows that the $\mathbb{Z}/|\pi_1|$-valued primary obstruction we mentioned above always vanishes, independently of the orientation character of X. We apply this theorem in Section 3.1 to correct the proof of a theorem in [Wall 3, Cor.5.4.1] which states that the signature is not necessarily multiplicative in coverings of Poincaré complexes. Moreover, in Section 3.2 we give an example of a free orientation preserving $\mathbb{Z}/2$-action on a 4-dimensional simply-connected Poincaré complex and show that it cannot be equivariantly homotopy equivalent to a manifold. Finally, in Section 3.4 we conclude from our Main Theorem that the π_1-*fundamental class* of a 4-dimensional Poincaré complex is determined by its quadratic 2-type. Here the π_1-fundamental class of a Poincaré complex X is the element

$$u_*[X] \in H_4^{w_1}(\pi_1) / \mathrm{Out}(\pi_1)_{w_1}$$

where $u : X \longrightarrow K(\pi_1, 1)$ is an arbitrary 2-equivalence.

Unfortunately, the quadratic 2-type is in general not a useful invariant because it is algebraically much too complicated. In fact, there are almost no results on the algebraic classification of equivariant forms, even if one only considers indefinite forms (which are quite simple in the 1-connected case). But if one could understand which quadratic 2-types are realized by closed 4-manifolds with a given fundamental group, the algebra of such realizable equivariant forms might become considerably easier. The main idea how to determine the realizable quadratic 2-types is the following corollary to Freedman's *Disc Embedding Theorem*.

Theorem . *Let π_1 be a finite group. Then a quadratic 2-type (π_1, π_2, k, S) is realized by a closed oriented 4-manifold if and only if the quadratic 2-type*

$$(\pi_1, \pi_2 \oplus (\mathbb{Z}\pi_1)^{2r}, i_*(k), S \perp r \cdot H(\mathbb{Z}\pi_1))$$

is realizable for some $r \in \mathbb{N}$. Here $H(\mathbb{Z}\pi_1)$ denotes the hyperbolic form on the integral group ring $\mathbb{Z}\pi_1$.

More precisely, Freedman shows that an algebraic decomposition

$$(\pi_2 M, S_M) \cong (\pi_2 \oplus (\mathbb{Z}\pi_1)^{2r}, S \perp r \cdot H(\mathbb{Z}\pi_1))$$

is always induced from a geometric decomposition

$$M \approx M' \# r \cdot (S^2 \times S^2).$$

Therefore, the first step for understanding the realizable quadratic 2-types is to understand the stable homeomorphism classification of 4-manifolds. Here the word *stable* means up to connected sum with copies of $S^2 \times S^2$. The following theorem was proved in [Kreck 1].

Theorem . *Two closed 4-manifolds M_1 and M_2 are stably homeomorphic if and only if there exist 2-equivalences $\tilde{\nu}_i$ with the following properties:*

(i) *If ν_i are the topological stable normal Gauß maps for the manifolds M_i then the diagrams*

$$\begin{array}{ccc} & & B \\ & \overset{\tilde{\nu}_i}{\nearrow} & \downarrow \xi \\ M_i & \underset{\nu_i}{\longrightarrow} & BTOP \end{array}$$

commute. Here ξ is the <u>normal 1-type</u> of the manifolds M_i, which is a 2-coconnected fibration over $BTOP$ <u>determined by the</u> existence of $\tilde{\nu}_i$.

(ii) *The pairs $(M_1, \tilde{\nu}_1)$ and $(M_2, \tilde{\nu}_2)$ represent the same element in the bordism group $\Omega_4^{TOP}(\xi)$.*

In Part(II) of this thesis we will first determine and describe all possible normal 1-types ξ which can appear in the above theorem. The new result being that the total space B of ξ is always a classifying space of a topological group G (determined by $\pi_1 M, w_1 M, w_2 M$ and $w_2 \widetilde{M}$) with

$$\pi_0 G \cong \pi_1 BG \cong \pi_1 B \cong \pi_1 M$$

and with connected component G_0 isomorphic either to $SpinTOP$ or $STOP$. Moreover, $\xi = B\rho$ for some representation

$$\rho : G \longrightarrow TOP.$$

Turning around the point of view, we also show that given a topological group G with connected component $SpinTOP$ or $STOP$ and $\pi_0 G$ finitely presentable, there exists a representation $\rho : G \longrightarrow TOP$ such that $B\rho$ is the normal 1-type of some closed 4-manifold. This classifying space description of a normal 1-type ξ then allows a computation of the group $\mathrm{Aut}(\xi)$ of all fiber homotopy classes of fiber homotopy self-equivalences of ξ. This group is important since it determines the possible choices of $\tilde{\nu}_i$, once ν_i and ξ are fixed.

The second new result in Part(II) is the construction of a spectral sequence for the case $w_1 M = 0 = w_2\widetilde{M}$. It converges to $\Omega_*^{TOP}(\xi)$ with E_2-term being equal to

$$E^2_{p,q} = H_p(\Pi; \Omega_q^{SpinTOP}).$$

This spectral sequence was well known (compare Section 5.3) if

$$B \simeq BSpinTOP \times K(\pi_1, 1),$$

but such a decomposition does not exist for all finite groups, see Remark 3.3.3. More generally, in Section 5.3 we give a condition on the real representation theory of π_1 which decides whether or not the total space of a given normal 1-type ξ can be decomposed as above. In Section 6.3 we exploit the fact that our spectral sequence can be defined in a more general setting to prove new results about possible signatures of closed 4-manifolds with a prescribed normal 1-type. For example, we show that given an arbitrary finitely presented group π_1 with $H^2(\pi_1; \mathbb{Z}/2) \neq 0$ there always exists a closed differentiable 4-manifold with fundamental group π_1, signature 8 and universal covering spin. A remarkable consequence of this example is the fact (which we prove in Section 8.1) that for an orientable normal 1-type ξ the Kirby-Siebenmann invariant

$$\mathfrak{ks} : \Omega_4^{\mathrm{TOP}}(\xi) \longrightarrow \mathbb{Z}/2$$

is a split surjection if and only if $w_2(\xi) \neq 0$. Another interesting example we present in the same Section is a closed differentiable 4-manifold with finite (metacyclic) fundamental group, signature 1 and universal covering spin. This example shows that even if the universal covering is spin, there is no hope for a Rohlin type theorem about the divisibility of the signature.

The third main result in Part(II) is the stable homeomorphism classification of closed oriented 4-manifolds with finite fundamental group whose 2-Sylow subgroups have periodic cohomology, see Section 7. Since we have determined all possible normal 1-types ξ in Section 5, by Kreck's Theorem above one only has to compute the corresponding bordism groups $\Omega_4^{TOP}(\xi)$ and to divide out the action of the groups $\mathrm{Aut}(\xi)$. The main technical problem in this computation is the determination of a certain d_3-differential in the above spectral sequence. We first state the spin case.

Theorem (4.4.10)**.** *Two 4-dimensional closed spin manifolds with the same finite fundamental group whose 2-Sylow subgroups have periodic cohomology are stably homeomorphic if and only if they have the same signature, π_1-fundamental class and $\mathfrak{sec}$-invariant.*

Here the $\mathfrak{sec}$-invariant of a 4-manifold M is the $\mathbb{Z}/2$-valued invariant defined by

$$\mathfrak{sec}(\mathfrak{M}) := \begin{cases} 0 & \text{if } S_M = q + q^* \text{ for some } q \in \mathrm{Hom}_{\mathbb{Z}\pi_1}(\pi_2 \otimes \pi_2, \mathbb{Z}), \\ 1 & \text{else.} \end{cases}$$

Note that since $w_2\widetilde{M} = 0$, the main point is that q also has to be π_1-equivariant. For an oriented 4-manifold M with arbitrary fundamental group and $w_2\widetilde{M} \neq 0$, it is easy to see that signature and π_1-fundamental class determine the stable homeomorphism class, compare Section 4. So let me assume that $w_2\widetilde{M} = 0$. Then the element

$$w := (u^*)^{-1}(w_2 M) \in H^2(\pi_1; \mathbb{Z}/2)/_{\mathrm{Out}(\pi_1)}$$

(where $u : M \longrightarrow K(\pi_1, 1)$ is again an arbitrary 2-equivalence) is a well-defined stable homeomorphism invariant which we call the *w_2-type* of M, compare Definition 4.4.1. Recall that $w = 0$ corresponds to the spin case. For two 4-manifolds with the same w_2-type $w \neq 0$, the stable classification result above changes in that one has to add the $\mathfrak{ks}$-invariant since it is not any more determined by the signature as in the spin case. Moreover, the $\mathfrak{sec}$-invariant is always trivial and one has to add, instead, a new $\mathbb{Z}/2$-valued invariant (which can be nontrivial only for generalized quaternion 2-Sylow subgroups). This invariant is a tertiary bordism invariant with respect to our spectral sequence and therefore we call it the $\mathfrak{ter}$-*invariant*.

The main difference between the $\mathfrak{sec}$- and the $\mathfrak{ter}$-invariant is that the former is by definition a homotopy invariant whereas we show in Section 8.2 that the $\mathfrak{ter}$-invariant can take different values on homotopy equivalent manifolds. Our proof uses the fact that for the fundamental groups in question the surgery obstruction for closed 4-manifold problems is detected by the signature, see [Hambleton et al.]. We obtain the first examples of two closed oriented differentiable 4-manifolds which are homotopy equivalent but not stably diffeomorphic. These examples have generalized quaternion fundamental groups and signature $16 \cdot k$ for any given $k \in \mathbb{Z}$.

Let me point out that Part(II) is the heart of my thesis and I have therefore given a more detailed *Outline of the Strategy* in Section 4.

In Part(III) we try to remove the $S^2 \times S^2$-summands and obtain from the stable classification a classification up to homeomorphism. The main tool we use is the following *Cancellation Theorem* from [Hambleton-Kreck 3].

Theorem . *Let M and N be two closed oriented 4-manifolds with finite fundamental group. Suppose that the connected sum $M\#r \cdot (S^2 \times S^2)$ is homeomorphic to $N\#r \cdot (S^2 \times S^2)$. If $N \approx N_0\#(S^2 \times S^2)$ then M is homeomorphic to N.*

Let me discuss the implications of this theorem only for manifolds with universal covering spin. The assumption on N implies that the intersection form on $H_2(N;\mathbb{Z})$ is indefinite. The 1-connected case shows that this assumption is necessary since stabilization with $S^2 \times S^2$ kills all definite forms and leaves the signature as the only invariant. Now suppose we are given two stably homeomorphic spin manifolds M and N with finite fundamental group, same Euler characteristic and indefinite intersection form on $H_2(N;\mathbb{Z})$. Denoting by $|E_8|$ the simply-connected closed 4-manifold with positive definite even intersection form of rank 8, one possibility to show that M and N are homeomorphic is to assume that the signature of N is divisible by 8 and that

$$N\#\tfrac{-\sigma(N)}{8} \cdot |E_8|$$

is stably homeomorphic to a rational homology 4-sphere Σ. Then M and N are both homeomorphic to

$$\Sigma\#\tfrac{\sigma(N)}{8} \cdot |E_8|\#\tfrac{\chi(N)-2-|\sigma(N)|}{2} \cdot (S^2 \times S^2)$$

because the assumption of the cancellation theorem is satisfied for this manifold. (Recall that the intersection form on $H_2(N;\mathbb{Z})$ is indefinite if and only if $\chi(N) - 2 > |\sigma(N)|$.) It follows that for a homeomorphism classification with the above methods, it is very useful to have a rational homology 4-sphere in every stable homeomorphism class (of signature 0). In Section 9.3 we describe several methods for constructing such rational homology 4-spheres and apply these constructions to obtain in Section 9.4 a homeomorphism classification for manifolds with special finite fundamental groups. For example, we prove the following

Theorem (6.4.2)**.** *Two 4-dimensional closed spin manifolds with the same 4-periodic fundamental group and indefinite intersection forms on $H_2(.;\mathbb{Z})$ are homeomorphic if and only if they have the same signature, $\mathfrak{sec}$-invariant and Euler characteristic.*

The same result holds for two oriented manifolds with the same w_2-type $w \neq 0$ if we add the $\mathfrak{ks}$-invariant and replace the $\mathfrak{sec}$- by the $\mathfrak{ter}$-invariant.

Let me close this introduction by pointing out that every Section in this thesis starts with a short summary which also includes all relevant assumptions. Finally we remark that this volume slightly differs from my original PhD thesis in that we left out the proof that the Poincaré complex Y with the quadratic 2-type of $\mathbb{RP}^4 \# \mathbb{CP}^2$ (see Section 3.3) cannot be homotopy equivalent to a manifold. This result is a corollary to the classification of nonorientable closed 4-manifolds with fundamental group $\mathbb{Z}/2$ which is to appear as a joint paper with Ian Hambleton and Matthias Kreck.

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Part 1. **Stable Classification of 4-Dimensional Manifolds**

1. An Outline of the Strategy

As described in the introduction, we divide the problem of classifying 4 -dimensional manifolds into two parts. The first part is the stable classification and the second part considers the cancellation problem which will be described in Part(III) of this thesis.

Definition . Let M, N be two locally oriented, closed $2m$-dimensional manifolds. They are called *stably homeomorphic* if there exist natural numbers r and s such that

$$M \# r \cdot (S^m \times S^m) \approx N \# s \cdot (S^m \times S^m).$$

Here the connected sum has to be formed compatibly with the local orientations. For differentiable manifolds there is an analogous notion of a stable diffeomorphism.

Remark . Although we are looking for a stable homeomorphism classification of topological 4 -manifolds, we will first consider the differentiable category. In Section 8 we will then explain the necessary changes in the topological category. It will turn out that the difference is measured by the Kirby-Siebenmann invariant only. The reason being that the striking results on exotic structures on 4 -manifolds all disappear if one allows stabilization in our sense because the s-cobordism theorem is stably true in the differentiable category, see [Quinn]. For an exact statement see Corollary 5.1.3.

We will now outline the methods of [Kreck 1] to determine the stable diffeomorphism type of a closed differentiable $2m$-dimensional manifold. Let M be such a manifold which we assume to be oriented at a chosen base point. For simplicity, we also assume all manifolds to be connected.

1.Step: Determining the normal $(m-1)$-type of M.

This is the fiber homotopy type of a fibration $\xi : B \longrightarrow BO$ which is determined by the following properties:

(i) ξ is m-coconnected, i.e. $\xi_* : \pi_i B \longrightarrow \pi_i BO$ is an isomorphism for $i > m$ and injective for $i = m$.

(ii) The stable normal Gauß map $\nu : M \longrightarrow BO$ given by some embedding of M into Euclidean space, lifts to an m-equivalence over B, i.e. there exists an m-equivalence $\tilde{\nu} : M \longrightarrow B$ such that the following diagram commutes:

$$\begin{tikzcd} & B \arrow[d, "\xi"] \\ M \arrow[ur, "\tilde{\nu}"] \arrow[r, "\nu"'] & BO \end{tikzcd}$$

Note that property (ii) does not depend on the specific choice of the stable normal Gauß map (i.e. of the embedding), because any two of them are stably isotopic and we assumed the map $\xi : B \longrightarrow BO$ to be a fibration.

In [Kreck 1] $\tilde{\nu}$ is called a *normal* $(m-1)$*-smoothing in* ξ. Moreover, a fibration $\xi : B \longrightarrow BO$ satisfying only property (i) is called $(m-1)$*-universal*. Existence and uniqueness of the normal $(m-1)$-type follow from the theory of Moore-Postnikov decompositions (see [Baues, p.306.311]) applied to the map $\nu : M \longrightarrow BO$. For 4-dimensional manifolds we will prove the following

Theorem (see 2.1.1 and 2.2.1)**.** *The normal* 1*-type of a* 4*-manifold* M *is completely determined by the fundamental group* $\pi_1 M$ *and the Stiefel-Whitney classes* $w_1 M, w_2 M, w_2 \widetilde{M}$.

We will also describe four explicit methods of construction for a 1-universal fibration with given fundamental group and Stiefel-Whitney classes. Thus we will see that this first step is easy in dimension 4, contrary to the ordinary surgery approach where the corresponding first step would be to determine the normal homotopy type of the manifold which, as we saw in Part(I), can be very hard. Roughly, we are just fixing the normal homotopy type of an m-skeleton in M, hoping that Poincaré duality will do the job in the upper half of the dimensions.

2.Step: Stabilizing with $S^m \times S^m$.

If $(M, \tilde{\nu})$ is a normal $(m-1)$-smoothing in ξ then there is an essentially unique normal $(m-1)$-smoothing of $M \# S^m \times S^m$ in ξ.

Definition . Two normal $(m-1)$-smoothing s $(M, \tilde{\nu})$ and $(M', \tilde{\nu}')$ in ξ are called ξ-*diffeomorphic* if there exists a base point and local orientation preserving diffeomorphism $f : M \longrightarrow M'$ such that $\tilde{\nu}$ and $\tilde{\nu}' \circ f$ are fiber homotopic. We define

$$\mathrm{NSt}_{2m}(\xi) := \{ \text{ Stable diffeomorphism classes of normal } (m-1)\text{-smoothings } (M, \tilde{\nu}) \text{ in } \xi \text{ } \}.$$

In fact, we do not want to distinguish between two manifolds which are diffeomorphic but possibly not ξ-diffeomorphic with respect to some normal $(m-1)$-smoothings in ξ. In other words, given a fixed manifold M we do not want to distinguish between different normal $(m-1)$-smoothings $(M, \tilde{\nu})$ in ξ. Therefore, we define

$$\mathrm{MSt}_{2m}(\xi) := \{ \text{ Stable diffeomorphism classes of } 2m\text{-manifolds with } (m-1)\text{-type } \xi \text{ } \}$$

and try to compute this set. Using obstruction theory one proves that two different normal $(m-1)$-smoothings in ξ differ only by a fiber homotopy equivalence of ξ, [Kreck 1, Proposition 1.9]. This shows that if we define $\mathrm{Aut}(\xi)$ to be the group of all fiber homotopy classes of fiber homotopy equivalences of ξ, the forgetful map induces a 1-1 correspondence

$$\mathrm{NSt}_{2m}(\xi)/_{\mathrm{Aut}(\xi)} \longleftrightarrow \mathrm{MSt}_{2m}(\xi).$$

3.Step: Translating into a Bordism Problem.

A normal $(m-1)$-smoothing $(M, \tilde{\nu})$ in ξ determines an element in $\Omega_{2m}(\xi)$, the bordism group of all manifolds with a normal ξ-structure in the sense of [Switzer] or [Stong]. These are bordism classes of manifolds M together with a lift of the stable normal Gauß map over ξ, i.e. the objects are again commutative triangles

$$\begin{array}{ccc} & \overset{\tilde{\nu}}{\nearrow} & B \\ & & \downarrow \xi \\ M & \underset{\nu}{\rightarrow} & BO \end{array}$$

but there are no conditions on the connectivity of $\tilde{\nu}$ whatsoever. Such a map $\tilde{\nu}$ is called a ξ-*structure* on M and the pair $(M, \tilde{\nu})$ is called a ξ-*manifold*. Since $(S^m \times S^m, \tilde{\nu})$ is zero-bordant, there is a well-defined map $\mathrm{NSt}_{2m}(\xi) \longrightarrow \Omega_{2m}(\xi)$. The main ingredient in the whole program is the following

Main Theorem ([Kreck 1, Theorem 2.4])**.** *Let $\xi : B \longrightarrow BO$ be an $(m-1)$-universal fibration and suppose that B has (up to homotopy equivalence) a finite m-skeleton and is connected. Then for $m \geq 2$ the map $\mathrm{NSt}_{2m}(\xi) \longrightarrow \Omega_{2m}(\xi)$ is bijective.*

Here the easy part is surjectivity which can be proven by surgery below the middle dimension. For injectivity one needs the (stable) s-cobordism theorem and a new concept developed by M.Kreck, namely the so called *subtraction of handles* in dimension m.

Note that $\mathrm{Aut}(\xi)$ also acts (linearly!) on $\Omega_n(\xi)$ by composition, so that under the above assumptions we get a 1-1 correspondence

$$\mathrm{MSt}_{2m}(\xi) \longleftrightarrow \Omega_{2m}(\xi)/_{\mathrm{Aut}(\xi)}.$$

4.Step: Computing $\Omega_{2m}(\xi)$.

On the one hand, for $m \geq 3$ we are running into difficulties because, firstly, there are a lot of $(m-1)$-universal fibrations and, secondly, these bordism groups are quite complicated for certain ξ. On the other hand, if one restricts to certain subcategories of manifolds (e.g. complete intersections) then one can obtain quite powerful classification results, see [Kreck 1].

In dimension 4 all possible ξ are known, moreover the special form of these normal 1-types allows an explicit computation of the groups $\Omega_4(\xi)$ in a lot of cases. We shall illustrate this in the following

Example . Let M be oriented with $w_2\widetilde{M} \neq 0$ and set $K := K(\pi_1 M, 1)$. Then the normal 1-type of M is the composition

$$Bi \circ p_1 : BSO \times K \longrightarrow BO$$

of the projection onto the first factor and the standard double covering $Bi : BSO \to BO$. It can easily be seen that in this case $\Omega_n(\xi) \cong \Omega_n^{SO}(K)$, the singular bordism group of maps $u : M \longrightarrow K$. Using the Atiyah-Hirzebruch spectral sequence $H_p(K; \Omega_q^{SO}) \Longrightarrow \Omega_{p+q}^{SO}(K)$ and the low dimensional information

$$\Omega_i^{SO} \cong \begin{cases} 0 & \text{if i=1,2,3,} \\ \mathbb{Z} & \text{if i=0,4} \end{cases}$$

we get an isomorphism

$$\begin{aligned} \Omega_4^{SO}(K) &\longrightarrow \mathbb{Z} \oplus H_4(K) \\ [M, u] &\longmapsto (\mathrm{sign}(M), u_*[M]). \end{aligned}$$

Since a normal 1-smoothing in ξ is just the choice of an orientation and of a 2-equivalence $u : M \longrightarrow K$, we finally obtain the stable classification

$$\mathrm{MSt}_4(\xi) \cong \mathbb{N} \oplus H_4(\pi_1)/_{\mathrm{Out}(\pi_1)}. \qquad \blacksquare$$

In Section 6.1, we shall describe a spectral sequence which generalizes the Atiyah-Hirzebruch spectral sequence from the above example in a way that it applies to all possible orientable normal 1-types.

5.Step: Computing $\mathrm{Aut}(\xi)$ and its action on $\Omega_{2m}(\xi)$.

In a lot of special cases, this step is as easy as in the example above. But in general, the group $\mathrm{Aut}(\xi)$ is quite complicated. In Theorems 2.1.9 and 2.2.6, we compute this group for all 1-universal fibrations and we also give an explicit description of its elements using the fact proven beforehand, namely that all total spaces of such fibrations are classifying spaces of certain topological groups. Moreover, in Section 6.2, we can partially answer the question of how these elements act on $\Omega_4(\xi)$.

Finally, in Section 7, we will put all this information together to give a complete stable classification result in the cases where the fundamental group is finite and has cyclic or quaternion 2-Sylow subgroups.

2. Normal 1-types

In this Section we want to determine all 1-universal fibrations $\xi : B \longrightarrow BO$ up to fiber homotopy type. For the sake of notation, we always assume that B is connected and we have chosen a base point. The homotopy groups of B are then well-defined and have to be those of BO except that $\pi_2 B$ only injects into $\pi_2 BO = \mathbb{Z}/2$ and $\pi_1 B$ can be arbitrary. This naturally leads to the two cases we are going to consider. Since the first case is notably simpler than the second but formally the theorems and proofs are similar, it should be quite convenient for the reader to read Sections 5.1 and 5.2 parallel.

Before we start to develop the details, we have to say some words about the category we are going to work in in this Section. The objects will be the so called CW-spaces which are topological spaces homotopy equivalent to CW-complexes. The morphisms are just continuous maps. In this category the Whitehead Theorem holds, additionally one doesn't leave the category by making maps into fibrations or taking pullbacks of fibrations. It will be understood without mentioning that if we construct a continuous map $\xi : B \longrightarrow BO$, we make it into a fibration by the usual process using the free path space of BO. This is certainly necessary, for example in order that $\Omega_n(\xi)$ becomes a group: To construct an inverse element, one has to lift a homotopy, i.e. a map from the cylinder to BO over ξ.

In the course of constructing 1-universal fibrations, we will use a classifying space functor B from topological groups to CW-spaces. By this we will always mean the simplicial construction given for example in [May] because it has all the properties we are going to need. It will therefore happen that the space BO constructed is only homotopy equivalent to the model obtained through the Grassmanian which we used to define the stable normal Gauß map $\nu M : M \longrightarrow BO$. Thus it is necessary at this point to choose a fixed homotopy equivalence between these two models for BO.

Recall that we have assumed all manifolds to be connected (if the converse is not stated explicitly).

2.1. The Case $\pi_2 B \neq 0$.

Theorem 2.1.1.

a) *The fiber homotopy type of a 1-universal fibration $\xi : B \longrightarrow BO$ with $\pi_2 B \neq 0$ is determined by the isomorphism class of the pair $(\pi_1 B, w_1(\xi))$.*

b) *Given a pair (Π, w_1), where Π is an arbitrary group and $w_1 \in H^1(\Pi; \mathbb{Z}/2)$, one has the following possibilities to describe a 1-universal fibration $\xi : B \longrightarrow BO$ with $\pi_2 B \neq 0$ and $(\pi_1 B, w_1(\xi)) \cong (\Pi, w_1)$:*

(I) *Using the isomorphism $H^1(\Pi; \mathbb{Z}/2) \cong [K(\Pi, 1), K(\mathbb{Z}/2, 1)]$ we define ξ_I by the pullback*

$$\begin{array}{ccc} B_I & \xrightarrow{u_I} & K(\Pi,1) \\ {\scriptstyle \xi_I}\downarrow & & \downarrow{\scriptstyle w_1} \\ BO & \xrightarrow{w_1(\gamma)} & K(\mathbb{Z}/2,1). \end{array}$$

Here γ is the stable universal bundle corresponding to id_{BO} and w_1 is chosen as to be a fibration.

(II) *Using the isomorphism $H^1(\Pi; \mathbb{Z}/2) \cong \mathrm{Hom}(\Pi, \mathbb{Z}/2)$ we form the semidirect product $G := SO \rtimes \Pi$ where the action of Π on SO is given by*

$$\begin{array}{cccccc} w_1 : \Pi \longrightarrow & \mathbb{Z}/2 & \hookrightarrow & O & \cong & \mathrm{Aut}(SO). \\ & \bar{1} & \mapsto & \begin{pmatrix} -1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, A & \mapsto & \textit{conjugation by } A \end{array}$$

Moreover, we have the representation

$$\begin{aligned} \rho : SO \rtimes \Pi &\longrightarrow O \\ (x, g) &\longmapsto x \cdot w_1(g) \end{aligned}$$

which allows us to define $\xi_{II} := B\rho : BG \longrightarrow BO$.

(III) *Using the isomorphisms $H^1(\Pi; \mathbb{Z}/2) \cong \mathrm{Vect}_1^{\mathbb{R}}(K(\Pi, 1)) \cong [K(\Pi, 1), BO(1)]$ we can define*

$$\xi_{III} := Bi \oplus w_1 : BSO \times K(\Pi, 1) \longrightarrow BO$$

where $i : SO \hookrightarrow O$ is the inclusion and thus Bi classifies the universal oriented vector bundle over BSO. Furthermore $\oplus$ denotes the H-space structure on BO given by the Whitney sum of vector bundles.

(IV) *Define B_{IV} to be the double covering of $BO \times K(\Pi, 1)$ determined by the subgroup*

$$\mathrm{Ker}\,(\mathfrak{w}_* : \pi_1(BO \times K(\Pi, 1)) \longrightarrow \pi_1 K(\mathbb{Z}/2, 1))$$

where $\mathfrak{w}$ is the composition

$$\mathfrak{w} : BO \times K(\Pi, 1) \xrightarrow{w_1(\gamma) \times w_1} K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 1) \xrightarrow{+} K(\mathbb{Z}/2, 1)$$

and " + " is the H-space structure on $K(\mathbb{Z}/2, 1)$. Then let

$$\xi_{IV} : B_{IV} \twoheadrightarrow BO \times K(\Pi, 1) \overset{p_1}{\twoheadrightarrow} BO.$$

Proof. a) The above pullback construction(I) clearly gives for any pair (Π, w_1) a 1-universal fibration $\xi_I : B_I \longrightarrow BO$ with $\pi_2 B_I \neq 0$ and $(\Pi, w_1) \cong (\pi_1 B_I, w_1(\xi_I))$. We use the 2-equivalence $u_I : B_I \longrightarrow K(\Pi, 1)$ to identify $\pi_1 B_I$ with Π and also elements of $H^1(B_I; \mathbb{Z}/2)$ with those in $H^1(\Pi; \mathbb{Z}/2)$.
Now take any 1-universal fibration $\xi : B \longrightarrow BO$ with $(\Pi, w_1) \cong (\pi_1 B, w_1(\xi))$ and $\pi_2 B \neq 0$.

These data give a 2-equivalence $u : B \longrightarrow K(\Pi, 1)$ such that $u^*(w_1) = w_1(\xi)$ and by the universal property of the pullback we get commutative diagrams

$$\begin{array}{ccccccccc} B & \xrightarrow{\phi} & B_I & \text{and} & B & \xrightarrow{\phi} & B_I. \\ {\scriptstyle u}\searrow & & \swarrow{\scriptstyle u_I} & & {\scriptstyle \xi}\searrow & & \swarrow{\scriptstyle \xi_I} \\ & K(\Pi,1) & & & & BO & \end{array}$$

Since both B and B_I are 1-universal with $\pi_2 \neq 0$ and u, u_I are 2-equivalences, we see that ϕ is an isomorphism on homotopy groups. We conclude that ϕ is a homotopy equivalence over BO. By a theorem of [Dold] this implies that ϕ is a fiber homotopy equivalence over BO.
b) Similarly as for (I) it is easy to check that ξ_{II}, ξ_{III} and ξ_{IV} are 1-universal with $\pi_2 \neq 0$ and that they represent the given pair (Π, w_1). The corresponding 2-equivalences $B \longrightarrow K(\Pi, 1)$ are obviously given by the canonical projections in all three cases. For example, the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{p_2} & \Pi \\ {\scriptstyle \rho}\big\downarrow & & \big\downarrow{\scriptstyle w_1} \\ O & \xrightarrow{\det} & \mathbb{Z}/2 \end{array}$$

shows that $w_1(\rho) = \det \circ \rho = w_1 \circ p_2 = p_2^*(w_1) \in H^1(G; \mathbb{Z}/2)$. ■

Example 2.1.2. If $w_1 = 0$ then

$$\xi = Bi \circ p_1 : BSO \times K(\Pi, 1) \longrightarrow BO$$

and if $(\Pi, w_1) = (\mathbb{Z}/2, \mathrm{id}_{\mathbb{Z}/2})$ we get $\xi = \mathrm{id}_{BO}$.

We will now determine the manifolds with normal 1-type $\xi : B \longrightarrow BO$ and $\pi_2 B \neq 0$.

Lemma 2.1.3. *Let $\xi : B \longrightarrow BO$ be the normal 1-type of a closed, differentiable 4-manifold M with universal covering $\widetilde{M}$. Then the following conditions are equivalent:*

(i) $\pi_2 B \neq 0$,
(ii) $\nu_* : \pi_2 M \longrightarrow \pi_2 BO = \mathbb{Z}/2$ *is nontrivial,*
(iii) $w_2(\widetilde{M}) \neq 0$.

Proof.
(i)⇔(ii) Let $\tilde{\nu} : M \longrightarrow B$ be a normal 1-smoothing in ξ. Since $\nu = \xi \circ \tilde{\nu}$ the statement follows directly from the properties of the normal 1-type which say that

$$\xi_* : \pi_2 B \longrightarrow \pi_2 BO = \mathbb{Z}/2 \text{ is injective and } \tilde{\nu}_* : \pi_2 M \longrightarrow \pi_2 B \text{ is onto.}$$

(ii)⇔(iii) Using the universal example $BSO = \widetilde{BO} \longrightarrow BO$ one easily checks that under the isomorphisms

$$\mathrm{Hom}(\pi_2 M; \mathbb{Z}/2) \cong \mathrm{Hom}(H_2 \widetilde{M}; \mathbb{Z}/2) \cong H^2(\widetilde{M}; \mathbb{Z}/2)$$

ν_* maps to $w_2(\widetilde{M})$. ■

Corollary 2.1.4. *differentiable* 4 *-manifold with* $w_2(\widetilde{M}) \neq 0$.

a) *The normal 1-type* ξ *of* M *is given by the pair* $(\pi_1 M, w_1 M)$ *under the correspondence of Theorem 2.1.1.*
b) *A pair* (Π, w_1) *occurs as the normal 1-type of such a manifold if and only if* Π *is finitely presentable.*

Proof. a) One can construct a normal 1-smoothing $\tilde{\nu} : M \longrightarrow B_I$ as follows:
Take any 2-equivalence $u : M \longrightarrow K(\pi_1 M, 1)$ and a stable normal Gauß map $\nu : M \longrightarrow BO$. Then $w_1(\nu) = u^*(w_1)$ where $u_I^*(w_1) = w_1(\xi_I)$. This gives a map $\tilde{\nu} : M \longrightarrow B_I$ by the universal property of the pullback. It is easy to see that $\tilde{\nu}$ is a 2-equivalence and by construction one has $\xi_I \circ \tilde{\nu} = \nu$.
b) follows from [Kreck 1, Theorem 2.4] (which I stated in Section 4) by noting that a CW-space has a finite 2-skeleton up to homotopy if and only if its fundamental group is finitely presentable.(Recall that a closed manifold is homotopy equivalent to a finite CW-complex.) ■

In the last part of this Section we want to compute the group $\mathrm{Aut}(\xi)$ of fiber homotopy classes of fiber homotopy equivalences of a 1-universal fibration $\xi : B \longrightarrow BO$ with $\pi_2 B \neq 0$.
We first choose a base point $b_0 \in B$ and denote by $Aut(\xi)^0$ the set of pointed fiber homotopy equivalences modulo pointed fiber homotopies. Also let $[X, Y]^0$ stand for the set of all continuous maps between (X, x_0) and (Y, y_0) modulo pointed homotopies.

Definition 2.1.5. Suppose given a fibration $\xi : B \longrightarrow K$ and a map $f : X \longrightarrow K$. We denote by $[X, B]_f$ the fiber homotopy classes of liftings $\tilde{f} : X \longrightarrow B$ of f over ξ. Similarly $[X, B]_f^0$ stands for the pointed homotopy classes of such liftings.

For an arbitrary fibration ξ clearly $\mathrm{Aut}(\xi) \subseteq [B, B]_\xi$ and $\mathrm{Aut}(\xi)^0 \subseteq [B, B]_\xi^0$. Using the same theorem of [Dold] as in the proof of Theorem 2.1.1, we see that for a 1-universal fibration ξ a lifting of ξ over ξ is a fiber homotopy equivalence if and only if it induces an isomorphism of fundamental groups.

Proposition 2.1.6. *Let* $\xi : B \longrightarrow K$ *be a fibration such that the fiber* F *is a* $K(\pi_1 F, 1)$ *and* $\pi_1 K$ *acts trivially on* $\pi_2 K$*. Furthermore, fix* $f : X \longrightarrow K$ *and let* $f_0, f_1 \in [X, B]_f^0$ *induce the same homomorphism of fundamental groups. Then there is an obstruction*

$$d(f_0, f_1) \in \mathrm{Hom}(\pi_1 X, \mathrm{Ker}(i_* : \pi_1 F \longrightarrow \pi_1 B)) \text{ such that } d(f_0, f_1) = 0 \Longleftrightarrow f_0 = f_1.$$

Proof. Two liftings f_0, f_1 of f are the same as two sections s_0, s_1 of the fibration p obtained by the pullback

$$\begin{array}{ccc} F & = & F \\ \downarrow j & & \downarrow i \\ \hat{B} & \xrightarrow[q]{} & B \\ \downarrow p & & \downarrow \xi \\ X & \xrightarrow{f} & K. \end{array} \tag{II.1}$$

Since the fiber F is connected and $\pi_i F = 0 \quad \forall i \geq 2$, it follows from [Baues, Cor.4.3.2] that s_0 and s_1 are section homotopic (i.e. $f_0 = f_1$) if and only if $s_{0*} = s_{1*} \in \mathrm{Hom}(\pi_1 X, \pi_1 \hat{B})$. The homotopy exact sequence for p breaks up into a family of short exact sequences because of the existence of a section. We thus obtain a commutative diagram

$$\begin{array}{ccc}
0 & \longrightarrow & \mathrm{Ker}(i_*) \\
\downarrow & & \downarrow \\
\pi_1 F & = & \pi_1 F \\
\downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle i_*} \\
\pi_1 \hat{B} & \xrightarrow{q_*} & \pi_1 B \\
\downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle \xi_*} \\
\pi_1 X & \xrightarrow{f_*} & \pi_1 K
\end{array}$$

which shows that for $a \in \pi_1 X$ the quotient

$$s_{0*}(a)^{-1} \cdot s_{1*}(a)$$

lies in the image of j_* since both s_i are sections of p. From the relations $f_i = q \circ s_i$ and $f_{0*} = f_{1*}$, it further follows that this quotient comes from a unique element in $\mathrm{Ker}(i_*)$. Thus the above quotient defines a map

$$d(f_0, f_1) : \pi_1 X \longrightarrow \mathrm{Ker}(i_*).$$

This map will be the desired obstruction once we have shown that it is a homomorphism. It is clear that $d(f_0, f_1)$ is a homomorphism if we can show that $j_*(\mathrm{Ker}(i_*))$ is central in $\pi_1 \hat{B}$. For any fibration $\mathrm{Ker}(i_*)$ is central in $\pi_1 F$ (see [G.W.Whitehead, p.166]) but since we assumed $\pi_1 K$ to act trivially on $\pi_2 K$ we can conclude that $\pi_1 X$ acts trivially on $\mathrm{Ker}(i_*)$ as well. This finishes the proof because the vanishing of $d(f_0, f_1)$ is clearly equivalent to $s_{0*} = s_{1*}$. ■

In our situation, the fiber of a 1-universal fibration $\xi : B \longrightarrow BO$ with $\pi_2 B \neq 0, \Pi := \pi_1 B$ is a $K(\pi_1 F, 1)$ if and only if $w_1(\xi) \neq 0$. In this case the fiber inclusion induces a monomorphism on fundamental groups and thus Proposition 2.1.6 shows that $\mathrm{Aut}(\xi)^0$ injects into $\mathrm{Aut}(\Pi)$.
If ξ is orientable, we can choose an orientation $\bar{\xi} : B \longrightarrow BSO$ whose fiber then is a $K(\Pi, 1)$. Again Proposition 2.1.6 shows that $\mathrm{Aut}(\bar{\xi})^0$ injects into $\mathrm{Aut}(\Pi)$ because the inclusion of the fiber induces a monomorphism on fundamental groups in this case, too. To determine the images in $\mathrm{Aut}(\Pi)$ of theses injections, we use the fact that by description(II)

$$\xi = B\rho : BG \longrightarrow BO \quad \text{and similarly} \quad \bar{\xi} = B\bar{\rho} : BG \longrightarrow BSO$$

for representations $\rho : G = SO \rtimes \Pi \longrightarrow O$ respectively $\bar{\rho} : G \longrightarrow SO$.
To fit the two cases into one notation, we let $\rho' : G \longrightarrow G'$ be defined by the following table:

	$w_1(\xi) \neq 0$	$w_1(\xi) = 0$
$G' :=$	O	SO
$\rho' :=$	ρ	$\bar{\rho}$

Moreover, we define

$$\xi' := B\rho' : B \longrightarrow B' := BG'.$$

With this notation at hand we look at the functor B which on morphism level gives us a homomorphism

$$\mathfrak{B} : \mathrm{Aut}(\rho') := \{\varphi \in \mathrm{Aut}(G) \mid \rho' \circ \varphi = \rho'\} \longrightarrow \mathrm{Aut}(\xi')^0.$$

Lemma 2.1.7. *The image of* $\pi_1 : \mathrm{Aut}(\xi')^0 \longrightarrow \mathrm{Aut}(\Pi)$ *is equal to*

$$\mathrm{Aut}(\Pi)_{w_1} := \{\alpha \in \mathrm{Aut}(\Pi) \mid \alpha^*(w_1) = w_1\}.$$

In particular, $\quad \mathrm{Aut}(\xi')^0 \cong \mathrm{Aut}(\Pi)_{w_1}$.

Proof. On the one hand, since $\phi \in \mathrm{Aut}(\xi')^0$ lies over B' the induced map $\pi_1(\phi)$ lies over $\pi_1 B'$ which is equivalent to $\phi^*(w_1(\xi)) = w_1(\xi)$. On the other hand, given $\alpha \in \mathrm{Aut}(\Pi)_{w_1}$ we get the automorphism $\alpha_G \in \mathrm{Aut}(\rho')$ defined by

$$\alpha_G(x, a) := (x, \alpha(a)).$$

The isomorphism $\pi_1(BG) \cong \pi_0(G) \cong G/_{G_0} \xrightarrow[p_2]{\cong} \Pi$ then shows that $(\pi_1 \circ \mathfrak{B})(\alpha_G) = \alpha$, finishing the proof. ■

Remark . It is very easy to see that in fact $\mathfrak{B}$ is an isomorphism but since we do not need this fact, we skip the proof.

The next step in our computation is to get rid of the base point, i.e. to compare $\mathrm{Aut}(\xi')^0$ with $\mathrm{Aut}(\xi')$. To this end, we again consider a more general setting.

Lemma 2.1.8. *Let* $\xi : B \longrightarrow K$ *be a fibration with connected fiber* F. *If we fix a map* $f : X \longrightarrow K$ *there is an action of* $\pi_1 F$ *on the set* $[X, B]_f^0$ *with orbit space equal to* $[X, B]_f$.

Proof. If K is a point, this is well known fact, see for example [G.W.Whitehead, III(1.11)]. The proof of the relative version is exactly the same as in the absolute case if one uses Theorems I(7.16,7.18) in [G.W.Whitehead]. Here we shall just describe the action of $\pi_1 F$:
Let $\omega : I \longrightarrow F$ be a closed path at $b_0 \in F$ and let $f_0 : (X, x_o) \longrightarrow (B, b_0)$ be a lifting of f over ξ. This gives a commutative diagram

$$\begin{array}{ccc} I \times \{x_0\} \cup \{0\} \times X & \xrightarrow{i \circ \omega \cup f_0} & B \\ {\scriptstyle inclusion}\downarrow & & \downarrow{\scriptstyle \xi} \\ I \times X & \xrightarrow{f \circ p_1} & K. \end{array}$$

By [G.W.Whitehead, I(7.16)] this homotopy lifting extension problem has a solution unique up to homotopy relative $I \times \{x_0\} \cup \{0\} \times X$. Calling such a solution $F : I \times X \longrightarrow B$, we define the action of $[\omega] \in \pi_1 F$ on $[f_0] \in [X, B]_f^0$ by

$$[f_0]^{[\omega]} := [F_1].$$

Note that f_0 and F_1 are freely homotopic over ξ, but the homotopy F between them runs through the path ω. ■

One directly computes that the induced map on the fundamental group is given by

$$[f_0]_*^a = c_{i_*(a)} \circ [f_0]_*$$

where $c_{i_*(a)}$ denotes conjugation by $i_*(a)$ for $a \in \pi_1 F$. We are now able to prove the main result:

Theorem 2.1.9. *If* $\xi : B \longrightarrow BO$ *is a 1-universal fibration with* $\pi_2 B \neq 0$ *and* $(\pi_1 B, w_1(\xi)) \cong (\Pi, w_1)$ *then there is an exact sequence of groups*

$$1 \to Z \to \mathrm{Aut}(\xi) \xrightarrow{\pi_1} \mathrm{Out}(\Pi)_{w_1} \to 1$$

$$\textit{with } Z = \begin{cases} \mathbb{Z}/2 & \textit{if the center of } \Pi \textit{ is contained in } \mathrm{Ker}(w_1), \\ 1 & \textit{else }. \end{cases}$$

Moreover, if $w_1 = 0$ (and thus $Z = \mathbb{Z}/2$) this sequence splits and we obtain an isomorphism

$$\mathrm{Aut}(\xi) \cong \mathbb{Z}/2 \times \mathrm{Out}(\Pi).$$

In particular, the choice of a normal 1-smoothing $\tilde{\nu} : M \longrightarrow B$ consists of the choice of an isomorphism $\varphi : \pi_1 M \longrightarrow \Pi$ such that $\varphi^*(w_1) = w_1 M$ and independently the choice of an orientation of M in the orientable case. In the nonorientable case, one can choose something similar to an orientation if and only if $C(\Pi) \leq \mathrm{Ker}(w_1)$.

Proof. First note that if (in the notation introduced in the table before Lemma 2.1.7) F is the fiber of ξ' then $\pi_1 F \cong \mathrm{Ker}(w_1)$. In the case $w_1 \neq 0$ (i.e. $\xi' = \xi$), the considerations preceding the theorem thus show that $\mathrm{Aut}(\xi)$ is isomorphic to the factor group

$$\mathrm{Aut}(\Pi)_{w_1} / \mathrm{Inn}(\Pi, \mathrm{Ker}(w_1))$$

where $\mathrm{Inn}(\Pi, U) := \{c_u \in \mathrm{Aut}(\Pi) \mid u \in U\}$ for any subgroup U of Π. In our context, we set $U := \mathrm{Ker}(w_1)$ which is then a subgroup of index two. The exact sequence of the statement is given by the following exact sequence:

$$1 \to Z := \mathrm{Inn}(\Pi)/\mathrm{Inn}(\Pi, U) \to \mathrm{Aut}(\Pi)_{w_1}/\mathrm{Inn}(\Pi, U) \to \mathrm{Aut}(\Pi)_{w_1}/\mathrm{Inn}(\Pi) = \mathrm{Out}(\Pi)_{w_1} \to 1.$$

We can read off the isomorphism type of the group Z by looking at the following commutative diagram of exact sequences:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & C(\Pi) \cap U \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & \mathrm{Inn}(\Pi, U) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & C(\Pi) \arrow[r] \arrow[d] & \Pi \arrow[r] \arrow[d] & \mathrm{Inn}(\Pi) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & C(\Pi)/C(\Pi) \cap U \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

We see that

$$C(\Pi)/C(\Pi) \cap U \cong C(\Pi) \cdot U/U = \begin{cases} 1 & \text{if } C(\Pi) \leq U, \\ \mathbb{Z}/2 & \text{else,} \end{cases}$$

which proves the assertion for $w_1 \neq 0$.

In the orientable case, we conclude from the considerations before the theorem that $\mathrm{Aut}(\bar{\xi}) \cong \mathrm{Out}(\Pi)$. The fact that $BSO \longrightarrow BO$ is a double covering shows that there is an exact sequence

$$1 \to \mathrm{Aut}(\bar{\xi}) \to \mathrm{Aut}(\xi) \to \mathbb{Z}/2$$

and we want to show that this gives in fact a decomposition

$$\mathrm{Aut}(\xi) \cong \mathrm{Aut}(\bar{\xi}) \times \mathbb{Z}/2.$$

Consider the map
$$c \times \mathrm{id} : B = BSO \times K(\Pi, 1) \longrightarrow B$$
where c is the covering translation of the double covering. Clearly $c \times \mathrm{id}$ has order 2 and commutes with all elements in $\mathrm{Aut}(\bar{\xi})$ since these are all of the form $\mathrm{id}_{BSO} \times B\alpha, \alpha \in \mathrm{Aut}(\Pi)$. ■

Example 2.1.10. We want to describe an infinite family of examples in which the exact sequence of the above theorem does not split:
Let $U := SL_2(\mathbb{F}_{2^{2n}})$ with $n > 1$ be one of the finite simple groups of Lie type. One knows (compare [Atlas]) that
$$\mathrm{Aut}(U) = \mathrm{Inn}(U) \rtimes \langle \varphi \rangle$$
where φ is the Frobenius automorphism (of order $2n$) of the Galois extension $(\mathbb{F}_{2^{2n}} : \mathbb{F}_2)$. We now form the semidirect product
$$\Pi := U \rtimes \mathbb{Z}/2$$
where $\mathbb{Z}/2$ acts via φ^n on U. Let $w_1 : \Pi \longrightarrow \mathbb{Z}/2$ be the projection with kernel U. Since φ^n is not an inner automorphism of U it follows that $C(\Pi) = 1$ and thus the group Z in Theorem 2.1.9 is nontrivial. If we can show that $\mathrm{Aut}(\Pi)_{w_1}/_{\mathrm{Inn}(\Pi, U)}$ is cyclic of order $2n$ then the exact sequence in question can certainly not split.
To this end, consider the restriction homomorphism
$$\mathfrak{res} : \mathrm{Aut}(\Pi)_{\mathfrak{w}_1} \longrightarrow \mathrm{Aut}(\mathfrak{U}).$$
This map is surjective because inner automorphisms of U can certainly be extended to Π and also the Frobenius extends to Π via the formula
$$\bar{\varphi}(x, a) := (\varphi(x), a).$$
The fact that $C(U) = 1$ easily shows that $\mathfrak{res}$ is also injective and thus we obtain a commutative diagram
$$\begin{tikzcd}
\mathrm{Inn}(\Pi, U) \arrow[r, "\mathfrak{res}", "\cong"'] \arrow[d] & \mathrm{Inn}(U) \arrow[d] \\
\mathrm{Aut}(\Pi)_{w_1} \arrow[r, "\mathfrak{res}", "\cong"'] & \mathrm{Aut}(U).
\end{tikzcd}$$
But this shows that
$$\mathrm{Aut}(\Pi)_{w_1}/_{\mathrm{Inn}(\Pi, U)} \cong \mathrm{Aut}(U)/_{\mathrm{Inn}(U)} \cong \langle \varphi \rangle \cong \mathbb{Z}/2n. \quad ■$$

2.2. The Case $\pi_2 B = 0$.

Theorem 2.2.1.

a) *The fiber homotopy type of a 1-universal fibration $\xi : B \longrightarrow BO$ with $\pi_2 B = 0$ is determined by the isomorphism class of the triple $(\pi_1 B, w_1(\xi), w_2(\xi))$.*

b) *Given a triple (Π, w_1, w_2), where Π is an arbitrary group and $w_i \in H^i(\Pi; \mathbb{Z}/2)$, one has the following possibilities to describe a 1-universal fibration $\xi : B \longrightarrow BO$ with $\pi_2 B = 0$ and $(\pi_1 B, w_1(\xi), w_2(\xi)) \cong (\Pi, w_1, w_2)$:*

(I) *Using the isomorphisms $H^i(\Pi; \mathbb{Z}/2) \cong [K(\Pi, 1), K(\mathbb{Z}/2, i)]$ we define ξ_I by the pullback*

$$\begin{tikzcd} B_I \arrow[r, "u_I"] \arrow[d, "\xi_I"'] & K(\Pi,1) \arrow[d, "w_1 \times w_2"] \\ BO \arrow[r, "w_1(\gamma)\times w_2(\gamma)"] & K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2). \end{tikzcd}$$

Here γ is again the stable universal bundle corresponding to id_{BO} and $w_1 \times w_2$ is chosen as to be a fibration.

(II) *Using the isomorphism $H^1(\Pi; \mathbb{Z}/2) \cong \mathrm{Hom}(\Pi, \mathbb{Z}/2)$ and the fact that the center of the group Spin is $\mathbb{Z}/2$, form the extension $Spin \rightarrowtail G \twoheadrightarrow \Pi$ determined by*

- *The action*

$$\begin{array}{ccccccccc} w_1 : \Pi \to & \mathbb{Z}/2 & \hookrightarrow & O & & \cong & \mathrm{Aut}(SO) & \cong & \mathrm{Aut}(Spin). \\ & \bar{1} & \mapsto & \begin{pmatrix} -1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} & =: r_1 & \mapsto & \text{conj. by } r_1 & \mapsto & \text{conj. by } e_1 \end{array}$$

- *The extension class $w_2 \in H^2(\Pi; \mathbb{Z}/2) = H^2(\Pi; C(Spin))$.*

Moreover, in the proof we will write down an explicit representation $\rho : G \longrightarrow O$ which allows us to define $\xi_{II} := B\rho : BG \longrightarrow BO$.

(III) *If there exists a stable vector bundle η over $K(\Pi, 1)$ with $w_i(\eta) = w_i$ we can define*

$$\xi_{III} := Bp \oplus \eta : BSpin \times K(\Pi, 1) \longrightarrow BO.$$

Here $p : Spin \longrightarrow O$ is the composition of the universal covering $Spin \twoheadrightarrow SO$ and the inclusion $i : SO \hookrightarrow O$.

(IV) *Define B_{IV} to be the pullback of the path-fibration*

$$\begin{tikzcd} B_{IV} \arrow[r] \arrow[d, "\mathfrak{p}"'] & P(K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2)) \arrow[d] \\ BO \times K(\Pi,1) \arrow[r, "\mathfrak{w}"] & K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2). \end{tikzcd}$$

Here $\mathfrak{w} : BO \times K(\Pi, 1) \xrightarrow{(w_1(\gamma) +_1 w_1) \times (w_2(\gamma) +_2 w_2)} K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2)$ and " $+_i$ " are the H-space structures on $K(\mathbb{Z}/2, i), i = 1, 2$. Then let

$$\xi_{IV} : B_{IV} \overset{\mathfrak{p}}{\twoheadrightarrow} BO \times K(\Pi, 1) \overset{p_1}{\twoheadrightarrow} BO.$$

Remark . Since $H^2(.; \mathbb{Z}/2)$ does not correspond directly to any vector bundle information, the assumption in (III) is not always fulfilled. We will discuss necessary and sufficient conditions for the existence of the vector bundle η for a given group Π in Section 5.3.

Proof. We can follow the lines of the proof of Theorem 2.1.1:
a) Again the above pullback construction(I) clearly gives for any triple (Π, w_1, w_2) a 1-universal fibration $\xi_I : B_I \longrightarrow BO$ with $\pi_2 B_I = 0$ and $(\Pi, w_1, w_2) \cong (\pi_1 B_I, w_1(\xi_I), w_2(\xi_I))$. In this case we use that $\pi_2 B_I = 0$ implies that we get a **3**-equivalence $u_I : B_I \longrightarrow K(\Pi, 1)$ to identify $\pi_1 B_I$ with Π and also elements of $H^i(B_I; \mathbb{Z}/2)$ with those in $H^i(\Pi; \mathbb{Z}/2), i = 1, 2$.

Now take any 1-universal fibration $\xi : B \longrightarrow BO$ with $(\Pi, w_1, w_2) \cong (\pi_1 B, w_1(\xi), w_2(\xi))$ and $\pi_2 B = 0$. These data give a 3-equivalence $u : B \longrightarrow K(\Pi, 1)$ such that $u^*(w_i) = w_i(\xi)$ and by the universal property of the pullback we again get commutative diagrams

$$\begin{array}{ccccccc} B & \xrightarrow{\phi} & B_I & \text{and} & B & \xrightarrow{\phi} & B_I. \\ {\scriptstyle u}\searrow & & \swarrow{\scriptstyle u_I} & & {\scriptstyle \xi}\searrow & & \swarrow{\scriptstyle \xi_I} \\ & K(\Pi,1) & & & & BO & \end{array}$$

Since both B and B_I are 1-universal with $\pi_2 = 0$ and u, u_I are 3-equivalences, we see that ϕ is an isomorphism on homotopy groups. Finally we can again conclude that ϕ is a fiber homotopy equivalence over BO.
b) De novo, we only have to check that the descriptions(II),(III) and (IV) give vector bundles with the correct Stiefel-Whitney classes because it is obvious from the constructions that they are 1-universal with $\pi_2 B = 0$.
For (IV) this follows from the fact that $\mathfrak{p}^*(w_i(\gamma)) = \mathfrak{p}^*(w_i) \in H^i(B_{IV}; \mathbb{Z}/2), i = 1, 2$. For (III) this is also clear since $w(Bp \oplus \eta) = w(Bp) \times w(\eta)$ and $w_i(Bp) = 0 \quad \forall i \leq 3$.
For (II) we first remark that one defines the Stiefel-Whitney classes of a representation $\rho : H \longrightarrow O(n)$ by applying the B-functor to get $B\rho : BH \longrightarrow BO(n)$ and then setting $w_i(\rho) := w_i(B\rho) \in H^i(BH; \mathbb{Z}/2)$. If H is a discrete group this is the so called *flat bundle construction* and gives $w_i(\rho) \in H^i(H; \mathbb{Z}/2) := H^i(BH; \mathbb{Z}/2)$. As an example, note that $w_1(\rho) = \det \circ \rho \in \mathrm{Hom}(H; \mathbb{Z}/2) \cong H^1(H; \mathbb{Z}/2)$ which we already used in the proof of Theorem 2.1.1, part b), (II).

To continue the proof, let $w_2 \in H^2(\Pi; \mathbb{Z}/2)$ be represented by the central extension

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \widetilde{\Pi} \xrightarrow{q} \Pi \longrightarrow 1.$$

Since $q^*(w_2) = 0$ and w_2 also determines the extension

$$1 \longrightarrow Spin \longrightarrow G \longrightarrow \Pi \longrightarrow 1$$

from the theorem, we see that pulling back this extension via q, we get the semidirect product $Spin \rtimes \widetilde{\Pi}$ where the action is given by

$$\begin{array}{rccl} w_1 \circ q : \widetilde{\Pi} \to & \mathbb{Z}/2 & \hookrightarrow & \mathrm{Aut}(Spin). \\ & \bar{1} & \mapsto & \text{conj. by } e_1 \end{array}$$

Moreover, if $\tau \in \widetilde{\Pi}$ denotes the central involution with $q(\tau) = 1$ then

$$G \cong Spin \rtimes \widetilde{\Pi} /_{\langle(-1, \tau)\rangle}$$

which can be read off from the following commutative diagram of exact sequences:

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & & \mathbb{Z}/2 \arrow[d] \arrow[r, equal] & \mathbb{Z}/2 \arrow[d] & \\
1 \arrow[r] & Spin \arrow[r, "i_1"] \arrow[d, equal] & Spin \rtimes \widetilde{\Pi} \arrow[r, "p_2"] \arrow[d] & \widetilde{\Pi} \arrow[r] \arrow[d, "q"] & 1 \\
1 \arrow[r] & Spin \arrow[r] & G \arrow[r, "u"] \arrow[d] & \Pi \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd}$$

Here $u : G \cong Spin \rtimes \widetilde{\Pi}/_{\langle(-1,\tau)\rangle} \twoheadrightarrow \Pi$ is given by $u[x,g] = q(g) \quad \forall x \in Spin, g \in \Pi$. Furthermore, we can define a representation

$$\begin{array}{rclcl} \rho : G \cong & Spin \rtimes \widetilde{\Pi}/_{\langle(-1,\tau)\rangle} & \longrightarrow & O \\ & [x,g] & \longmapsto & p(x)\cdot w_1(g). \end{array} \tag{II.2}$$

The commutative diagram

$$\begin{tikzcd}
G \arrow[r, "u"] \arrow[d, "\rho"'] & \Pi \arrow[d, "w_1"] \\
O \arrow[r, "\det"] & \mathbb{Z}/2
\end{tikzcd}$$

shows that $w_1(\rho) = \det\circ\rho = w_1 \circ u = u^*(w_1) \in H^1(G;\mathbb{Z}/2)$ and we are finished if we show $w_2(\rho) = u^*(w_2)$:

By naturality, the 2^{nd} Stiefel-Whitney class of a representation $\rho : H \longrightarrow O(n)$ is represented by the following pullback extension:

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}/2 \arrow[r] \arrow[d, equal] & \widetilde{H} \arrow[r] \arrow[d, "\tilde{\rho}"] & H \arrow[r] \arrow[d, "\rho"] & 1 \\
1 \arrow[r] & \mathbb{Z}/2 \arrow[r] & Pin(n) \arrow[r, "r"] & O(n) \arrow[r] & 1
\end{tikzcd}$$

Here $Pin(n)$ is the usual subgroup of the units of the Clifford algebra $Cl(n)$ corresponding to the standard *positive* definite scalar product on $\mathbb{R}^n$. This means that $Cl(n)$ has $\mathbb{R}$-algebra generators $e_1, \ldots, e_n$ with $e_i^2 = 1$. Some authors denote this algebra by $Cl(n)^+$ (and the above subgroup by $Pin(n)^+$), whereas they write $Pin(n)^-$ for the subgroup in the $\mathbb{R}$-algebra $Cl(n)^-$ with $e_i^2 = -1$ corresponding to the negative definite scalar product on $\mathbb{R}^n$. The point here is that $Pin(n)^+$ corresponds to w_2 whereas $Pin(n)^-$ corresponds to $w_2 + w_1^2 \in H^2(BO(n);\mathbb{Z}/2)$. This can be seen by taking

$$\rho : \mathbb{Z}/2 \longrightarrow O(n), \quad \rho(\bar{1}) := \begin{pmatrix} -1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

and noting that $w_2(\rho) = 0$ because ρ is the stabilization of a 1-dimension representation. But the pullback of ρ under $Pin(n)^+$ is clearly trivial ($e_1^2 = 1$) whereas the pullback under $Pin(n)^-$ is the group $\mathbb{Z}/4$ ($e_1^2 = -1$) and thus is nontrivial.

Returning from the general discussion to our special case, we see that we get a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}/2 \arrow[r] \arrow[d, equal] & Spin(n) \rtimes \widetilde{\Pi} \arrow[r] \arrow[d, "\tilde{\rho}"] & G(n) \arrow[r] \arrow[d, "\rho|_{G(n)}"] & 1 \\
1 \arrow[r] & \mathbb{Z}/2 \arrow[r] & Pin(n) \arrow[r, "r"] & O(n) \arrow[r] & 1
\end{tikzcd}$$

by setting

$$G(n) := Spin(n) \rtimes \widetilde{\Pi}/_{\langle(-1,\tau)\rangle} \quad \text{and} \quad \tilde{\rho}(x,g) := j(x) \cdot w(g).$$

Here $j : Spin(n) \hookrightarrow Pin(n)^+$ is the inclusion (since $w_1 = 0$ on $SO(n)$ one has $Spin(n)^+ \cong Spin(n)^-$) and

$$w : \widetilde{\Pi} \xrightarrow{q} \Pi \xrightarrow{w_1} \mathbb{Z}/2 = \langle e_1 \rangle \subset Pin(n)^+.$$

Taking the union over all $n \in \mathbb{N}$ completes the proof of the theorem. ■

Example 2.2.2. If $w_1 = w_2 = 0$ then

$$\xi = Bp \circ p_1 : BSpin \times K(\Pi, 1) \longrightarrow BO$$

and if $(\Pi, w_1, w_2) = (\mathbb{Z}/2, \mathrm{id}_{\mathbb{Z}/2}, w_2)$ we get

$$\xi^\pm = Br^\pm : BPin^\pm \longrightarrow BO$$

depending on $w_2 \in H^2(\Pi; \mathbb{Z}/2) = \mathbb{Z}/2$.

Corollary 2.2.3. *Let M be a closed,differentiable 4 -manifold with $w_2(\widetilde{M}) = 0$.*

a) *The normal 1-type ξ of M is given by the triple $(\pi_1 M, w_1(\nu M), w_2(\nu M))$ under the correspondence of Theorem 2.2.1.*
b) *A pair (Π, w_1, w_2) occurs as the normal 1-type of such a manifold if and only if Π is finitely presentable.*

The proof of this corollary is identical to the proof of Corollary 2.1.4, so we skip it.

In accordance to Section 5.1, the last part of this Section is devoted to the computation of $\mathrm{Aut}(\xi)$ for a 1-universal fibration $\xi : B \longrightarrow BO$ with $\pi_2 B = 0, \pi_1 B = \Pi$. Again we choose a base point in B and start with the computation of $\mathrm{Aut}(\xi)^0$, recalling that $\mathrm{Aut}(\xi)^0 \subseteq [B,B]^0_\xi$ is the subset of those liftings which induce an isomorphism on the fundamental group. As in Section 5.1, we let $\xi' : B \longrightarrow B'$ be equal to ξ for $w_1(\xi) \neq 0$ respectively to a lifting $\bar{\xi} : B \longrightarrow BSO$ if $w_1(\xi) = 0$.
Then the fiber F of ξ' is a $K(\pi_1 F, 1)$ and hence Proposition 2.1.6 applies. Note however that the inclusion of the fiber is no more injective on fundamental groups, but now has kernel $\mathbb{Z}/2$. This shows that the kernel $\mathrm{Aut}(\xi')_{\pi_1}$ of the homomorphism

$$\pi_1 : \mathrm{Aut}(\xi')^0 \longrightarrow \mathrm{Aut}(\Pi)$$

admits an injective obstruction map

$$\mathfrak{o} : \mathrm{Aut}(\xi')_{\pi_1} \rightarrowtail H^1(\Pi; \mathbb{Z}/2)$$

defined by $\mathfrak{o}(\varphi) := d(\mathrm{id}_B, \varphi)$, compare Proposition 2.1.6 for the definition of d.

Lemma 2.2.4. *The image of $\pi_1 : \mathrm{Aut}(\xi')^0 \longrightarrow \mathrm{Aut}(\Pi)$ is contained in*

$$\mathrm{Aut}(\Pi)_{w_1,w_2} := \{\alpha \in \mathrm{Aut}(\Pi) \mid \alpha^*(w_i) = w_i, i = 1,2\}.$$

Proof. This can be read off directly from description(I) in Theorem 2.2.1. ∎

To determine the exact image of the maps π_1 and $\mathfrak{o}$ and to find the group structure of $\mathrm{Aut}(\xi')^0$, we will now use the fact that by description(II), we know that

$$\xi' = B\rho' \text{ with } \rho' : G \longrightarrow G', \quad G' = O \text{ or } SO \text{ respectively },$$

and thus we have the homomorphism

$$\mathfrak{B} : \mathrm{Aut}(\rho') \longrightarrow \mathrm{Aut}(\xi')^0.$$

Proposition 2.2.5. *There is a* realization homomorphism

$$\mathfrak{r} : H^1(\Pi;\mathbb{Z}/2) \times \mathrm{Aut}(\Pi)_{w_1,w_2} \longrightarrow \mathrm{Aut}(\rho') \textit{ such that:}$$

(i) *The image of* $(\mathfrak{B}\circ\mathfrak{r})|H^1(\Pi;\mathbb{Z}/2)\times\{\mathrm{id}_\Pi\}$ *lies in* $\mathrm{Aut}(\xi')_{\pi_1}$ *and composed with the obstruction map* $\mathfrak{o}$ *gives the identity on* $H^1(\Pi;\mathbb{Z}/2)$.
(ii) $(\pi_1 \circ \mathfrak{B} \circ \mathfrak{r})|\{0\} \times \mathrm{Aut}(\Pi)_{w_1,w_2} = \mathrm{id}_{\mathrm{Aut}(\Pi)_{w_1,w_2}}$.

In particular, the map

$$\mathfrak{B} \circ \mathfrak{r} : H^1(\Pi;\mathbb{Z}/2) \times \mathrm{Aut}(\Pi)_{w_1,w_2} \longrightarrow \mathrm{Aut}(\xi')^0$$

is an isomorphism of groups.

Remark . It is easy to see that the homomorphism $\mathfrak{r}$ (and thus also $\mathfrak{B}$) is an isomorphism. But since we do not need this fact, we shall skip the details.

Proof. Recall from (II.2) in the proof of Theorem 2.2.1 that ρ' is given by

$$\begin{array}{rccc} \rho' : G \cong & Spin \rtimes \widetilde{\Pi}/_{\langle(-1,\tau)\rangle} & \longrightarrow & G' \\ & [x,g] & \longmapsto & p(x)\cdot w_1(g). \end{array}$$

Let $\alpha \in \mathrm{Aut}(\Pi)_{w_1,w_2}$ and $\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{\Pi})_{w_1\circ q}$ be the unique automorphism which makes the diagram

$$\begin{tikzcd} 1 \arrow[r] & \langle\tau\rangle \arrow[r] \arrow[d, equal] & \widetilde{\Pi} \arrow[r, "q"] \arrow[d, "\widetilde{\alpha}"] & \Pi \arrow[r] \arrow[d, "\alpha"] & 1 \\ 1 \arrow[r] & \langle\tau\rangle \arrow[r] & \widetilde{\Pi} \arrow[r, "q"] & \Pi \arrow[r] & 1 \end{tikzcd}$$

commutative. Then the map $\alpha_G : G \longrightarrow G$ defined by

$$\alpha_G[x,a] := [x,\widetilde{\alpha}(a)]$$

is a well-defined group homomorphism which lies over ρ'.
Next let $\beta \in \mathrm{Hom}(\Pi,\mathbb{Z}/2)$ and set $\tilde{\beta} := \beta \circ p_2 \in \mathrm{Hom}(G,\mathbb{Z}/2)$. Since the element $\sigma := [1,\tau]$ is a central involution in G with $p_2(\sigma) = 0$, the map

$$\begin{array}{rccc} \beta_G : & G & \longrightarrow & G \\ & g & \longmapsto & g\cdot\sigma^{\tilde{\beta}(g)} \end{array}$$

also is a group homomorphism over ρ'. Note that $\tilde{\beta}(\sigma) = 0$ implies that $\beta_G(\sigma) = \sigma$ which in turn shows that the correspondence

$$\begin{array}{ccc} H^1(\Pi;\mathbb{Z}/2) & \longrightarrow & \mathrm{Aut}(G) \\ \beta & \longmapsto & \beta_G \end{array}$$

is a group homomorphism. By construction all α_G commute with all β_G and thus we obtain a homomorphism

$$\mathfrak{r} : H^1(\Pi;\mathbb{Z}/2) \times \mathrm{Aut}(\Pi)_{w_1,w_2} \longrightarrow \mathrm{Aut}(\rho') \qquad \text{by setting}$$

$$\mathfrak{r}(\beta, \alpha) := \beta_G \cdot \alpha_G.$$

The isomorphism $\pi_1 BG \cong \pi_0 G \cong G/_{G_0} \xrightarrow[p_2]{\cong} \Pi$ shows that

$$(\pi_1 \circ \mathfrak{B})(\beta_G) = \mathrm{id}_\Pi \text{ and } (\pi_1 \circ \mathfrak{B})(\alpha_G) = \alpha$$

which proves property(ii) of the assertion. Furthermore, we see that $H^1(\Pi; \mathbb{Z}/2) \times \{\mathrm{id}_\Pi\}$ is mapped to $\mathrm{Aut}(\xi')_{\pi_1}$ under $\mathfrak{B} \circ \mathfrak{r}$. Thus the proof is finished if we show that for all $\beta \in \mathrm{Hom}(\Pi, \mathbb{Z}/2)$ we have the relation $(\mathfrak{o} \circ \mathfrak{B})(\beta_G) =: \mathfrak{o}(B\beta_G) = \beta$.
To prove this, let $\Pi' := \mathrm{Ker}(\rho') = q^{-1}(\mathrm{Ker}(w_1) \leq \widetilde{\Pi}$ and note that $\Pi' \cong \pi_1 F$ if F is the fiber of $\xi' : B \longrightarrow B'$. Recall from Proposition 2.1.6 that the obstruction map $\mathfrak{o}$ was obtained by forming the pullback diagram(II.1). This pullback diagram can be obtained in our situation by applying the functor B to the following pullback diagram of groups:

$$\begin{tikzcd}
\Pi' \arrow[r, equal] \arrow[d, "j"] & \Pi' \arrow[d] \\
\hat{G} \arrow[r] \arrow[d] & G \arrow[d, "\rho'"] \\
G \arrow[r, "\rho'"] & G'
\end{tikzcd}$$

where $\hat{G} = \{(g_1, g_2) \in G \times G \mid \rho'(g_1) = \rho'(g_2)\}$.
We can now construct the sections s_0 and s_1 which correspond to the automorphisms id_G and β_G over ρ'. These sections are obviously given by the formulae

$$s_0(g) = (g, g) \quad \text{and} \quad s_1(g) = (\beta_G(g), g) \quad \forall g \in G.$$

Therefore, we have for all $g \in G$

$$\begin{aligned} s_0(g)^{-1} \cdot s_1(g) &= (g^{-1} \cdot \beta_G(g), 1) = (\sigma^{\tilde{\beta}(g)}, 1) \\ &= j(\tau^{\tilde{\beta}(g)}). \end{aligned}$$

Using again the isomorphism $\pi_1 BG \cong \pi_0 G \cong G/_{G_0} \xrightarrow[p_2]{\cong} \Pi$, we can conclude that for any $a \in \Pi$ the equation

$$\mathfrak{o}(B\beta_G)(a) = d(\mathrm{id}_B, B\beta_G)(a) = (Bs_0)_*(a)^{-1} \cdot (Bs_1)_*(a) = \beta(a)$$

holds true. ■

The next step in our computation is to compare $\mathrm{Aut}(\xi')^0$ with $\mathrm{Aut}(\xi')$. In Lemma 2.1.8 we have described an action of $\pi_1 F \cong \Pi'$ on $\mathrm{Aut}(\xi')^0$ such that the orbit space is equal to $\mathrm{Aut}(\xi')$. Our aim is to describe now an action of Π' on $\mathrm{Aut}(\rho')$ such that $\mathfrak{B}$ is a Π'-equivariant map. Such an action can be obtained as follows:
Recall that $\Pi' = \mathrm{Ker}(\rho')$ and thus conjugation by $a \in \Pi'$ gives an element $c_a \in \mathrm{Aut}(\rho')$. Let Π' act on $\mathrm{Aut}(\rho')$ via the formula

$$\varphi^a := c_a \circ \varphi, \qquad a \in \Pi', \varphi \in \mathrm{Aut}(\rho').$$

Under these actions, $\mathfrak{B}$ is Π'-equivariant because of the general fact that for any $g \in G$ the map $\mathfrak{B}(c_g)$ is freely homotopic to id_{BG} under a homotopy which at the base point runs through the path $g \cdot G_0 \in G/_{G_0} \cong \pi_0 G \cong \pi_1 BG$. Moreover, since in our situation $g = a \in \mathrm{Ker}(\rho')$ this homotopy can be chosen to lie over $B\rho'$. We are now able to prove the main result:

Theorem 2.2.6. *If $\xi : B \longrightarrow BO$ is a 1-universal fibration with $\pi_2 B = 0$ and $(\pi_1 B, w_1(\xi), w_2(\xi)) \cong (\Pi, w_1, w_2)$ then there is an exact sequence of groups*

$$1 \to Z \to \mathrm{Aut}(\xi) \xrightarrow{\mathfrak{o} \times \pi_1} H^1(\Pi;\mathbb{Z}/2) \times \mathrm{Out}(\Pi)_{w_1,w_2} \to 1$$

$$\text{with } Z = \begin{cases} \mathbb{Z}/2 & \text{if the center of } \widetilde{\Pi} \text{ is contained in } \Pi', \\ 1 & \text{else }. \end{cases}$$

Here $\Pi' := q^{-1}(\mathrm{Ker}(w_1))$ and $w_2 \in H^2(\Pi;\mathbb{Z}/2)$ classifies the extension

$$1 \to \mathbb{Z}/2 \to \widetilde{\Pi} \xrightarrow{q} \Pi \to 1$$

Moreover, if $w_1 = 0$ (and thus $Z = \mathbb{Z}/2$) this sequence splits and we obtain an isomorphism

$$\mathrm{Aut}(\xi) \cong \mathbb{Z}/2 \times H^1(\Pi;\mathbb{Z}/2) \times \mathrm{Out}(\Pi)_{w_2}.$$

In particular, the choice of a normal 1-smoothing $\tilde{\nu} : M \longrightarrow B$ consists of the choice of an isomorphism $\varphi : \pi_1 M \longrightarrow \Pi$ such that $\varphi^*(w_i) = w_i(\nu M), i = 1, 2$ and independently the choice of an orientation of M in the orientable case. In the nonorientable case, one can choose something similar to an orientation if and only if $C(\widetilde{\Pi}) \leq \Pi'$.
Finally, in all cases one is allowed to choose independently something similar to a spin structure in the sense that two such choices differ by an element of $H^1(\Pi;\mathbb{Z}/2) \cong H^1(M;\mathbb{Z}/2)$.

Proof. First recall that Π' is the fundamental group of the fiber of $\xi : B \longrightarrow B'$. In the case $w_1 \neq 0$ (i.e. $\xi' = \xi$ and thus Π' has index 2 in $\widetilde{\Pi}$), the considerations preceding the theorem together with the natural isomorphism $\mathrm{Aut}(\Pi)_{w_1,w_2} \cong \mathrm{Aut}(\widetilde{\Pi})_{w_1 \circ q}$ thus shows that

$$\mathrm{Aut}(\xi) \cong H^1(\Pi;\mathbb{Z}/2) \times \left(\mathrm{Aut}(\widetilde{\Pi})_{w_1 \circ q} / \mathrm{Inn}(\widetilde{\Pi}, \Pi') \right).$$

Replacing the pair $(\Pi, \mathrm{Ker}(w_1))$ in the proof of Theorem 2.1.9 by the pair $(\widetilde{\Pi}, \Pi')$, we see that in the exact sequence

$$1 \to Z := \mathrm{Inn}(\widetilde{\Pi}) / \mathrm{Inn}(\widetilde{\Pi}, \Pi') \to \mathrm{Aut}(\widetilde{\Pi})_{w_1 \circ q} / \mathrm{Inn}(\widetilde{\Pi}, \Pi') \to \mathrm{Out}(\widetilde{\Pi})_{w_1 \circ q} = \mathrm{Out}(\Pi)_{w_1,w_2} \to 1$$

the group Z has the isomorphism type described in the statement. This proves the theorem if $w_1 \neq 0$.
In the orientable case, we conclude from the considerations before the theorem that

$$\mathrm{Aut}(\bar{\xi}) \cong H^1(\Pi;\mathbb{Z}/2) \times \mathrm{Out}(\Pi)_{w_2}.$$

The fact that $BSO \longrightarrow BO$ is a double covering shows that there is an exact sequence

$$1 \to \mathrm{Aut}(\bar{\xi}) \to \mathrm{Aut}(\xi) \to \mathbb{Z}/2$$

and we want to show that this gives in fact a decomposition

$$\mathrm{Aut}(\xi) \cong \mathrm{Aut}(\bar{\xi}) \times \mathbb{Z}/2.$$

As in the corresponding situation in the proof of Theorem 2.1.9, we will describe an involution $c \in \mathrm{Aut}(\xi) \setminus \mathrm{Aut}(\bar{\xi})$ which commutes with all elements of $\mathrm{Aut}(\bar{\xi})$. To find c, we first note that the group $G = Spin \times \widetilde{\Pi} / \langle(-1,\tau)\rangle$ is a subgroup of index 2 in the group

$$H := Pin \times \widetilde{\Pi} / \langle(-1,\tau)\rangle.$$

Denoting the projection $Pin \longrightarrow O$ by r, we have a representation $\eta : H \longrightarrow O$ given by $\eta[x,a] := r(x)$. Then the following diagrams commute:

$$\begin{array}{ccc} G & \xrightarrow{i_G} & H \\ \downarrow{\scriptstyle \bar\rho} & & \downarrow{\scriptstyle \eta} \\ SO & \xrightarrow{i_{SO}} & O \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{i_G} & H \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \eta} \\ O & = & O \end{array}$$

This gives a double covering $BG \longrightarrow BH$ whose covering translation c lies over $B\rho$ but not over $B\bar\rho$. To check that c commutes with all automorphisms $\alpha_G, \beta_G \in \mathrm{Aut}(\bar\rho)$ constructed in Proposition 2.2.5, we note that there are automorphisms $\alpha_H, \beta_H \in \mathrm{Aut}(\eta)$ restricting to α_G respectively β_G. Namely, we define

$$\alpha_H[x,a] := [x, \widetilde\alpha(a)] \quad \text{ and } \quad \beta_H(g) := g \cdot \sigma^{\tilde\beta(g)},$$

i.e. we use exactly the same formulae as in Proposition 2.2.5. It is clear from the definition that all $\alpha_H \in \mathrm{Aut}(\eta)$, but since $\sigma = [1,\tau] \in G \le H$ is central in all of H and $\eta(\sigma) = 1$, we can conclude that also all $\beta_H \in \mathrm{Aut}(\eta)$.

Now let $\varphi_H \in \mathrm{Aut}(\eta)$ denote one of these automorphisms and denote by φ_G its restriction to G. We have a commutative diagram

$$\begin{array}{ccccccc} BG & \xrightarrow{B\varphi_G} & BG & \xrightarrow{c} & BG & \xrightarrow{B(\varphi_G^{-1})} & BG \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ BH & \xrightarrow{B\varphi_H} & BH & \xrightarrow{\mathrm{id}} & BH & \xrightarrow{B(\varphi_H^{-1})} & BH \end{array}$$

where all horizontal arrows stand for the double covering $BG \longrightarrow BH$.

This diagram shows that the map $B(\varphi_G^{-1}) \circ c \circ B\varphi_G$ is a covering translation. It must be equal to c since $c \neq \mathrm{id}_{BG}$. Therefore, $B\varphi_G$ commutes with c. ∎

2.3. When is $B \simeq BSpin \times K(\Pi, 1)$?

All four constructions from Theorems 2.1.1 and 2.2.1 do have their advantages: The pullback construction(I) already proved to be useful as a technical tool in the uniqueness statements of the theorems. The *classifying space* description(II) is interesting because it shows that a normal 1-smoothing is just a lift of the stable structure group O of the stable normal bundle of a manifold to a group which is uniquely determined by the manifold and can be any extension of SO respectively $Spin$ by a finitely presentable group. Also, this description was essential in the computation of $\mathrm{Aut}(\xi)$. The fiber construction(IV) will be used to describe an interesting invariant called $\mathfrak{sec}$ in Section 6.2. Finally, description(III) is very convenient in the computation of the corresponding bordism groups, as we shall show below.

The title of this Section is explained by observing that a 1-universal fibration $\xi : B \longrightarrow BO$ with $\pi_2 B = 0$ allows description(III) if and only if

$$B \simeq BSpin \times K(\Pi, 1).$$

Before we start with the computation of bordism groups, we want to recall the definition of the Thom spectrum of a stable vector bundle $\xi : B \longrightarrow BO$:

Definition 2.3.1. Let $B_n := \xi^{-1}(BO(n))$, $\xi_n := \xi|_{B_n} : B_n \longrightarrow BO(n)$ and

$$T(\xi_n) := \text{Thom space of } \xi_n := D(\xi_n)/S(\xi_n).$$

There are commutative diagrams

$$\begin{array}{ccc} \xi_n \oplus \epsilon_{\mathbb{R}} & \longrightarrow & \xi_{n+1} \\ \downarrow & & \downarrow \\ B_n & \longrightarrow & B_{n+1} \end{array}$$

which give maps $s_n : S^1 \wedge T(\xi_n) \longrightarrow T(\xi_{n+1})$ and one calls the corresponding spectrum $\{T(\xi_n), s_n \mid n \geq 0\}$ the *Thom spectrum* $M\xi$.

Remarks:

1. The homotopy type of $M\xi$ does not depend on the special choice of the vector bundles ξ_n as long as the direct limit of their bases B_n gives the space B ([Lewis et al.]). In particular we could have chosen B_n as some n-skeleton of B (up to homotopy) and would have obtained a CW-spectrum in the sense of [Adams].
2. In [Lewis et al.] it is also proved that the homotopy type of $M\xi$ only depends on the homotopy class of ξ.
3. If $\xi : B \longrightarrow BO(N)$ is an unstable vector bundle then our definition gives
$$M\xi \simeq \Sigma^{-N} T(\xi)$$
because $T(\xi)$ appears only as the N^{th} term of the spectrum $M\xi$. This simplifies the indexing because for example the cohomology Thom class then lies in $H^N(T(\xi); \mathbb{Z}/2) \cong H^0(M\xi; \mathbb{Z}/2)$.

If $\xi : B \longrightarrow BO$ is 1-universal with $\pi_2 B \neq 0$ then $\xi = Bi \oplus \eta : BSO \times K(\Pi, 1) \longrightarrow BO$ and there is a standard way for computing $\Omega_n(\xi)$:
First use the Pontrjagin-Thom isomorphism (see e.g. [Bröcker-tom Dieck]) and then observe that $M\xi \simeq MSO \wedge M\eta$ to obtain the isomorphisms

$$\Omega_n(\xi) \cong \pi_n(M\xi) \cong \pi_n(MSO \wedge M\eta) = MSO_n(M\eta)$$

where MSO_* is the homology theory corresponding to the spectrum MSO (see [Adams]), i.e. $MSO_*(S^0) \cong \Omega_*^{SO}$. The Atiyah-Hirzebruch spectral sequence

$$H_p(M\eta; \Omega_q^{SO}) \Longrightarrow MSO_{p+q}(M\eta)$$

applies and for $n = 4$ it is quite easy to evaluate because $\Omega_i^{SO} \cong \begin{cases} 0 & \text{if } i = 1, 2, 3 \\ \mathbb{Z} & \text{if } i = 0, 4. \end{cases}$

Similarly, if $\xi = Bp \oplus \eta : BSpin \times K(\Pi, 1) \longrightarrow BO$ then

$$\Omega_n(\xi) \cong \pi_n(M\xi) \cong \pi_n(MSpin \wedge M\eta) = MSpin_n(M\eta)$$

and for the corresponding Atiyah-Hirzebruch spectral sequence

$$H_p(M\eta; \Omega_q^{Spin}) \Longrightarrow MSpin_{p+q}(M\eta)$$

one knows that $\Omega_i^{Spin} \cong \begin{cases} 0 & \text{if } i = 3 \\ \mathbb{Z} & \text{if } i = 0, 4 \\ \mathbb{Z}/2 & \text{if } i = 1, 2 \end{cases}$.

Lemma 2.3.2. *Let X be a spectrum and $H_p(X; \Omega_q^{Spin}) \Longrightarrow MSpin_{p+q}(X)$ the Atiyah-Hirzebruch spectral sequence as above.*

1. *The differential $d_2 : H_p(X; \Omega_1^{Spin}) \longrightarrow H_{p-2}(X; \Omega_2^{Spin})$ is the dual of $Sq^2 : H^{p-2}(X; \mathbb{Z}/2) \longrightarrow H^p(X; \mathbb{Z}/2)$.*

2. *The differential* $d_2 : H_p(X;\Omega_0^{Spin}) \longrightarrow H_{p-2}(X;\Omega_1^{Spin})$ *is reduction mod 2 composed with the dual of* Sq^2.

Proof. Let $\iota : S^0 \longrightarrow MSpin$ be the unit of the ring spectrum $MSpin$ coming from the inclusions of the bottom cell $D^n/S^{n-1} \hookrightarrow T(\xi_n)$. One knows that $\iota_* : \pi_i(S^0) \longrightarrow \pi_i(MSpin)$ is an isomorphism for $i \leq 2$. Then the naturality of the Atiyah-Hirzebruch spectral sequence shows that in the range in question we can as well compute the differentials d_2 for the spectral sequence

$$H_p(X;\pi_q^{st}) \Longrightarrow \pi_{p+q}(X).$$

Now the differentials d_2 are stable homology operations and thus are induced from elements in

$$\begin{array}{rcccccl} [H\mathbb{Z}/2, \Sigma^2 H\mathbb{Z}/2] & \cong & H^2(H\mathbb{Z}/2;\mathbb{Z}/2) & = & \langle Sq^2 \rangle & \cong & \mathbb{Z}/2 \text{ in part (1) and} \\ [H\mathbb{Z}, \Sigma^2 H\mathbb{Z}/2] & \cong & H^2(H\mathbb{Z};\mathbb{Z}/2) & = & \langle Sq^2 \circ r_2 \rangle & \cong & \mathbb{Z}/2 \text{ in part (2).} \end{array}$$

Here for any abelian group A, HA denotes the spectrum associated to ordinary homology with coefficients in A. To finish the proof we have to show that in both cases $d_2 \neq 0$. For this just take $X := \Sigma^{p-2} H\mathbb{Z}/2$ and recall that $\pi_i(\Sigma^{p-2}H\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{if } i = p-2\ , \\ 0 & \text{else.} \end{cases}$ ■

Remark . In the same way one can show that the differential

$$d_3 : E^3_{p,0} \longrightarrow E^3_{p-3,2}$$

is dual to a secondary operation associated to the relation $Sq^2 \circ Sq^2 = 0$ on integral cohomology classes. But since this seems well-known and we shall not use it, we omit the details.

With all these information at hand, one can in many cases compute the groups $\Omega_4(\xi)$, provided the normal 1-type ξ can be written as in description(III) of Theorem 2.1.1.

Remark . This is the reason why we studied the question under which conditions description(III) is available. However, in the process of writing this thesis, I later found the spectral sequence described in Section 6, which is applicable to all normal 1-types and has the same E^2- and E^∞-term as the spectral sequence above. Nevertheless, for some future problems it still might be interesting to know in which cases description(III) is applicable.

In the remaining part of this Section, we will give a necessary and sufficient condition for the case of a finite group Π. The question is:
Which pairs $(w_1, w_2), w_i \in H^i(\Pi;\mathbb{Z}/2)$, *are realizable as the first two Stiefel-Whitney classes of a stable vector bundle over* $K(\Pi,1)$*?*

Lemma 2.3.3. *For any space* K *and classes* $w_i \in H^i(K;\mathbb{Z}/2), i = 1,2$, *the pair* (w_1,w_2) *is realizable if and only if the pair* $(0,w_2)$ *is realizable.*

Proof. Let η be a stable vector bundle over K with Stiefel-Whitney classes $w_i := w_i(\eta)$. Furthermore, let l be a real line bundle over K with $w_1(l) = w_1$. Then

$$\begin{aligned} w(\eta \oplus 3\cdot l) &= w(\eta)\cdot(1+w_1)^3 \\ &= (1 + w_1 + w_2 + \dots)\cdot(1 + w_1 + w_1^2 + w_1^3) \\ &= (1 + w_2 + w_2 \cdot w_1 + w_1^3 + \dots). \end{aligned}$$

On the other hand if $w_1(\eta) = 0, w_2(\eta) = w_2$ and $w_1 = w_1(l)$ is given then $w_i(\eta \oplus l) = w_i$ for $i = 1,2$. ■

We can thus reformulate our question as follows: *Determine the image of the homomorphism*

$$w_2 : [K, BSO] \longrightarrow H^2(K; \mathbb{Z}/2).$$

In the universal example $K = K(\mathbb{Z}/2, 2)$, the image is trivial because if $\iota_2 = w_2(\eta)$ then ι_2^2 would be the mod 2 reduction of the first Pontrjagin class $p_1(\eta)$ but this is impossible since $H^4(K(\mathbb{Z}/2, 2); \mathbb{Z}) = 0$. On the contrary for a given space K some parts of $H^2(K; \mathbb{Z}/2)$ are easily seen to be in the image of w_2:

Lemma 2.3.4.

1. *The image of the mod 2 reduction* $r_2 : H^2(K; \mathbb{Z}) \longrightarrow H^2(K; \mathbb{Z}/2)$ *lies in the image of* w_2.
2. *All cup products* $x_1 \cup x_2$ *of* 1*-dimensional classes* $x_i \in H^1(K; \mathbb{Z}/2)$ *lie in the image of* w_2.

Proof. (1) Let $w = r_2(c)$. Then there exists a complex line bundle L over K with first Chern class $c_1(L) = c$. Then clearly $w_2(L) = r_2(c_1(L)) = w$.
(2) Let $w = x_1 \cup x_2$ and let l_i be real line bundles over K with $w_1(l_i) = x_i$. Then $w_2(l_1 \oplus l_2) = x_1 \cup x_2$. ■

Corollary 2.3.5. *If* $K = K(\Pi, 1)$ *and* $H^3(\Pi)$ *has no* 2*-torsion or* Π *is a finite abelian group then* w_2 *is onto.*

Proof. If $H^3(\Pi)$ has no 2-torsion then r_2 is onto and we are done by Lemma 2.3.4(1). Note that this applies in particular for cyclic groups. Now if Π is a direct product of finite cyclic groups then the Künneth isomorphism gives the result by applying Lemma 2.3.4(2). ■

Example 2.3.6. If Π is a finite group then the 2-torsion of $H^3(\Pi) \cong H_2(\Pi)$ vanishes if and only if it vanishes for the 2-Sylow subgroup of Π. Examples of such 2-groups are the quaternion groups which have periodic cohomology implying that all odd-dimensional cohomology groups vanish [Brown].

Before we can state the general result, we have to consider one more special case in which w_2 is onto. Let

$$D_{2n} := (x, y \mid x^n = y^2 = 1,\ y^{-1}xy = x^{-1}) \cong \mathbb{Z}/n \rtimes \mathbb{Z}/2$$

be a dihedral group of order $2n$. There are classes $x_i \in H^1(D_{2n}; \mathbb{Z}/2) \cong \mathrm{Hom}(D_{2n}, \mathbb{Z}/2)$ defined by

$$x_i(x^\epsilon \cdot y^\delta) := \begin{cases} \delta \pmod 2 & \text{if } i = 1, \\ \epsilon \pmod 2 & \text{if } i = 2 \end{cases}$$

and one has the standard representation $\rho_D : D_{2n} \hookrightarrow O(2)$ which represents D_{2n} as the symmetry group of a regular n-gon. The following computations of the cohomology of D_{2n} can be found for example in [Snaith, page 38].

Proposition . *Define* $w := w_2(\rho_D)$ *then the following assertions hold:*

1. *If* n *is odd then* $H^*(D_{2n}; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1]$ *and* $H_2(D_{2n}) = 0$.
2. *If* $n \equiv 2$ (4) *then* $H^*(D_{2n}; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, x_2]$ *and* $H_2(D_{2n}) = \langle d \rangle = \mathbb{Z}/2$ *with Kronecker products* $\langle x_i^2, d \rangle = 0,\ \langle x_1 \cdot x_2, d \rangle \neq 0$.
3. *If* $n \equiv 0$ (4) *then* $H^*(D_{2n}; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, x_2, w]/_{(x_1 \cdot x_2 + x_2^2 = 0)}$ *and* $H_2(D_{2n}) = \langle d \rangle = \mathbb{Z}/2$ *with Kronecker products* $\langle x_i \cdot x_j, d \rangle = 0,\ \langle w, d \rangle \neq 0$.

Lemma 2.3.7. *In the short exact universal coefficient sequence*

$$0 \to \mathrm{Ext}(H_{i-1}K, \mathbb{Z}/n) \xrightarrow{i_n} H^i(K; \mathbb{Z}/n) \xrightarrow{k_n} \mathrm{Hom}(H_iK, \mathbb{Z}/n) \to 0$$

one has $\mathrm{Ker}(k_n) = \mathrm{Image}(r_n : H^i(K; \mathbb{Z}) \longrightarrow H^i(K; \mathbb{Z}/n))$ *if* H_iK *is a finite group.*

Proof. The short exact sequence $\mathbb{Z} \overset{\cdot n}{\rightarrowtail} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n$ of coefficients gives a commutative diagram of exact sequences

$$\begin{tikzcd}
& \arrow[d] & \arrow[d] & \arrow[d] & \\
0 \arrow[r] & \mathrm{Ext}(H_{i-1}K,\mathbb{Z}) \arrow[r, "i_0"] \arrow[d] & H^i(K;\mathbb{Z}) \arrow[r, "k_0"] \arrow[d, "r_n"] & \mathrm{Hom}(H_iK,\mathbb{Z}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{i-1}K,\mathbb{Z}/n) \arrow[r, "i_n"] \arrow[d] & H^i(K;\mathbb{Z}/n) \arrow[r, "k_n"] \arrow[d] & \mathrm{Hom}(H_iK,\mathbb{Z}/n) \arrow[r] \arrow[d] & 0 \\
& {} & {} & {} &
\end{tikzcd}$$

But by assumption $\mathrm{Hom}(H_iK,\mathbb{Z}) = 0$ which implies that i_0 is an isomorphism and thus the result follows from the fact that $\mathrm{Ext}^2_{\mathbb{Z}}(.,\mathbb{Z}) = 0$. ■

If Π is a finite group then $H_n\Pi$ is also finite and therefore by Lemmas 2.3.4 and 2.3.7 it is enough to compute the image of

$$k_2 \circ w_2 : [K(\Pi,1), BSO] \longrightarrow H^2(\Pi;\mathbb{Z}/2) \twoheadrightarrow \mathrm{Hom}(H_2\Pi;\mathbb{Z}/2).$$

Let $U/_N$ be a subquotient of Π, i.e. $N \trianglelefteq U \leq \Pi$ and denote by $t_{U/_N}$ the composition

$$t_{U/_N} : H_2\Pi \xrightarrow{tr_*} H_2U \xrightarrow{p_*} H_2(U/_N)$$

and the $\mathbb{Z}/2$-dual by

$$t^*_{U/_N} : \mathrm{Hom}(H_2(U/_N),\mathbb{Z}/2) \longrightarrow \mathrm{Hom}(H_2\Pi,\mathbb{Z}/2).$$

Theorem 2.3.8. *For any finite group Π, the image of $k_2 \circ w_2$ equals the image of the map*

$$\sum_{U/_N \text{ dihedral}} t^*_{U/_N} : \bigoplus_{U/_N \text{ dihedral}} \mathrm{Hom}(H_2(U/_N),\mathbb{Z}/2) \longrightarrow \mathrm{Hom}(H_2\Pi,\mathbb{Z}/2).$$

Remark . In this sum, which a priori runs over all dihedral subquotients of Π, we can obviously restrict to those with $|U/_N| \equiv 0 \ (4)$ and a transfer argument shows that it is enough to consider those $U/_N$ which are 2-groups. Moreover, only the maximal representatives give a nontrivial contribution because if $N \lhd U \lhd V$ and $V/_N$ is dihedral then $t_{U/_N}$ and $t_{V/_U}$ are both trivial. The reason for this is that $tr_* : H_2(V/_N) \longrightarrow H_2(U/_N)$ and $p_* : H_2(V/_N) \longrightarrow H_2(V/_U)$ are zero which one readily checks using the theorem on $H^*(D_{2n};\mathbb{Z}/2)$. Finally, one can select one conjugacy class of each such maximal dihedral subquotient.

Proof of Theorem 2.3.8. The *flat bundle construction* discussed in Section 5.2 gives a map

$$\mathfrak{FB} : \widetilde{\mathfrak{RSO}}(\Pi) \longrightarrow [\mathfrak{K}(\Pi,1), \mathfrak{BSO}]$$

where $\widetilde{RSO}(\Pi)$ denotes the Grothendieck group of all real virtuell representations of dimension 0 and determinant 1. It is an easy corollary to Atiyah's completion theorem [Atiyah-Segal] that the image of $w_2 : [K(\Pi,1), BSO] \longrightarrow H^2(\Pi;\mathbb{Z}/2)$ equals the image of $w_2 \circ \mathfrak{FB}$. Thus we need to control only the real representations of Π and due to Lemma 2.3.4 only those which do not come from complex representations. The proof of the theorem is finished by using the following two observations which were proved in [Deligne]:

Proposition .

a) *If* $U \leq \Pi$ *and* $\det(\rho) = 1$ *then* $\det(\mathrm{Ind}_U^\Pi(\rho)) = 1$ *and the following diagram commutes:*

$$\begin{tikzcd} \widetilde{RSO}(U) \arrow[r, "w_2"] \arrow[d, "\mathrm{Ind}_U^\Pi"] & H^2(U;\mathbb{Z}/2) \arrow[d, "tr^*"] \\ \widetilde{RSO}(\Pi) \arrow[r, "w_2"] & H^2(\Pi;\mathbb{Z}/2) \end{tikzcd}$$

b) *Every real virtuell representation* ρ *of dimension 0 and determinant 1 comes from a complex representation or can be written as a* $\mathbb{Z}$*-linear combination of representations which are induced from dihedral subquotients of* Π.

Warning:
The diagram in a) does not commute if one replaces $\widetilde{RSO}$ by either RSO or $\widetilde{RO}$, i.e. the dimension and determinant conditions are necessary.
Although it seems to be the right place now, we will not give an example of a group Π where w_2 is not onto using Theorem 2.3.8. Instead, we will use the following much easier criterion to give a counterexample in Remark 3.3.3. This criterion is only necessary but not sufficient, more exactly, it is the first obstruction for finding a lift of w in the diagram

$$\begin{tikzcd} & BSO \arrow[d, "w_2(\gamma)"] \\ K(\Pi,1) \arrow[r, "w"] & K(\mathbb{Z}/2,2). \end{tikzcd}$$

Using the method of the universal example, one shows that this obstruction is

$$\beta(w^2) \in H^5(\Pi;\pi_4(BSO)) \cong H^5(\Pi;\mathbb{Z}).$$

Here $\beta : H^4(\Pi;\mathbb{Z}/2) \longrightarrow H^5(\Pi;\mathbb{Z})$ is the Bockstein homomorphism corresponding to the coefficient sequence $\mathbb{Z} \rightarrowtail \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2$. Note that the vanishing of $\beta(w^2)$ is clearly necessary since $\beta(w_2(\eta)^2) = \beta(r_2(p_1(\eta))) = 0$.

While this obstruction theoretic approach works very well to find examples, it does not lead much further because analyzing the higher obstructions is very involved and impossible in practice. This is the reason why we have chosen the representation theoretic approach in Theorem 2.3.8. It gives a complete algebraic description, though I must admit that computing all these transfer maps is not yet standard for algebraic topologists.

3. The James Spectral Sequence

We will now concentrate on oriented manifolds M with $w_2\widetilde{M} = 0$, i.e. the normal 1-type of M is a 1-universal fibration $\xi : B \longrightarrow BSO$ with $\pi_2 B = 0$. We construct a spectral sequence converging to $\Omega_*(\xi)$, generalizing the Atiyah-Hirzebruch spectral sequence described in Section 5.3. This spectral sequence behaves very well under the action of $\mathrm{Aut}(\xi)$ on $\Omega_*(\xi)$ in that the action is filtration preserving and well-understood on the E^2-term, see Corollary 3.1.2. Through its edge-homomorphisms, the spectral sequence gives interesting invariants which we identify with geometric invariants in Section 6.2. As a consequence of the sole existence of the spectral sequence, we obtain in Theorem 3.3.2 a couple of new results on the image of the signature homomorphism $\sigma : \Omega_4(\xi) \longrightarrow \mathbb{Z}$.

3.1. **Construction of the Spectral Sequence.**

Let $\xi : B \longrightarrow BSO$ be a 1-universal fibration with $w_1(\xi) = 0, \pi_2 B = 0$ and $\Pi := \pi_1 B$. Our aim is to construct a spectral sequence with $E^2_{p,q} \cong H_p(K(\Pi, 1); \Omega_q^{Spin})$ which converges to $\Omega_{p+q}(\xi)$. In the preceding Section we obtained such a spectral sequence for the case that $w_2(\xi) = w_2(\eta)$ for a stable vector bundle η over $K(\Pi, 1)$. In general, such a bundle η does not exist but we still have a fibration

$$BSpin \xrightarrow{i} B \to K(\Pi, 1)$$

such that $\xi \circ i$ is the universal bundle over $BSpin$. Moreover, the orientability of ξ implies that the homotopy equivalences of the fiber $BSpin$ induced by elements of Π are all homotopic to the identity, see description(II) in Theorem 2.2.1. Therefore, the existence of a spectral sequence as above follows from the following more general result by applying it to the above fibration and using stable homotopy as the generalized homology theory.

Theorem 3.1.1. *Let h be a generalized homology theory which is connected, i.e. $\pi_i(h) = 0$ $\forall i < 0$. Furthermore, let $F \to B \xrightarrow{f} K$ be an h-orientable fibration and $\xi : B \longrightarrow BSO$ a stable vector bundle. Then there exists a spectral sequence*

$$E^2_{p,q} \cong H_p\left(K; h_q(M(\xi|F))\right) \Longrightarrow h_{p+q}(M\xi)$$

(which we shall call the <u>*James spectral sequence*</u> *for the fibration f, because I.M.James juggled around with Thom spaces in his book* [James] *in a similar way we are going to do it.)*

Remark . All spectral sequences we will consider will be 1.quadrant spectral sequences, so there won't occur any problems concerning their convergence. This is the reason why we assumed the generalized homology theory h to be connected.

Proof. Since $SO = \bigcup_{n\in\mathbb{N}} SO(n)$ is a topological group, there is a contractible space ESO on which SO acts freely and a model for BSO is the orbit space ESO/SO. As a subgroup of SO each $SO(n)$ also acts freely on this space ESO and if we define $BSO(n) := ESO/SO(n)$ then the maps $i_n : BSO(n) \longrightarrow BSO$ are fiber bundles with fibers $SO/SO(n)$. Similarly all maps $i_n^{n+1} : BSO(n) \longrightarrow BSO(n+1)$ are fiber bundles with fibers $SO(n+1)/SO(n)$. From the originally given stable vector bundle ξ over B we now construct a sequence of fiber bundles over B by the pullback

$$\begin{tikzcd} B_n \arrow[r, "b_n"] \arrow[d, "\xi_n"'] & B \arrow[d, "\xi"'] \\ BSO(n) \arrow[r, "i_n"] & BSO. \end{tikzcd}$$

Composing the maps b_n with the original fibration $f : B \longrightarrow K$ we get a sequence of fibrations $f_n : B_n \longrightarrow K$ with fibers F_n together with vector bundles ξ_n over each B_n such that the following diagrams commute:

$$\begin{tikzcd} F_n \arrow[r] \arrow[d] & B_n \arrow[r, "f_n"] \arrow[d] & K \arrow[d, equal] \\ F_{n+1} \arrow[r] & B_{n+1} \arrow[r, "f_{n+1}"] & K \end{tikzcd} \qquad \begin{tikzcd} B_n \arrow[r, "\xi_n"] \arrow[d] & BSO(n) \arrow[d, "i_n^{n+1}"] \\ B_{n+1} \arrow[r, "\xi_{n+1}"] & BSO(n+1) \end{tikzcd}$$

By definition (compare Section 5.3), the Thom spectrum $M\xi$ consists of the family of Thom spaces $\{T(\xi_n), s_n : S^1 \wedge T(\xi_n) \longrightarrow T(\xi_{n+1})\}$ and similarly $M\xi|F = \{T(\xi_n|F_n), s_n|S^1 \wedge T(\xi_n|F_n)\}$. We will obtain the desired spectral sequence converging to $h(M\xi)$ as a direct limit of relative

Serre spectral sequences as follows: For any $n \in \mathbb{N}$, the disc-sphere bundle pair $(D(\xi_n), S(\xi_n))$ is a relative fibration over B_n with relative fiber (D^n, S^{n-1}). Composed with $f_n : B_n \longrightarrow K$ this becomes a relative fibration over K with relative fiber $(D(\xi_n|F_n), S(\xi_n|F_n))$. This fibration is h-orientable because we are considering oriented vector bundles and the original fibration f was assumed to be h-orientable. Thus there is a relative Serre spectral sequence (see [Switzer, chapter 15, remark 2])

$$^nE : H_p(K; h_q(D(\xi_n|F_n), S(\xi_n|F_n))) \Longrightarrow h_{p+q}(D(\xi_n), S(\xi_n)).$$

Replacing ξ_n by $\xi_n \oplus \epsilon_{\mathbb{R}}$ we obtain a spectral sequence which is isomorphic to the above via the suspension isomorphism for the generalized homology theory h. We shall identify these two spectral sequences and obtain from the vector bundle homomorphism $\xi_n \oplus \epsilon_{\mathbb{R}} \longrightarrow \xi_{n+1}$ homomorphisms of spectral sequences $^nE \longrightarrow {}^{n+1}E$. More exactly, we obtain a family of commutative diagrams

$$\begin{tikzcd} {}^nE^i_{p,q} \arrow[r] \arrow[d, "{}^nd^i_r"'] & {}^{n+1}E^i_{p,q} \arrow[d, "{}^{n+1}d^i_r"'] \\ {}^nE^i_{p-r,q+r-1} \arrow[r] & {}^{n+1}E^i_{p-r,q+r-1} \end{tikzcd}$$

and we can define a spectral sequence E by the direct limit, i.e. we set

$$(E^i_{p,q}, d^i_r) := (\varinjlim_n {}^nE^i_{p,q}, \varinjlim_n {}^nd^i_r).$$

The exactness of the direct limit functor gives the isomorphism $E^{i+1}_{*,*} \cong H(E^i_{*,*}, d^i)$ which shows that we have in fact defined a spectral sequence. By definition we have

$$\begin{aligned} E^2_{p,q} &= \varinjlim_n H_p(K; h_q(D(\xi_n|F_n), S(\xi_n|F_n))) \\ &\cong H_p(K; \varinjlim_n h_q(T(\xi_n|F_n))) \\ &\cong H_p(K; h_q(M\xi|F)). \end{aligned}$$

Since all ^nE are 1.quadrant spectral sequences we obtain $E^\infty_{p,q} = \varinjlim_n {}^nE^\infty_{p,q}$ with graded object

$$\varinjlim_n h_{p+q}(D(\xi_n|F_n), S(\xi_n|F_n)) \cong \varinjlim_n h_{p+q}(T(\xi_n)) \cong h_{p+q}(M\xi). \quad \blacksquare$$

Remark . By construction, the James spectral sequence is natural with respect to commutative diagrams of fibrations

$$\begin{tikzcd} F' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d, "\varphi"'] & K' \arrow[d] \\ F \arrow[r] & B \arrow[r] & K \end{tikzcd}$$

such that $\xi' = \xi \circ \varphi$.
Moreover, if $f : (B', B) \longrightarrow K$ is a relative fibration with relative fiber (F', F) and $\xi : B' \longrightarrow BSO$ is a stable vector bundle then we obtain a relative version of the James spectral sequence

$$H_p(K; h_q(M(\xi|F'), M(\xi|F))) \Longrightarrow h_{p+q}(M\xi, M(\xi|B))$$

by the same direct limit process. This time we have to take at the n'th step the relative Serre spectral sequence of the relative fibration $(D(\xi_n), S(\xi_n) \cup D(\xi_n|B_n))$ over K.

Corollary 3.1.2. *If $\xi : B \longrightarrow BSO$ is a 1-universal fibration with $\Pi := \pi_1 B$ and $\pi_2 B = 0$, any $\varphi \in \mathrm{Aut}(\xi)$ has a representative which preserves the filtration*

$$\Omega_n^{Spin} = F_{0,n} \subseteq F_{1,n-1} \subseteq \cdots \subseteq F_{n,0} = \Omega_n(\xi)$$

given by the James spectral sequence. Moreover, φ acts on the E^2-term of the spectral sequence via the induced map $\pi_1(\varphi)$ and this action induces the action on the filtration quotients $E^\infty_{p,q}$.

Proof. This follows directly from the naturality of the James spectral sequence once we recall from Theorem 2.2.6 that any $\varphi \in \mathrm{Aut}(\xi)$ fits into a commutative diagram

$$\begin{tikzcd} BSpin \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, "\varphi"] & K(\Pi,1) \arrow[d, "\bar\varphi"] \\ BSpin \arrow[r] & B \arrow[r] & K(\Pi,1). \end{tikzcd}$$

Here the homotopy class of $\bar\varphi$ is completely determined by $\pi_1(\bar\varphi) = \pi_1(\varphi)$ and vice versa. ■

We now determine those differentials d_2 in the James spectral sequence which are interesting for the groups $\Omega_4(\xi)$. In this case the generalized homology theory is stable homotopy and the fibration f is the pullback

$$\begin{tikzcd} BSpin \arrow[r] \arrow[d] & B \arrow[r, "f"] \arrow[d] & K \arrow[d, "w"] \\ BSpin \arrow[r] & BSO \arrow[r, "w_2(\gamma)"] & K(\mathbb{Z}/2,2) \end{tikzcd}$$

as in decription(I) of Theorem 2.2.1.

Theorem 3.1.3. *Let $Sq^2_w : H^{p-2}(K;\mathbb{Z}/2) \longrightarrow H^p(K;\mathbb{Z}/2)$ denote the homomorphism given by $Sq^2_w(x) := Sq^2(x) + x \cup w$. Then the following assertions hold:*

1. *For $p \leq 4$, the differential $d_2 : H_p(K;\Omega_1^{Spin}) \longrightarrow H_{p-2}(K;\Omega_2^{Spin})$ is the dual of Sq^2_w.*
2. *For $p \leq 5$, the differential $d_2 : H_p(K;\Omega_0^{Spin}) \longrightarrow H_{p-2}(K;\Omega_1^{Spin})$ is reduction mod 2 composed with the dual of Sq^2_w.*

Proof. First note that for the fibration $\{pt\} \longrightarrow B \xrightarrow{\mathrm{id}} B$ the James spectral sequence

$$H_p(B; h_q(M\xi|\{pt\})) \Longrightarrow h_{p+q}(M\xi)$$

translates by construction into the Atiyah-Hirzebruch spectral sequence for $M\xi$ if one uses the Thom isomorphism $H_p(M\xi) \cong H_p(B)$ and the fact that $M\xi|\{pt\} \simeq S^0$ as spectra. By Lemma 2.3.2, the differentials d_2 in the Atiyah-Hirzebruch spectral sequence are given by the dual of Sq^2 on $H^*(M\xi;\mathbb{Z}/2)$, respectively the composition with the reduction mod 2. But under the Thom isomorphism these maps become $Sq^2_{w_2(\xi)}$ on $H^*(B;\mathbb{Z}/2)$. Now we use naturality of the James spectral sequence for the fibrations

$$\begin{tikzcd} \{pt\} \arrow[r] \arrow[d] & B \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & B \arrow[d, "f"] \\ BSpin \arrow[r] & B \arrow[r, "f"] & K \end{tikzcd}$$

to get for all $x \in H_p(K;\mathbb{Z}/2), p \leq 4$:

$$d_2(x) = d_2(f_*(y)) = (Sq^2_{w_2(\xi)})^*(f_*(y)) = (Sq^2_w)^*(x)$$

and the corresponding result for $x \in H_p(K;\mathbb{Z}), p \leq 5$.
Here we used that f_* is onto for $p \leq 4$ which follows from the Hurewicz Theorem since f is a 4-equivalence. Finally, we also used that $f_* : H_5(B;\mathbb{Z}) \longrightarrow H_5(K;\mathbb{Z})$ is onto. This is equivalent to the vanishing of the differential $d_5 : H_5(K;\mathbb{Z}) \longrightarrow H_4(BSpin) \cong \mathbb{Z}$ in the Serre spectral sequence for the fibration f. But since f is the pullback of the fibration $BSpin \longrightarrow BSO \longrightarrow K(\mathbb{Z}/2,2)$ and the corresponding differential d_5 vanishes ($H_5(K(\mathbb{Z}/2,2);\mathbb{Z})$ is finite !), we are done by the naturality of the Serre spectral sequence. ■

Remark . It is conceivable that the assertions in Theorem 3.1.3 also hold for arbitrary p although our proof only works in the range described.

3.2. **The Edge-Homomorphisms of the Spectral Sequence.**

If the generalized homology theory h occurring in the James spectral sequence is stable homotopy then by the Pontrjagin-Thom construction we have an isomorphism $\pi_n M\xi \cong \Omega_n(\xi)$ and we can ask for a geometrical interpretation of the edge-homomorphisms. If the base B of ξ is fibered as $F \longrightarrow B \xrightarrow{f} K$ then the edge-homomorphism of the James spectral sequence coming from the base-line is

$$\mathfrak{ed} : \pi_{\mathfrak{n}}(\mathfrak{M}\xi) \longrightarrow \mathfrak{H}_{\mathfrak{n}}(\mathfrak{K}; \pi_{\mathfrak{o}}(\mathfrak{M}\xi|\mathfrak{F}))$$

where by the Hurewicz- and Thom isomorphisms $\pi_0(M\xi|F) \cong \mathbb{Z}$ (assuming that F is connected) and thus

$$\mathfrak{ed} : \pi_{\mathfrak{n}}(\mathfrak{M}\xi) \longrightarrow \mathfrak{H}_{\mathfrak{n}}(\mathfrak{K}).$$

Proposition 3.2.1. *Let $[\tilde{\nu} : M \to B] \in \Omega_n(\xi)$, i.e. $\xi \circ \tilde{\nu}$ is a stable normal Gauß map for M. Then the above edge-homomorphism is given by*

$$\mathfrak{ed}[\tilde{\nu} : \mathfrak{M} \to \mathfrak{B}] = \mathfrak{f}_* \circ \tilde{\nu}_*[\mathfrak{M}] \in \mathfrak{H}_{\mathfrak{n}}(\mathfrak{K}),$$

where $[M] \in H_n(M)$ is the fundamental class given by the orientation determined by $\tilde{\nu}$.

Proof. Using the naturality of the James spectral sequence for the fibrations

$$\begin{tikzcd}
\{pt\} \arrow[r] \arrow[d] & B \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & B \arrow[d, "f"] \\
F \arrow[r] & B \arrow[r, "f"] & K
\end{tikzcd}$$

we are reduced to showing $\mathfrak{ed}[\tilde{\nu} : \mathfrak{M} \to \mathfrak{B}] = \tilde{\nu}_*[\mathfrak{M}]$ in the spectral sequence for the upper fibration. Let us now choose an embedding $M \hookrightarrow S^{n+k}$ for sufficiently large k. Then we obtain a commutative diagram

$$\begin{tikzcd}
H_{n+k}(S^{n+k}) \arrow[r] & H_{n+k}(T(\nu(M \hookrightarrow S^{N+k}))) \arrow[r, "T\tilde{\nu}_*"] \arrow[d, "\cong"'] & H_{n+k}(T(\xi_k)) \arrow[r, "\cong"] \arrow[d, "\cong"'] & H_n(M\xi) \\
 & H_n(M) \arrow[r, "\tilde{\nu}_*"] & H_n(B) &
\end{tikzcd}$$

where a generator of $H_{n+k}(S^{n+k})$ is mapped to $[M] \in H_n(M)$. This proves that the diagram

$$\begin{tikzcd}
\pi_n(M\xi) \arrow[r, "Hurewicz"] \arrow[d, "Pontrjagin-Thom"', "\cong"] & H_n(M\xi) \arrow[d, "\cong"', "Thom"] \\
\Omega_n(\xi) \arrow[r, "\phi"] & H_n(B)
\end{tikzcd} \tag{II.3}$$

commutes if we set $\phi[\tilde{\nu} : M \to B] := \tilde{\nu}_*[M]$.
Since the James spectral sequence for a fibration where the fiber is a point is isomorphic to the Atiyah-Hirzebruch spectral sequence under the Thom isomorphism (compare the proof of Theorem 3.1.3), diagram (II.3) shows that it suffices to show that the Hurewicz homomorphism is the edge-homomorphism for the Atiyah-Hirzebruch spectral sequence for $M\xi$. But this follows from the fact that this edge-homomorphism is a stable homology operation from stable homotopy to ordinary homology, i.e. an element of $[S^0, H\mathbb{Z}] \cong \mathbb{Z}$. Moreover, the Hurewicz homomorphism generates this group and if we take the spectrum $\Sigma^n H\mathbb{Z}$ as a test case we can conclude that the edge-homomorphism cannot be a nontrivial multiple of this generator. Finally, using the spheres S^n as a second test example, one sees that the sign is correct, too. ■

The other edge-homomorphism coming from the inclusion of the fiber is a map

$$\mathfrak{ed}' : \mathfrak{H}_\mathfrak{o}(\mathfrak{K}; \pi_\mathfrak{n}(\mathfrak{M}\xi|\mathfrak{F})) \longrightarrow \pi_\mathfrak{n}(\mathfrak{M}\xi)$$

and if we also assume K to be connected then $\mathfrak{ed}' : \Omega_\mathfrak{n}(\xi|\mathfrak{F}) \longrightarrow \Omega_\mathfrak{n}(\xi)$.

Proposition 3.2.2. *Let $[\tilde{\nu} : M \to F] \in \Omega_n(\xi|F)$, i.e. $\xi|F \circ \tilde{\nu}$ is a stable normal Gauß map for M. Then the above edge-homomorphism is given by*

$$\mathfrak{ed}'[\tilde{\nu} : \mathfrak{M} \to \mathfrak{F}] = [\mathfrak{M}, \mathfrak{i} \circ \tilde{\nu}] \in \Omega_\mathfrak{n}(\xi),$$

where $i : F \longrightarrow B$ is the inclusion of the fiber.

Proof. This is just the naturality of the Pontrjagin-Thom construction, combined with the fact that in the Serre spectral sequence this edge-homomorphism is given by the induced map i_*, see [Switzer, Ch.15, Rem.5]. ■

The last map we are interested in is not an edge-homomorphism any more but it is defined only on the kernel of the edge-homomorphism we called $\mathfrak{ed}$ above. More precisely, we consider now the pullback of the path fibration

$$\begin{array}{ccc} B & \longrightarrow & P(K(\mathbb{Z}/2,2)) \\ {\scriptstyle \mathfrak{p}}\big\downarrow & & \big\downarrow \\ BSO \times K & \xrightarrow{u} & K(\mathbb{Z}/2,2) \end{array} \qquad \text{(II.4)}$$

as in description(IV) of Theorem 2.2.1. Here

$$u : BSO \times K \xrightarrow{w_2 \times w} K(\mathbb{Z}/2,2) \times K(\mathbb{Z}/2,2) \xrightarrow{+} K(\mathbb{Z}/2,2), \quad w \in H^2(K;\mathbb{Z}/2) \text{ arbitrary,}$$

and $+$ denotes the H-space structure on $K(\mathbb{Z}/2,2)$. Then the fibration

$$f := p_2 \circ \mathfrak{p} : B \longrightarrow K$$

has fiber $BSpin$ and $\xi := p_1 \circ \mathfrak{p}$ is a stable vector bundle over B. As an abbreviation, we let

$$B' := BSO \times K \quad \text{and} \quad \xi' := p_1 : B' \longrightarrow BSO.$$

This gives a commutative diagram of fibrations

$$\begin{array}{ccccc} F = BSpin & \longrightarrow & B & \xrightarrow{f} & K \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{p}} & & \big\| \\ F' = BSO & \longrightarrow & B' & \xrightarrow{p_2} & K \end{array}$$

with $\xi = \mathfrak{p} \circ \xi'$.

Assumption: $[\tilde{\nu} : M \to B] \in \Omega_n(\xi)$ *maps to zero in* $\Omega_n(\xi')$.
This assumption has two important consequences:

First of all, we can choose an element $[\mu : W \to B'] \in \Omega_{n+1}(\xi', \xi) \cong \pi_{n+1}(M\xi', M\xi)$ mapping to $[\tilde{\nu} : M \to B]$ in the exact sequence of the pair $(M\xi', M\xi)$. The commutative diagram

$$\begin{array}{ccccc} M & \xrightarrow{\tilde{\nu}} & B & \longrightarrow & P(K(\mathbb{Z}/2,2)) \\ {\scriptstyle i}\downarrow & & {\scriptstyle \mathfrak{p}}\downarrow & & \downarrow \\ W & \xrightarrow{\mu} & B' & \xrightarrow{u} & K(\mathbb{Z}/2,2) \end{array}$$

then gives an obstruction for extending the B-structure $\tilde{\nu}$ to W. This obstruction

$$\mathfrak{o} \in \mathfrak{H}^2(\mathfrak{W}, \mathfrak{M}; \mathbb{Z}/2) \cong [\mathfrak{W}/_{\mathfrak{M}}, \mathfrak{K}(\mathbb{Z}/2,2)] \cong [\mathfrak{W} \cup_{\mathfrak{M}} \mathfrak{C}(\mathfrak{M}), \mathfrak{K}(\mathbb{Z}/2,2)]$$

is well-defined since maps from M to $P(K(\mathbb{Z}/2,2))$ are adjoint to maps from the cone $C(M)$ to $K(\mathbb{Z}/2,2)$. In other words, such a map gives a homotopy from $u \circ \mu \circ i$ to the constant map.
Secondly, the edge-homomorphism $\mathfrak{ed}$ vanishes on $[\tilde{\nu} : M \to B]$ because $\mathfrak{ed}$ factors over $\Omega_n(\xi')$ by Lemma 3.2.1. Thus the James spectral sequence gives an element

$$\mathfrak{sec}[\tilde{\nu} : \mathfrak{M} \to \mathfrak{B}] \in \mathfrak{H}_{\mathfrak{n}-1}(\mathfrak{K}; \mathbb{Z}/2)/\operatorname{Image}(\mathfrak{d}_2),$$

a *secondary edge-invariant.*

Proposition 3.2.3. *In the situation above we have*

$$\mathfrak{sec}[\tilde{\nu} : \mathfrak{M} \to \mathfrak{B}] = (\mathfrak{p}_2 \circ \mu)_*(\mathfrak{o} \cap [\mathfrak{W}, \mathfrak{M}]) \in \mathfrak{H}_{\mathfrak{n}-1}(\mathfrak{K}; \mathbb{Z}/2)/\operatorname{Image}(\mathfrak{d}_2).$$

Proof. The exact sequence of the pair $(M\xi', M\xi)$ induces a commutative diagram of James spectral sequences

$$\begin{array}{ccc} H_p(K; \pi_{q+1}(M\xi'|F', M\xi|F)) & \Longrightarrow & \pi_{p+q+1}(M\xi', M\xi) \\ \downarrow \partial & & \downarrow \partial \\ H_p(K; \pi_q(M\xi|F)) & \Longrightarrow & \pi_{p+q}(M\xi) \end{array}$$

and the homomorphism ∂ on the right hand side maps $[\mu : W \to B']$ to $[\tilde{\nu} : M \to B]$ for $p + q = n$. Since

$$\pi_i(M\xi'|F', M\xi|F) = \pi_i(MSO, MSpin) = 0 \quad \forall i \le 1 \text{ and}$$
$$\pi_2(MSO, MSpin) \xrightarrow[\partial]{\cong} \pi_1(MSpin) \cong \mathbb{Z}/2$$

the edge-homomorphism in the upper spectral sequence is a map

$$\mathfrak{ed}_{\mathfrak{rel}} : \pi_{\mathfrak{n}+1}(\mathfrak{M}\xi', \mathfrak{M}\xi) \longrightarrow \mathfrak{H}_{\mathfrak{n}-1}(\mathfrak{K}; \mathbb{Z}/2)$$

and we are done if we show that we get

$$\mathfrak{ed}_{\mathfrak{rel}}[\mu : \mathfrak{W} \to \mathfrak{B}'] = (\mathfrak{p}_2 \circ \mu)_*(\mathfrak{o} \cap [\mathfrak{W}, \mathfrak{M}]).$$

Let me remark that at this point the indeterminacy

$$\operatorname{Image}(d_2) \subseteq H_{n-1}(K; \mathbb{Z}/2)$$

of $\mathfrak{sec}[\tilde{\nu} : \mathfrak{M} \to \mathfrak{B}]$ has vanished. This results form the fact that we have chosen a particular element $[\mu : W \to B']$ mapping under ∂ to $[\tilde{\nu} : M \to B]$. Two such choices differ by an element in $\Omega_{n+1}(\xi')$ whose fundamental class in $H_{n+1}(K; \mathbb{Z})$ maps under the differential d_2 to the difference of the $\mathfrak{ed}_{\mathfrak{rel}}$'s in $H_{n-1}(K; \mathbb{Z}/2)$.

To compute $\mathfrak{ed}_{\mathfrak{rel}}$ we want to compare with a certain Atiyah-Hirzebruch spectral sequence. Let $E := \Sigma^2 H\mathbb{Z}/2$. Via the Thom isomorphism we can view the map $u : B' \longrightarrow K(\mathbb{Z}/2, 2)$ from diagram (II.4) as a map of spectra $\bar{u} : M\xi' \longrightarrow E$. This gives us the map

$$M\xi' \xrightarrow{\Delta} M\xi' \wedge B'_+ \xrightarrow{\bar{u}\wedge p_{2+}} E \wedge K_+$$

and because $u \circ \mathfrak{p}$ is zero homotopic (via a given homotopy to the constant map) we get a map of pairs

$$\begin{array}{ccc} (M\xi', M\xi) & \xrightarrow{U} & (E \wedge K_+, *) \\ \uparrow & & \uparrow \\ (M\xi'|F', M\xi|F) & \xrightarrow{U|(M\xi'|F', M\xi|F)} & (E, *). \end{array}$$

Here $U|(M\xi'|F', M\xi|F)$ is by construction the nontrivial element in $H^2(MSO, MSpin; \mathbb{Z}/2) \cong \mathbb{Z}/2$. Applying the functor π_*^{st} to the diagram above, the left hand side gives the relative James spectral sequence

$$H_p(K; \pi_q(M\xi'|F', M\xi|F)) \Longrightarrow \pi_{p+q}(M\xi', M\xi)$$

whereas the right hand side gives the Atiyah-Hirzebruch spectral sequence for the generalized homology theory E:

$$H_p(K; \pi_q(E)) \cong H_p(K; E_q(S_0)) \Longrightarrow E_{p+q}(K) \cong \pi_{p+q}(E \wedge K_+).$$

Since both spectral sequences are derived from the skeletal filtration of K, the map U can be defined on every level of the spectral sequences and we obtain a commutative diagram

$$\begin{array}{ccc} H_p(K; \pi_q(M\xi'|F', M\xi|F)) & \Longrightarrow & \pi_{p+q}(M\xi', M\xi) \\ \downarrow U|(M\xi'|F', M\xi|F)* & & \downarrow U_* \\ H_p(K; \pi_q(E)) & \Longrightarrow & \pi_{p+q}(E \wedge K_+) \cong H_{p+q-2}(K; \mathbb{Z}/2). \end{array}$$

But E is an Eilenberg-MacLane spectrum, so the Atiyah-Hirzebruch spectral sequence collapses and by naturality the edge-homomorphism $\mathfrak{ed}_{\mathfrak{rel}} : \pi_{\mathfrak{n}+1}(\mathfrak{M}\xi', \mathfrak{M}\xi) \longrightarrow \mathfrak{H}_{\mathfrak{n}-1}(\mathfrak{K}; \mathbb{Z}/2)$ in question is just the map U_* for $p + q = n + 1$.
To understand this map U_*, let $T : M\xi' \longrightarrow H\mathbb{Z}/2$ be the Thom class of the stable vector bundle ξ'. If $u : B' \longrightarrow H\mathbb{Z}/2$ represents the class $u \in H^2(B'; \mathbb{Z}/2)$ from diagram (II.4) then by definition the class $\bar{u} \in H^2(M\xi'; \mathbb{Z}/2)$ is represented by the map

$$\bar{u} : M\xi' \xrightarrow{\Delta} M\xi' \wedge B'_+ \xrightarrow{T\wedge u} H\mathbb{Z}/2 \wedge H\mathbb{Z}/2 \xrightarrow{m} H\mathbb{Z}/2$$

where $m : H\mathbb{Z}/2 \wedge H\mathbb{Z}/2 \longrightarrow H\mathbb{Z}/2$ is the multiplication map for the ring spectrum $H\mathbb{Z}/2$. A longer diagram chase which only uses the definitions of all objects involved then shows that the following diagram commutes:

$$\begin{array}{ccccc} M\xi' & \xrightarrow{\Delta} & M\xi' \wedge B'_+ & \xrightarrow{T\wedge\Delta_{B'_+}} & H\mathbb{Z}/2 \wedge B'_+ \wedge B'_+ \\ \downarrow\Delta & & & & \downarrow 1\wedge u\wedge 1 \\ M\xi' \wedge B'_+ & \xrightarrow{\bar{u}\wedge 1} & \Sigma^2 H\mathbb{Z}/2 \wedge B'_+ & \xleftarrow{\Sigma^2 m\wedge 1} & H\mathbb{Z}/2 \wedge \Sigma^2 H\mathbb{Z}/2 \wedge B'_+ \end{array}$$

The definition of the cap product with u as the homomorphism of stable homotopy groups induced by

$$H\mathbb{Z}/2 \wedge B'_+ \xrightarrow{1\wedge\Delta_{B'_+}} H\mathbb{Z}/2 \wedge B'_+ \wedge B'_+ \xrightarrow{1\wedge u\wedge 1} H\mathbb{Z}/2 \wedge \Sigma^2 H\mathbb{Z}/2 \wedge B'_+ \xrightarrow{\Sigma^2 m\wedge 1} \Sigma^2 H\mathbb{Z}/2 \wedge B'_+$$

then gives the commutative diagram

$$\begin{array}{ccccc}
\pi_{n+1}(M\xi') & \xrightarrow{Hurewicz} & H_{n+1}(M\xi';\mathbb{Z}/2) & \xrightarrow[\cong]{Thom} & H_{n+1}(B';\mathbb{Z}/2) \\
\downarrow{\scriptstyle \Delta_*} & & & & \downarrow{\scriptstyle .\cap u} \\
\pi_{n+1}(M\xi' \wedge B'_+) & \xrightarrow{(\bar{u}\wedge 1)_*} & \pi_{n+1}(\Sigma^2 H\mathbb{Z}/2 \wedge B'_+) & \xrightarrow[\cong]{} & H_{n-1}(B')
\end{array}$$

which finally leads by naturality to the commutative diagram

$$\begin{array}{ccccc}
\pi_{n+1}(M\xi', M\xi) & \xrightarrow{Hurewicz} & H_{n+1}(M\xi', M\xi;\mathbb{Z}/2) \cong H_{n+1}(B', B;\mathbb{Z}/2) & \xleftarrow{\mu_*} & H_{n+1}(W, M;\mathbb{Z}/2) \\
\downarrow{\scriptstyle U_*} & & \downarrow{\scriptstyle .\cap u} & & \downarrow{\scriptstyle .\cap \mathfrak{o}} \\
H_{n+1}(K;\mathbb{Z}/2) & \xleftarrow{(p_2)_*} & H_{n-1}(B';\mathbb{Z}/2) & \xleftarrow{\mu_*} & H_{n-1}(W;\mathbb{Z}/2).
\end{array}$$

From this diagram we can read off in the same way we proved the commutativity of diagram (II.3) in Proposition 3.2.1 that the composition

$$\Omega_{n+1}(\xi', \xi) \xrightarrow[\cong]{Pontrjagin-Thom} \pi_{n+1}(M\xi', M\xi) \xrightarrow{U_*} H_{n-1}(K;\mathbb{Z}/2)$$

maps $[\mu : W \to B']$ to $(p_2 \circ \mu)_*(\mathfrak{o} \cap [\mathfrak{W}, \mathfrak{M}])$. ■

Remark . From the definition of $\mathfrak{sec}$ through the James spectral sequence, it is clear that the image of $\mathfrak{sec}$ lies in the subquotient of $H_{n-1}(K;\mathbb{Z}/2)$ on which all differentials vanish. In Theorems 2.3.2 and 3.1.3 we have computed the differential d_2 on this group to be the dual of Sq^2_w if $w = w_2(\eta)$ for a stable vector bundle η over K respectively if $n \leq 4$. The following computation is on the one hand a consistency check for the formula in Proposition 3.2.3 in the case $d_2 = (Sq^2_w)^*$. On the other hand, it underlines our conjecture that this differential is in all dimensions the dual of Sq^2_w.
Let $x \in H^{n-3}(K;\mathbb{Z}/2)$ and set $f := p_2 \circ \mu : W \longrightarrow K$. Then we claim that

$$\langle Sq^2_w(x), f_*(\mathfrak{o} \cap [\mathfrak{W}, \mathfrak{M}]\rangle = \langle \mathfrak{f}^*(\mathfrak{Sq}^2(\mathfrak{x}) + \mathfrak{w} \cup \mathfrak{x}) \cup \mathfrak{o}, [\mathfrak{W}, \mathfrak{M}]\rangle = \mathfrak{o}.$$

The reason is that $\mathfrak{o} \in \mathfrak{H}^2(\mathfrak{W}, \mathfrak{M}; \mathbb{Z}/2)$ maps to $f^*(w) + w_2 W \in H^2(W;\mathbb{Z}/2)$ and thus we obtain from the relation

$$Sq^2(y) = w_2 W \cup y \quad \forall y \in H^{n-1}(W;\mathbb{Z}/2)$$

the following equations:

$$\begin{aligned}
f^*(x) \cup Sq^2(\mathfrak{o}) &= f^*(x) \cup \mathfrak{o} \cup \mathfrak{o} \\
&= f^*(x) \cup \mathfrak{o} \cup (\mathfrak{f}^*(\mathfrak{w}) + \mathfrak{w}_2\mathfrak{W}) \\
&= f^*(x \cup w) \cup \mathfrak{o} + \mathfrak{Sq}^2(\mathfrak{f}^*(\mathfrak{x}) \cup \mathfrak{o}) \\
&= f^*(x \cup w) \cup \mathfrak{o} + Sq^2(f^*(x)) \cup \mathfrak{o} + \mathfrak{Sq}^1(\mathfrak{f}^*(\mathfrak{x}) \cup \mathfrak{Sq}^1(\mathfrak{o})) + \mathfrak{f}^*(\mathfrak{x}) \cup \mathfrak{Sq}^2(\mathfrak{o}).
\end{aligned}$$

This proves the assertion

$$f^*(Sq^2(x) + w \cup x) \cup \mathfrak{o} = \mathfrak{o}$$

because of the relation

$$Sq^1(y) = w_1 W \cup y = 0 \quad \forall y \in H^n(W, M;\mathbb{Z}/2). \qquad ■$$

Using Proposition 3.2.3, we can make the following nice observation about the effect of changing the ξ-structure of a fixed n-dimensional manifold M. To this end we first describe an obstruction theoretically convenient way for obtaining elements of $\mathrm{Aut}(\xi)$.
By definition the fibration $\mathfrak{p} : B \longrightarrow B' = BSO \times K$ is a principal fibration with fiber

$\Omega(K(\mathbb{Z}/2,2)) = K(\mathbb{Z}/2,1)$ in the sense of [Baues, Def.1.3.1]. It follows that there is an action

$$+ : [B, K(\mathbb{Z}/2,1)]^0 \times [B,B]^0_{\mathfrak{p}} \longrightarrow [B,B]^0_{\mathfrak{p}}$$

and thus we can define for a given $\beta \in H^1(B;\mathbb{Z}/2)$ an element $\bar{\beta} := \beta + \mathrm{id}_B \in [B,B]_{\mathfrak{p}}$. Note that by [Baues, Thm.1.3.8], the correspondence $\beta \mapsto \bar{\beta}$ is a bijection between $H^1(B;\mathbb{Z}/2)$ and $[B,B]_{\mathfrak{p}}$.
Moreover, if $\xi : B \longrightarrow BSO$ is 1-universal then K is a $K(\Pi,1)$, $H^1(B;\mathbb{Z}/2) \cong H^1(\Pi;\mathbb{Z}/2)$ and $\bar{\beta}$ is an automorphism corresponding to the element $B\beta_G \in \mathrm{Aut}(\xi)_{\pi_1}$ which we constructed in Proposition 2.2.5. This follows from the observation that in this case $B = BG, \xi = B\bar{\rho}$ and the action $+$ above comes from the homomorphism

$$\begin{array}{ccc} \mathbb{Z}/2 \times G & \longrightarrow & G \\ (t,g) & \longmapsto & g \cdot \sigma^t \end{array}$$

which we used to define $\beta_G \in \mathrm{Aut}(\bar{\rho})$.
Now given a ξ-structure $\tilde{\nu} : M \longrightarrow B$, we can form another ξ-structure $\bar{\beta} \circ \tilde{\nu} : M \longrightarrow B$ and get the ξ'-bordism

$$\mu : W := M \times I \xrightarrow{p_1} M \xrightarrow{\mathfrak{p}\circ\tilde{\nu}} B'$$

between $[\tilde{\nu} : M \to B]$ and $-[\bar{\beta} \circ \tilde{\nu} : M \to B]$. Thus the secondary invariant $\mathfrak{sec}$ of this difference is defined.

Proposition 3.2.4. *The following formula holds:*

$$\mathfrak{sec}\left([\tilde{\nu} : \mathfrak{M} \to \mathfrak{B}] - [\bar{\beta} \circ \tilde{\nu} : \mathfrak{M} \to \mathfrak{B}]\right) = \mathfrak{p}_{2*}(\beta \cap \tilde{\nu}_*[\mathfrak{M}]) \in \mathfrak{H}_{\mathfrak{n}-1}(\mathfrak{K};\mathbb{Z}/2)/\,\mathrm{Image}(\mathfrak{d}_2).$$

Proof. By [Baues, (1.3.7),Thm.1.3.8], the obstruction for finding a solution of the homotopy lifting problem

$$\begin{array}{ccc} B \times \{0,1\} & \xrightarrow{\mathrm{id}_B \cup \bar{\beta}} & B \\ {\scriptstyle inclusion}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{p}} \\ B \times I & \xrightarrow{\mathfrak{p}\circ p_1} & B' \end{array}$$

is given by $\delta(0,\beta)$ in the long exact sequence

$$\ldots \to H^1(B;\mathbb{Z}/2) \oplus H^1(B;\mathbb{Z}/2) \cong H^1(B \times \{0,1\};\mathbb{Z}/2) \xrightarrow{\delta} H^2(B \times I, B \times \{0,1\};\mathbb{Z}/2) \to \ldots$$

of the pair $(B \times I, B \times \{0,1\})$. By naturality this implies that the obstruction for finding a solution of the homotopy lifting problem

$$\begin{array}{ccc} M \times \{0,1\} & \xrightarrow{\tilde{\nu}\cup\bar{\beta}\circ\tilde{\nu}} & B \\ {\scriptstyle inclusion}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{p}} \\ W = M \times I & \xrightarrow{\mu} & B' \end{array}$$

is given by $\delta(0,\tilde{\nu}^*(\beta)) \in H^2(W, M \times \{0,1\};\mathbb{Z}/2)$. (In Proposition 3.2.3 this obstruction was called $\mathfrak{o}$.) We can conclude that

$$\begin{aligned} \mathfrak{sec}\left([\tilde{\nu} : \mathfrak{M} \to \mathfrak{B}] - [\bar{\beta} \circ \tilde{\nu} : \mathfrak{M} \to \mathfrak{B}]\right) &= (p_2 \circ \mu)_*(\mathfrak{o} \cap [\mathfrak{W}, \partial\mathfrak{W}]) \\ &= (p_2 \circ \mu \circ i_M)_*((0,\tilde{\nu}^*(\beta)) \cap [M \times \{0,1\}]) \\ &= (p_2 \circ \tilde{\nu})_*(\tilde{\nu}^*(\beta) \cap [M]) \\ &= p_{2*}(\beta \cap \tilde{\nu}_*[M]) \end{aligned}$$

which proves the formula. ■

Remark . We can again make a consistency check for the above formula by proving directly that $d_2(p_{2*}(\beta \cap \tilde{\nu}_*[M])) = 0 \quad \forall \beta \in H^1(B; \mathbb{Z}/2)$. Recall that this is only possible for us if we assume that $d_2 = (Sq^2_w)^*$.
Let $x \in H^{n-3}(K; \mathbb{Z}/2)$ and set $f := p_2 \circ \tilde{\nu} : M \longrightarrow K$. Then we claim that

$$\langle Sq^2_w(x), p_{2*}(\beta \cap \tilde{\nu}_*[M]) \rangle = \langle \tilde{\nu}^* \left(p_2^*(Sq^2_w(x)) \cup \beta \right), [M] \rangle = 0.$$

The reason is that we have the relations $f^*(w) = w_2 M, w_1 M = 0$ and thus obtain (writing $\beta' := \tilde{\nu}^*(\beta)$):

$$\begin{aligned} \tilde{\nu}^*(p_2^*(Sq^2_w(x)) \cup \beta) &= f^*(Sq^2_w(x)) \cup \beta' \\ &= f^*(Sq^2(x)) \cup \beta' + f^*(w \cup x) \cup \beta' \\ &= Sq^2(f^*(x) \cup \beta') + w_2 M \cup (f^*(x) \cup \beta') \\ &= 0. \qquad \blacksquare \end{aligned}$$

3.3. **Applications to Signature Questions.**

In this Section we want to use the James spectral sequence to give partial results to the following question: If $\xi : B \longrightarrow BSO$ is a 1-universal fibration and $\sigma : \Omega_4(\xi) \longrightarrow \mathbb{Z}$ is given by the signature, what is the image of σ?
Because $\sigma(\mathbb{CP}^{\nVdash}) = \nVdash$, the signature is onto if $\pi_2 B \neq 0$ so this case is uninteresting. Therefore, we can again assume $\pi_2 B = 0$ in the following. By Theorem 2.2.1, $\xi = \xi(\Pi, w)$ then only depends on Π and $w \in H^2(\Pi; \mathbb{Z}/2) \cong H^2(B; \mathbb{Z}/2)$ where w is mapped to $w_2(\xi)$ under the isomorphism.

Definition 3.3.1. For $w \in H^2(\Pi; \mathbb{Z}/2)$, Π a finitely presentable group, define $\sigma(\Pi, w) \in \mathbb{N}$ by the requirement that $\sigma(\Omega_4(\xi(\Pi, w))) = \sigma(\Pi, w) \cdot \mathbb{Z}$.

Recall that every bordism class in $\Omega_4(\xi)$ is represented by a 4-manifold with normal 1-type ξ, so we are making statements about the possible signatures of such manifolds. In the next Theorem we will prove some results on the image of the signature just by using the existence and naturality of the James spectral sequence. These results include Rohlin's well-known theorem (see part (2)) as well as a couple of new results.

Theorem 3.3.2.
1. $\sigma(\Pi, w)$ *divides* 16.
2. $\sigma(\Pi, 0) = 16$.
3. *If* $w \neq 0$ *then* $\sigma(\Pi, w)$ *divides* 8.
4. $\sigma(\Pi, w) = 8$ *if* $0 \neq w \in \operatorname{Ext}(H_1\Pi; \mathbb{Z}/2) \hookrightarrow H^2(\Pi; \mathbb{Z}/2)$.
5. *If* $Sq^1 w \notin \{a \cup w \mid a \in H^1(\Pi; \mathbb{Z}/2)\}$ *then* $\sigma(\Pi, w)$ *divides* 4.
6. *If the multiplication by* w *is injective on* $H^1(\Pi; \mathbb{Z}/2)$ *then* $\sigma(\Pi, w) = 1$ *if and only if* $w^2 \notin \{b^2 + b \cup w \mid b \in H^2(\Pi; \mathbb{Z}/2)\} + \operatorname{Ker}(k_2 : H^4(\Pi; \mathbb{Z}/2) \longrightarrow \operatorname{Hom}(H_4(\Pi), \mathbb{Z}/2))$.

Proof. (1) follows from the existence of the Kummer surface which is a simply-connected spin manifold with signature 16.
(2) Our aim is to prove Rohlin's Theorem using only the knowledge of Ω_i^{SO} for $i \leq 4$ and the fact that $\Omega_3^{Spin} = 0$ which was proven in an elementary way in [Kirby]. If we apply the James spectral sequence to the fibration $BSpin \longrightarrow BSO \longrightarrow K(\mathbb{Z}/2, 2)$ and use the fact that $\Omega_i^{SO} = 0$ for $i = 1, 2, 3$, we can conclude that $\Omega_i^{Spin} \cong \mathbb{Z}/2$ for $i = 1, 2$ and we obtain a filtration

$$\Omega_4^{Spin}/\operatorname{Image}(d_i) \underbrace{\subseteq}_{\mathbb{Z}/2} F_{2,2} \underbrace{\subseteq}_{\mathbb{Z}/2} F_{3,1} \underbrace{\subseteq}_{\mathbb{Z}/4} \Omega_4^{SO} \xrightarrow[\sigma]{\cong} \mathbb{Z}.$$

Because $\mathrm{Image}(d_i) \subseteq \Omega_4^{Spin}$ is a torsion group and thus the signature vanishes on this subgroup, the divisibility of the signature on Ω_4^{Spin} equals the product of the orders of the three subquotients in the filtration. But since the homology of $K(\mathbb{Z}/2,2)$ is given by

$$\begin{aligned} H_i(K(\mathbb{Z}/2,2);\mathbb{Z}/2) &\cong \mathbb{Z}/2 \qquad \text{for } i=2,3 \\ H_4(K(\mathbb{Z}/2,2);\mathbb{Z}) &\cong \mathbb{Z}/4 \end{aligned}$$

the above subquotients are as claimed once we show that there are no differentials involved. All differentials leaving from $H_5(K(\mathbb{Z}/2,2);\mathbb{Z}) \cong \mathbb{Z}/2$ are trivial because the edge-homomorphism $\Omega_5^{SO} \longrightarrow H_5(K(\mathbb{Z}/2,2);\mathbb{Z})$ (compare Lemma 3.2.1) is onto. This follows from the fact that the nontrivial element $z \in H_5(K(\mathbb{Z}/2,2);\mathbb{Z})$ is realized by the oriented 5-manifold $M := SU(3)/SO(3)$. To verify this assertion, note that M is simply-connected and non-spin with $H_2(M;\mathbb{Z}) \cong \pi_2 M \cong \mathbb{Z}/2$. Thus by Poincaré duality

$$0 \neq \langle w_2(M) \cup Sq^1(w_2(M)), [M] \rangle = \langle \iota_2 \cup Sq^1(\iota_2), (w_2)_*[M] \rangle, \text{ i.e. } (w_2)_*[M] = z.$$

The only other possible differentials in dimension 4 are

$$d_2 : E^2_{4,i} \longrightarrow E^2_{2,i+1}, i = 0,1.$$

But using Theorem 3.1.3 these are given by $(Sq^2_{\iota_2})^*$ respectively by $(Sq^2_{\iota_2})^* \circ r_2$. Now in our situation $(Sq^2_{\iota_2})^* \equiv 0$ because $Sq^2_{\iota_2}(\iota_2) = Sq^2(\iota_2) + \iota_2 \cup \iota_2 = 0$ and thus Rohlin's Theorem follows.
(3) There is a commutative diagram of fibrations

$$\begin{array}{ccccc} BSpin & \longrightarrow & B & \longrightarrow & K(\Pi,1) \\ \Vert & & \big\downarrow \xi(\Pi,w) & & \big\downarrow w \\ BSpin & \longrightarrow & BSO & \longrightarrow & K(\mathbb{Z}/2,2) \end{array}$$

and $w \neq 0$ implies that $w_* : H_2(\Pi;\mathbb{Z}/2) \longrightarrow H_2(K(\mathbb{Z}/2,2);\mathbb{Z}/2) \cong \mathbb{Z}/2$ is onto.
We now apply the James spectral sequence to both fibrations. The lower fibration was discussed in (2) and for the upper fibration, the James spectral sequence gives a filtration

$$\Omega_4^{Spin} \subseteq F_{2,2}(\xi) \subseteq F_{3,1}(\xi) \subseteq \Omega_4(\xi).$$

Here certainly differentials can exist and thus the subquotients of this filtration are only subquotients of $E^2_{p,4-p}$. Now consider an element $x \in H^2(\Pi;\mathbb{Z}/2) \cong E^2_{2,2}$ with $w_*(x) \neq 0$. Then x cannot be hit by a differential since $x = d_i(y)$ would give the contradiction $w_*(x) = d_i(\xi_*(y))$ to the arguments in (2). Therefore, x survives to infinity to give an element $\bar{x} \in F_{2,2}(\xi) \subseteq \Omega_4(\xi)$ mapping to $\xi_*(\bar{x}) \in F_{2,2} \setminus \Omega_4^{Spin}$. Thus the corresponding manifold has signature 8 (mod 16).
(4) If M is a manifold with normal 1-type ξ and $u : M \longrightarrow K(\pi_1 M, 1)$ is a 2-equivalence then the universal coefficient sequence shows that under our assumption one has:

$$\langle w_2(M), x \rangle = \langle u^*(w), x \rangle = \langle w, u_*(x) \rangle = 0 \quad \forall x \in H_2(M;\mathbb{Z})$$

which implies that the intersection form on $H_2(M;\mathbb{Z})$ is even. But since $\sigma(M)$ is just the signature of this form, it follows that $\sigma(M)$ is divisible by 8. Since by (3) there also exists a manifold with signature 8 in this normal 1-type, we are done.
(5) To prove $\sigma(\Pi, w) \mid 4$, we want to find an element in $F_{3,1}(\xi) \subseteq \Omega_4(\xi)$ which maps to $F_{3,1} \setminus F_{2,2}$ under ξ_*.
Take an arbitrary element $x \in H_3(\Pi;\mathbb{Z}/2)$. Then $w_*(x) \neq 0$ if and only if $\langle Sq^1 w, x \rangle \neq 0$ because $H^3(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$ is generated by $Sq^1(\iota_2)$. Such an element x cannot be hit by a differential since again this would contradict our knowledge about the spectral sequence for the

fibration with total space BSO. Therefore, x survives to infinity if and only if $d_2(x) = 0$ which is equivalent to

$$\langle a \cup w, x\rangle = 0 \quad \forall a \in H^1(\Pi; \mathbb{Z}/2)$$

because by Theorem 3.1.3 the dual of d_2 is multiplication by w (Sq^2 vanishes on 1-dimensional classes). Obviously our assumption is equivalent to the existence of an element x satisfying the above two conditions.

(6) We have a commutative diagram

$$\begin{array}{ccccc} \Omega_4(\xi) & \xrightarrow{\xi_*} & \Omega_4^{SO} & \xrightarrow{\sigma} & \mathbb{Z} \\ {\scriptstyle q}\downarrow & & {\scriptstyle q'}\downarrow & & {\scriptstyle r_4}\downarrow \\ H_4(\Pi) & \xrightarrow{w_*} & H_4(K(\mathbb{Z}/2,2)) & \xrightarrow{\cong} & \mathbb{Z}/4 \end{array}$$

which shows that $w_* \circ q$ is the signature mod 4. Recall that $H^4(K(\mathbb{Z}/2,2);\mathbb{Z}/4)$ is generated by the Pontrjagin square $\wp$ and thus as a map into $\mathbb{Z}/4$, w_* is given by

$$w_*(x) = \langle \wp, w_*(x)\rangle = \langle w^*(\wp), x\rangle = \langle \wp(w), x\rangle, \quad x \in H_4(\Pi).$$

Reducing further mod 2 gives $\langle w^2, x\rangle$. Moreover, x lies in the image of q if and only if $d_i(x) = 0$ for $i = 2, 3$.
By assumption multiplication by w is injective on $H^1(\Pi; \mathbb{Z}/2)$ which is equivalent to the surjectivity of $d_2 : E^2_{3,1} \longrightarrow E^2_{1,2}$. But this implies the vanishing of d_3 on $E^3_{4,0}$.
Thus if the image of

$$w_* : \{x \in H_4(\Pi) \mid d_2(x) = 0\} \longrightarrow \mathbb{Z}/4$$

is $n \cdot \mathbb{Z}/4$ with $n \in \{0, 1, 2\}$ then $\sigma(\Pi, w) \equiv n \mod 4$.
In particular, $\sigma(\Pi, w) \equiv 1 \mod 4$ if and only if there exists an $x \in H_4(\Pi)$ with $\langle w^2, x\rangle \neq 0$ and $d_2(x) = 0$ ($\Longleftrightarrow \quad \langle b^2 + b \cup w, x\rangle = 0 \quad \forall b \in H^2(\Pi; \mathbb{Z}/2)$ by Theorem 3.1.3).

Claim: *These two conditions are equivalent to our second assumption*

$$w^2 \notin \{b^2 + b \cup w \mid b \in H^2(\Pi; \mathbb{Z}/2)\} + \mathrm{Ker}(k_2 : H^4(\Pi; \mathbb{Z}/2) \longrightarrow \mathrm{Hom}(H_4(\Pi), \mathbb{Z}/2)).$$

To see this equivalence, first observe that $\mathrm{Ker}(k_2)$ is by definition the subgroup of $H^4(\Pi; \mathbb{Z}/2)$ which annihilates all of $H_4(\Pi)$. Dividing out $\mathrm{Ker}(k_2)$ and defining $A := H_4(\Pi)/2 \cdot H_4(\Pi)$,

$$U := k_2(\{b^2 + b \cup w \mid b \in H^2(\Pi; \mathbb{Z}/2)\}) \subseteq \mathrm{Hom}(H_4(\Pi), \mathbb{Z}/2) = \mathrm{Hom}(A, \mathbb{Z}/2)$$

and $\mu := k_2(w^2) \in \mathrm{Hom}(A, \mathbb{Z}/2)$, we get the following statement which is equivalent to the claim:

$$\exists x \in A \text{ with } \mu(x) \neq 0 \text{ and } u(x) = 0 \quad \forall u \in U \Longleftrightarrow \mu \notin U.$$

In other words, we want to proof $U = \mathrm{Ann}(\mathrm{Ann}(U))$ if we define the annihilator by

$$\mathrm{Ann}(U) := \{y \in A \mid u(y) = 0 \ \forall u \in U\}.$$

Clearly we have the inclusion $U \subseteq \mathrm{Ann}(\mathrm{Ann}(U))$ and since both sides are finite dimensional $\mathbb{Z}/2$-vector spaces for the equality we have to show that their dimensions agree. But this follows directly from the nondegeneracy of the bilinear form

$$\begin{aligned} \mathrm{Hom}(A, \mathbb{Z}/2) \times A &\longrightarrow \mathbb{Z}/2 \\ (u, a) &\longmapsto u(a). \quad \blacksquare \end{aligned}$$

Remark . Under the assumption of part (6) of the Theorem one can show with the same methods that if $\sigma(\Pi, w) \neq 1$ then $\sigma(\Pi, w) = 2$ if and only if

$$\wp(w) \notin i_{4,2}\left(\{b^2 + b \cup w \mid b \in H^2(\Pi; \mathbb{Z}/2)\}\right) + \mathrm{Ker}\left(k_4 : H^4(\Pi; \mathbb{Z}/4) \longrightarrow \mathrm{Hom}(H_4(\Pi), \mathbb{Z}/4)\right).$$

where $i_{4,2} : \mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4$. This can be used to show that for example $\sigma((\mathbb{Z}/4)^6, w) = 2$ for a certain class w, but we will not give any details here.

Examples:

a) If Π is a finite group with cyclic or quaternion 2-Sylow subgroup, then $\sigma(\Pi, w) = 8 \quad \forall w \neq 0$.

b)

$$\sigma(\mathbb{Z}/2 \times \mathbb{Z}/2, w) = \begin{cases} 4 & \text{if } w = x_1^2 + x_1 \cdot x_2 + x_2^2 =: y, \\ 8 & \text{if } w \neq 0, y, \\ 16 & \text{if } w = 0. \end{cases}$$

where $\{x_1, x_2\}$ is the usual basis for $H^1(\mathbb{Z}/2 \times \mathbb{Z}/2; \mathbb{Z}/2)$.

c) Let $\Pi := \mathbb{Z}/16 \rtimes \mathbb{Z}/8$ with action of $\mathbb{Z}/8$ on $\mathbb{Z}/16$ given by $t \mapsto t^5$. Then there exists a class $w \in H^2(\Pi; \mathbb{Z}/2)$ such that $\sigma(\Pi, w) = 1$.

Proof. a) follows directly from Theorem 3.3.2(4) because the groups in question fulfill the assumption there.
b) If $w \neq y$, there exists an inclusion $i : \mathbb{Z}/2 \hookrightarrow \mathbb{Z}/2 \times \mathbb{Z}/2$ such that $i^*(w) = 0$. Now if M is a 4-manifold with normal 1-type $\xi(\mathbb{Z}/2 \times \mathbb{Z}/2, w)$ then the double covering corresponding to $i(\mathbb{Z}/2)$ is a spin manifold and thus its signature is divisible by 16. Therefore, $\sigma(M) \equiv 0 \mod 8$ and the assertion follows from part (3) of Theorem 3.3.2. The case $w = y$ is handled by part (5) of the theorem because one computes that

$$Sq^1(y) = x_1^2 \cdot x_2 + x_1 \cdot x_2^2 \notin \{a \cdot y \mid a \in H^1(\mathbb{Z}/2 \times \mathbb{Z}/2; \mathbb{Z}/2)\}.$$

c) We want to apply part(6) of Theorem 3.3.2 and will make use of the following computation of the $\mathbb{Z}/2$-cohomology ring of the group Π given in [Diethelm]:

$$H^*(\Pi; \mathbb{Z}/2) \cong \mathbb{Z}/2[a, b, v, w]/_{(a^2 = b^2 = 0)}$$

with $\deg(a) = \deg(b) = 1$ and $\deg(v) = \deg(w) = 2$ and

$$b = p^*(\beta), v = p^*(\nu) \text{ and } i^*(a) = \alpha, i^*(w) = \mu.$$

Here

$$\mathbb{Z}/16 \cong \langle t \rangle \xrightarrow{i} \Pi \xrightarrow{p} \mathbb{Z}/8$$

is the split extension in question and α, μ respectively β, ν are the generators for the corresponding cyclic groups.
Obviously multiplication by w is injective on $H^1(\Pi; \mathbb{Z}/2)$ and thus we have to show that

$$w^2 \notin \{y \cdot w + y^2 \mid y \in H^2(\Pi; \mathbb{Z}/2)\} + \mathrm{Image}(r_2).$$

Assume that w^2 can be written as

$$\begin{aligned} w^2 &= l_1(y \cdot w + y^2) + l_2 \cdot r_2(x), \quad l_i \in \mathbb{Z}/2 \text{ with } x \in H^4(\Pi) \text{ and} \\ y &= \lambda_1 \cdot w + \lambda_2 \cdot v + \lambda_3 \cdot a \cdot b, \quad \lambda_i \in \mathbb{Z}/2. \end{aligned}$$

Now observe that the map i^* factors through the fixed point set of the action $t \mapsto t^5$, i.e. through the group $H^4(\langle t \rangle)^{\mathbb{Z}/8}$.

If the element n generates $H^2(\langle t\rangle)$ then n^2 generates the group $H^4(\langle t\rangle)$ and thus we see that $\mathbb{Z}/8$ acts as $n \mapsto 5 \cdot n$ respectively as $n^2 \mapsto 25 \cdot n^2 = 9 \cdot n^2$. This shows that

$$H^4(\langle t\rangle)^{\mathbb{Z}/8} = \langle 2 \cdot n^2\rangle.$$

and therefore

$$r_2(i^*(x)) = 0 \quad \forall x \in H^4(\Pi).$$

If we use the relations

$$i^*(v) = i^*(b) = 0 \qquad \text{and thus } i^*(y) = \lambda_1 \cdot w$$

we get the following contradiction:

$$0 \neq \mu^2 = i^*(w^2) = l_1 \cdot \big(i^*(y) \cdot i^*(w) + i^*(y)^2\big) = l_1 \cdot (\lambda_1 \cdot w^2 + \lambda_1^2 \cdot w^2) = 0. \qquad \blacksquare$$

Remark 3.3.3. For the metacyclic group from example c) above, there cannot exist a stable vector bundle $\eta : K(\Pi, 1) \longrightarrow BO$ with $w_2(\eta) = w$. Otherwise $r_2(p_1(\eta)) = w_2(\eta)^2 = w^2$ would give a contradiction to the above proof. This gives the long promised example of a pair (Π, w) where description(III) from Theorem 2.2.1 is not applicable.

4. The Bordism Groups $\Omega_4(\xi)$ for Special Fundamental Groups

Since we will be interested in oriented manifolds, the bordism group $\Omega_4(\xi)$ for a 1-universal fibration $\xi : B \longrightarrow BSO$ with $\pi_2 B \neq 0, \Pi := \pi_1 B$ was computed in the example of Section 4. We showed that signature and π_1-fundamental class give an isomorphism to $\mathbb{Z} \times H_4(\Pi)$.

In this Section we can thus concentrate on 1-universal fibrations with $\pi_2 B = 0$. We will compute $\Omega_4(\xi)$ for all finite Π with periodic 2-Sylow subgroups. We prove that in this case the bordism group either equals $\mathbb{Z} \times H_4(\Pi)$, again classified by signature and π_1-fundamental class or is equal to $\mathbb{Z} \times \mathbb{Z}/2 \times H_4(\Pi)$ with a new $\mathbb{Z}/2$-valued bordism invariant. In the spin case, this invariant is essentially the $\mathfrak{sec}$-invariant from Section 6.2, however in the nonspin case it comes from the $E^\infty_{2,2}$-term of the James spectral sequence for $\Omega_4(\xi)$ and gets therefore the name $\mathfrak{ter}$ for *tertiary* bordism invariant. The main difference between these two invariants is the fact that $\mathfrak{sec}$ is a homotopy invariant (see Theorem 6.4.1), whereas $\mathfrak{ter}$ takes different values on homotopy equivalent manifolds as we show in Section 8.2.

As a guideline for the reader, we remark that Sections 7.1 to 7.3 cover the case of periodic 2-groups and contain all interesting computations. In Section 7.4 we then put all information together to obtain the classification results.

As a matter of convenience, we shall write in the future $B\Pi$ for an Eilenberg-MacLane space of type $(\Pi, 1)$. This is consistent with the classifying space notation, since Π is always a discrete group.

4.1. **The $\mathbb{Z}/2$-Cohomology Ring for Periodic 2-Groups.**

We first recall from [Brown, Thm.VI,9.3] that a 2-group with periodic cohomology is either a cyclic or a (generalized) quaternion group. To fix the notation, let n be a power of 2 and let

$$C = C_{2n} := (z \mid z^{2n} = 1) \quad , \quad Q = Q_{8n} = (x, y \mid x^{4n} = y^2, y^{-1}xy = x^{-1})$$

be the cyclic and quaternion groups of orders $2n$ respectively $8n$. (We omitted Q_4 because it is cyclic.) There are the well-known faithful representations

$$\rho_C : C \longrightarrow U(1) \quad \text{and} \quad \rho_Q : Q \longrightarrow SU(2)$$

given by

$$\rho_C(z) := \epsilon_{2n} \quad \text{respectively} \quad \rho_Q(x) = \begin{pmatrix} \epsilon_{4n}^{-1} & 0 \\ 0 & \epsilon_{4n} \end{pmatrix}, \rho_Q(y) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $\epsilon_k \in \mathbb{C}$ denotes a primitive k'th root of unity. Furthermore we have the following 1-dimensional orthogonal representations:

$$a(z^i) := (-1)^i \quad \text{respectively} \quad x_k(x^i \cdot y^j) := \begin{cases} (-1)^j & \text{if } k = 1, \\ (-1)^i & \text{if } k = 2 \end{cases}$$

which define cohomology classes $a \in H^1(C; \mathbb{Z}/2)$ respectively $x_1, x_2 \in H^1(Q; \mathbb{Z}/2)$. The following computation of the $\mathbb{Z}/2$-cohomology rings for C and Q can be found for example in [Snaith]. The periodicity of the $\mathbb{Z}$-homology is a direct consequence of the existence of the representations ρ_C respectively ρ_Q above.

Lemma 4.1.1. *Let $b := w_2(\rho_C) \in H^2(C; \mathbb{Z}/2)$ and $p := w_4(\rho_Q) \in H^4(Q; \mathbb{Z}/2)$. Then*

a)

$$H^*(C_{2n}; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2[a,b]/(a^2 = b) = \mathbb{Z}/2[a] & \text{if } n = 1, \\ \mathbb{Z}/2[a,b]/(a^2 = 0) & \text{if } n > 1, \end{cases}$$

in particular $\dim_{\mathbb{Z}/2} H_i(C; \mathbb{Z}/2) = \dim_{\mathbb{Z}/2} H^i(C; \mathbb{Z}/2) = 1 \quad \forall i \in \mathbb{N}_0$.
Moreover, the $\mathbb{Z}$-homology of C is given by

$$H_i(C; \mathbb{Z}) \cong \begin{cases} C & \text{if } i \text{ is odd}, \\ 0 & \text{if } 0 < i \text{ is even}. \end{cases}$$

b)

$$H^*(Q_{8n}; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2[x_1, x_2, p]/(x_1^2 + x_1x_2 + x_2^2 = 0 = x_1^3) & \text{if } n = 1, \\ \mathbb{Z}/2[x_1, x_2, p]/(x_1x_2 + x_2^2 = 0 = x_1^3) & \text{if } n > 1, \end{cases}$$

in particular, $\dim_{\mathbb{Z}/2} H_i(Q; \mathbb{Z}/2) = \dim_{\mathbb{Z}/2} H^i(Q; \mathbb{Z}/2) = \begin{cases} 1 & \text{if } i \equiv 0, 3 \ (4), \\ 2 & \text{if } i \equiv 1, 2 \ (4). \end{cases}$

Moreover, the $\mathbb{Z}$-homology of Q is given by

$$H_i(Q_{8n}; \mathbb{Z}) \cong \begin{cases} Q^{ab} \cong \mathbb{Z}/2 \times \mathbb{Z}/2 & \text{if } i \equiv 1 \quad (4) \\ C_{8n} & \text{if } i \equiv 3 \quad (4) \\ 0 & \text{if } 0 < i \text{ is even}. \end{cases}$$

Remark . It clearly follows from the relations in $H^*(Q;\mathbb{Z}/2)$ that all products of x_1 and x_2 vanish in dimensions 4 and higher. Moreover, the main difference between Q_8 and $Q_{8n}, n > 1$ is the fact that for Q_8 one has

$$x_1^2 \cdot x_2 = x_1 \cdot x_2^2 \neq 0 \quad , \quad x_1^3 = 0 = x_2^3,$$

whereas for $n > 1$ the relations in $H^3(Q_{8n};\mathbb{Z}/2)$ look like

$$x_1^2 \cdot x_2 = x_1 \cdot x_2^2 = x_2^3 \neq 0 \quad , \quad x_1^3 = 0.$$

In order to determine the d_2-differentials in the James spectral sequence, we will need the following information about the action of Sq^1 and Sq^2 on the $\mathbb{Z}/2$-cohomology rings.

Lemma 4.1.2. *With the notation from lemma 4.1.1 one has*

a) $Sq^k(a^i) = \binom{i}{k} \cdot a^{i+k}$
and for $n > 1$: $\quad Sq^1(a^i \cdot b^j) = i \cdot a^{i+1} \cdot b^j \quad , \quad Sq^2(a^i \cdot b^j) = j \cdot a^i \cdot b^{j+1}$.

b) $Sq^1(x_1^i \cdot x_2^j \cdot p^k) = \begin{cases} x_1^2 \cdot p^k & \textit{if } i = 1, j = 0, \\ x_2^2 \cdot p^k & \textit{if } j = 1, i = 0, \\ 0 & \textit{else}, \end{cases}$

$Sq^2(x_1^i \cdot x_2^j \cdot p^k) = \begin{cases} x_1^2 \cdot x_2^j \cdot p^k & \textit{if } i = 1 \neq j, \\ x_1^i \cdot x_2^2 \cdot p^k & \textit{if } j = 1 \neq i, \\ 0 & \textit{else}. \end{cases}$

Proof. If we use the so called *Wu formula* (see [Milnor-Stasheff])

$$Sq^k(w_m) = \sum_{i=0}^{k} \binom{k-m}{i} w_{k-i} \cup w_{m+i}$$

and note that

$$w_i(\rho_C) = 0 \quad \forall i \neq 2 \qquad \text{and} \qquad w_i(\rho_Q) = 0 \quad \forall i \neq 4,$$

we can deduce

$$Sq^1(b) = 0 = Sq^1(p) = Sq^2(p).$$

Using the Cartan product formula and the fact that Sq^2 squares 2-dimensional and vanishes on 1-dimensional classes, one easily obtains the assertions. ■

4.2. **The Spin Case.**

If $\xi : B = BSpin \times B\Pi \xrightarrow{Bpop_1} BSO$ is the normal 1-type for spin manifolds then $\Omega_*(\xi) \cong \Omega_*^{Spin}(B\Pi)$ and we can apply the Atiyah-Hirzebruch spectral sequence for the generalized homology theory Ω_*^{Spin} to the computation of $\Omega_*(\xi)$ as described in Section 5.3. Since the relevant d_2-differentials are given by the duals of Sq^2 (see Lemma 2.3.2), Lemmas 4.1.1 and 4.1.2 show that the natural map

$$\Omega_4^{Spin} \longrightarrow \Omega_4^{Spin}(BC)$$

is an isomorphism because $E^3_{i,4-i} = 0 \quad \forall i \neq 0$. In particular, the bordism groups are isomorphic to the integers via the signature divided by 16.
Quite to the contrary, in the quaternion case all relevant d_2-differentials vanish by Lemma 4.1.2 and in order to determine $\Omega_4^{Spin}(BQ)$, we have to look at the next differential.

Proposition 4.2.1. *The differential*

$$d_3 : E^3_{5,0} = H_5(BQ;\mathbb{Z}) \longrightarrow H_2(BQ;\mathbb{Z}/2) = E^3_{2,2}$$

is an isomorphism.

Proof. Let $C = C_{4n} \subseteq Q_{8n} = Q$ be the cyclic subgroup of order $4n$ generated by $x \in Q$. Then $C = \mathrm{Ker}(x_1)$ and if η is a 1-dimensional vector bundle over BQ with $w_1(\eta) = x_1$ then its sphere bundle $S\eta$ is a space of type $(C,1)$. Clearly the disc bundle $D\eta \simeq BQ$ is a space of type $(Q,1)$ and thus we obtain a cofiber sequence

$$BC \longrightarrow BQ \xrightarrow{p} T\eta$$

where $T\eta := D\eta/S\eta$ is the Thom space of η.
Let $u \in H^1(T\eta;\mathbb{Z}/2)$ be the Thom class. Then by definition of the first Stiefel-Whitney class as an unoriented version of the Euler class we get

$$p^*(u \cdot x_i) = w_1(\eta)\cdot x_i = x_1 \cdot x_i \in H^2(Q;\mathbb{Z}/2).$$

Let $\{a_1, a_2\}$ respectively $\{b_1, b_2\}$ denote the $\mathbb{Z}/2$-basis of $H_2(Q;\mathbb{Z}/2)$ respectively $H_2(T\eta;\mathbb{Z}/2)$ which are dual to the basis $\{x_1^2, x_2^2\}$ respectively $\{u\cdot x_1, u\cdot x_2\}$ of $H^2(Q;\mathbb{Z}/2)$ respectively $H^2(T\eta;\mathbb{Z}/2)$. From the relations in $H^2(Q;\mathbb{Z}/2)$ we deduce that

$$\langle u\cdot x_i, p_*(a_2)\rangle = \langle x_1 \cdot x_i, a_2\rangle = \begin{cases} 0 & \text{if } i = 1,\\ 1 & \text{if } i = 2,\end{cases}$$

and thus $p_*(a_2) = b_2$.
Now the idea is to compare the Atiyah-Hirzebruch spectral sequences for BQ and $T\eta$. Let

$$z \in E^2_{4,1}(T\eta) = H_4(T\eta;\mathbb{Z}/2) \cong H_3(Q;\mathbb{Z}/2) \cong \mathbb{Z}/2$$

denote a generator. Then we compute that

$$\begin{aligned}\langle u\cdot x_i, d_2^{T\eta}(z)\rangle &= \langle Sq^2_{T\eta}(u\cdot x_i), z\rangle\\ &= \langle Sq^2(u)\cdot x_i + Sq^1(u)\cdot Sq^1(x_i) + u\cdot Sq^2(x_i), z\rangle\\ &= \langle w_2(\eta)\cdot u\cdot x_i + w_1(\eta)\cdot u\cdot x_i^2, z\rangle\\ &= \langle u\cdot x_1\cdot x_i^2, z\rangle\end{aligned}$$

which implies that $d_2^{T\eta}(z) = b_2$ via the relations $x_1^3 = 0 \neq x_1\cdot x_2^2$. If we consider the Atiyah-Hirzebruch spectral sequence

$$\widetilde{H}_*(X;\Omega_*^{Spin}) \Longrightarrow \widetilde{MSpin}_*(X)$$

for the reduced theories, we get a commutative diagram

$$\begin{tikzcd} E^\infty_{2,2}(BQ) \arrow[r, "p_*"] \arrow[d] & E^\infty_{2,2}(T\eta) \arrow[d] \\ \widetilde{MSpin}_4(BQ) \arrow[r, "p_*"] & \widetilde{MSpin}_4(T\eta) \end{tikzcd}$$

where both vertical arrows are inclusions. Now observe that the bottom map p_* is also injective because $\widetilde{MSpin}_4(BC) = 0$ as we noted at the beginning of this Section. This shows that p_* is injective on $E^\infty_{2,2}$-terms as well.

Claim: $a_2 \in \mathrm{Image}(d_3^{BQ})$.
Proof: If not then $0 \neq \bar{a}_2 \in E^\infty_{2,2}(BQ)$ and therefore also $p_*(\bar{a}_2) \neq 0$. But this contradicts the above computations $p_*(a_2) = b_2 = d_2^{T\eta}(z)$.
This finishes the proof of the proposition in the case $n = 1$ because Q_8 has an automorphism which interchanges x and y and thus x_1 and x_2 respectively a_1 and a_2. Therefore, also $a_1 \in \mathrm{Image}(d_3^{BQ_8})$ and thus $d_3^{BQ_8}$ is an isomorphism.
If $n > 1$, we consider the inclusion

$$\begin{array}{rccc} i: & Q_8 & \longrightarrow & Q_{8n} \\ & x[8] & \longmapsto & x[8n]^n \\ & y[8] & \longmapsto & y[8n]. \end{array}$$

Then we have $i^*(x_{1[8n]}) = x_{1[8]}$ and $i^*(x_{2[8n]}) = 0$, implying $i_*(a_{1[8]}) = a_{1[8n]}$. The naturality of the differential d_3 gives a commutative diagram

$$\begin{tikzcd} H_5(Q_8) \arrow[r, "i_*"] \arrow[d, "\cong"', "d_3^{BQ_8}"] & H_5(Q_{8n}) \arrow[d, "d_3^{BQ_{8n}}"] \\ H_2(Q_8;\mathbb{Z}/2) \arrow[r, "i_*"] & H_2(Q_{8n};\mathbb{Z}/2). \end{tikzcd}$$

This diagram shows that $a_{1[8n]} = i_*(a_{1[8]}) \in \mathrm{Image}(d_3^{BQ_{8n}})$ which together with the above claim proves the proposition in the cases $n > 1$. ■

Theorem 4.2.2. *There is an isomorphism of groups*

$$(\sigma, \mathfrak{sec}) : \Omega_4^{Spin}(BQ) \longrightarrow 16 \cdot \mathbb{Z} \times \mathbb{Z}/2.$$

Proof. Putting the information about the differentials together, the Atiyah-Hirzebruch spectral sequence gives an exact sequence

$$0 \to \Omega_4^{Spin} \to \Omega_4^{Spin}(BQ) \to E^\infty_{3,1} \cong \mathbb{Z}/2 \to 0.$$

Moreover, by forgetting the map into BQ one obtains a splitting of this sequence. Finally, following Section 6.2 we see that the secondary edge-homomorphism to $E^\infty_{3,1}$ is given by the map we called $\mathfrak{sec}$ there. ■

Remark . To be precise, we should have mentioned that we can apply $\mathfrak{sec}$ only to elements which are zero-bordant in $\Omega_4^{SO}(BQ) \cong \mathbb{Z}$. Thus the above map to $\mathbb{Z}/2$ is in fact given by

$$[M] \longmapsto \mathfrak{sec}([M] - \frac{\sigma(M)}{16} \cdot [K]),$$

where K denotes the Kummer surface.

4.3. **The Non-Spin Case.**

As in the spin case, the cyclic groups are easy to handle. First note that there is only one case to consider since $H^2(C;\mathbb{Z}/2) = \langle b \rangle$. Since multiplication by b is injective on $H^1(C;\mathbb{Z}/2)$, we see that the differential

$$d_2 : E^2_{3,1} \longrightarrow E^2_{1,2}$$

is injective and thus the James spectral sequence for $\Omega_*(\xi(C,b))$ gives an exact sequence

$$0 \to \Omega_4^{Spin} \to \Omega_4(\xi(C,b)) \to E^\infty_{2,2} \cong \mathbb{Z}/2 \to 0.$$

Since by Theorem 3.3.2 there exists a manifold with signature 8 in this normal 1-type, we proved that

$$\frac{\sigma}{8} : \Omega_4(\xi(C,b)) \longrightarrow \mathbb{Z}$$

is an isomorphism.
Before we start to work on the quaternion groups, we need a general

Lemma 4.3.1 ([James])**.** *Let E, F be real vector bundles over a space X. If $q : S(F) \longrightarrow X$ denotes the sphere bundle projection then there is a cofiber sequence of Thom spaces*

$$T(q^*E) \longrightarrow T(E) \longrightarrow T(E \oplus F).$$

Proof. One first notes that $T(E) = DE/SE$ is a deformation retract of

$$\frac{DE \times DF}{SE \times DF},$$

where $\times$ stands for the fiber product over X. Then there is an inclusion

$$T(q^*E) = \frac{DE \times SF}{SE \times SF} \hookrightarrow \frac{DE \times DF}{SE \times DF}$$

with cofiber

$$\frac{DE \times DF}{DE \times SF \cup SE \times DF} = T(E \oplus F). \quad \blacksquare$$

Remark 4.3.2. Recalling from Definition 2.3.1 that for an N-dimensional vector bundle E the Thom spectrum ME equals the suspension spectrum $\Sigma^{-N}T(E)$, we get the following cofiber sequence of Thom spectra:

$$Mq^*E \to ME \xrightarrow{p} \Sigma^{\dim F} M(E \oplus F).$$

If we apply the generalized homology theory Ω_*^{Spin} to this cofiber sequence, we get a long exact sequence

$$\ldots \to \Omega_n^{Spin}(Mq^*E) \xrightarrow{q_*} \Omega_n^{Spin}(ME) \xrightarrow{.\cap[F]} \Omega_{n-\dim F}^{Spin}(M(E \oplus F)) \xrightarrow{\partial} \Omega_{n-1}^{Spin}(Mq^*E) \to \ldots$$

Interpreting the group $\Omega_n^{Spin}(ME) \cong \Omega_4(\xi_E)$ as the bordism group of ξ_E-manifolds, where

$$\xi_E := Bp \oplus E : BSpin \times X \longrightarrow BO,$$

(and similarly the other bordism groups) then the maps in the above exact sequence have the following geometrical interpretation (compare [Bröcker-tom Dieck, p.21]):

- q_* is given by composing the ξ_{q^*E}-structure with q to obtain a ξ_E-structure on the same manifold.
- If $\tilde{\nu} : M \longrightarrow BSpin \times X$ is a ξ_E-manifold, we can pull back the vector bundle F over X to M and can take the self-intersection of the zero-section in this bundle. This gives the map $. \cap [F]$.
- To get the image under ∂ for an $(n - \dim F)$-dimensional $\xi_{E \oplus F}$-manifold $\mu : N \longrightarrow BSpin \times X$, one just takes the pullback of the sphere bundle $q : S(F) \longrightarrow X$ under μ. This comes naturally equipped with a ξ_{q^*E}-structure.

We now consider the quaternion groups $Q = Q_{8n}$, n a power of 2. We fix a class $w \in H^2(Q; \mathbb{Z}/2)$ and want to use the James spectral sequence for the computation of $\Omega_4(\xi(Q,w))$. As in the cyclic case, there is no $\mathfrak{sec}$-invariant problem, since the differential

$$d_2 : E^2_{3,1} \longrightarrow E^2_{1,2}$$

is injective for all w and thus $E^{\infty}_{3,1}$ is always trivial. On the contrary, all other relevant d_2-differentials vanish. Therefore, the only open question about differentials is answered in the following

Proposition 4.3.3. *The differential*

$$d_3(w) : E^3_{5,0} = H_5(BQ;\mathbb{Z}) \longrightarrow H_2(BQ;\mathbb{Z}/2) = E^3_{2,2}$$

is nontrivial if $w = x_2^2$ or $x_1^2 + x_2^2$.
If $w = x_1^2$ then $d_3 \neq 0$ if and only if Q is the ordinary quaternion group of order 8.

Proof. First note that there is an automorphism of Q_{8n} which maps y to $x{\cdot}y$ leaving x fixed. This shows that we only have to consider the two cases $w = x_i^2$, because the answer for $w = x_1^2 + x_2^2$ will be the same as the one for $w = x_2^2$.

<u>1.Case:</u> $w = x_2^2$.
Since $w \in \operatorname{Image}(r_2 : H^2(Q;\mathbb{Z}) \longrightarrow H^2(Q;\mathbb{Z}/2))$, we can choose a complex line bundle L with $w_2(L) = w$. Then description(III) of Theorem 2.2.1 is applicable, i.e.

$$\xi(Q,w) = BSpin \times BQ \xrightarrow{Bp\oplus L} BSO$$

and $\Omega_*(\xi) \cong \pi_*(M\xi) \cong MSpin_*(ML)$ can be computed by Section 5.3 via the Atiyah-Hirzebruch spectral sequence with E^2-term equal to $H_*(ML;\Omega_*^{Spin})$. It is clear that it suffices to study the corresponding d_3-differential in this spectral sequence. We will use a similar method as in Proposition 4.2.1, so let again $C = C_{4n} \subseteq Q_{8n} = Q$ be the cyclic subgroup of order $4n$ generated by $x \in Q$. Then $C = \operatorname{Ker}(x_1)$ and if η is a 1-dimensional vector bundle over BQ with $w_1(\eta) = x_1$ then its sphere bundle $S\eta$ is a space of type $(C,1)$. If we call $q : S\eta \longrightarrow BQ$ the projection, Remark 4.3.2 gives us a cofiber sequence of Thom spectra

$$Mq^*L \longrightarrow ML \xrightarrow{p} \Sigma M(L\oplus\eta). \tag{II.5}$$

Since the order of C is $4n$, the square of the generator of $H^1(C;\mathbb{Z}/2)$ is trivial and thus

$$w_2(q^*L) = q^*(w) = (q^*(x_2))^2 = 0,$$

which shows that

$$MSpin_4(Mq^*L) \cong MSpin_4(BC) \cong 16\cdot\mathbb{Z}.$$

Then the commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & MSpin_4(S^0) \arrow[r, "\cong"] \arrow[d, equal] & MSpin_4(Mq^*L) \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & MSpin_4(S^0) \arrow[r] \arrow[d] & MSpin_4(ML) \arrow[r] \arrow[d, "p_*"] & E^{\infty}_{2,2}(ML) \arrow[r] & 0 \\
 & 0 \arrow[r] & MSpin_4(\Sigma M(L\oplus\eta)) & &
\end{tikzcd}$$

shows that the composition

$$E^{\infty}_{2,2}(ML) \xrightarrow{E^{\infty}_{2,2}(p_*)} E^{\infty}_{2,2}(\Sigma M(L\oplus\eta)) \subseteq MSpin_4(\Sigma M(L\oplus\eta))$$

is injective, in particular $E^{\infty}_{2,2}(p_*)$ is a monomorphism. Since

$$\dim_{\mathbb{Z}/2} H_2(ML;\mathbb{Z}/2) = 2 = \dim_{\mathbb{Z}/2} H_2(\Sigma M(L\oplus\eta);\mathbb{Z}/2)$$

and $E^{\infty}_{2,2}(ML) = H_2(ML;\mathbb{Z}/2)/\,\mathrm{Image}(d_3)$, the proof is finished if we can show that

$$E^{\infty}_{2,2}(\Sigma M(L\oplus\eta)) \neq E^{2}_{2,2}(\Sigma M(L\oplus\eta)).$$

This would be certainly guaranteed if the differential

$$d_2 : H_4(\Sigma M(L\oplus\eta);\mathbb{Z}/2) \longrightarrow H_2(\Sigma M(L\oplus\eta);\mathbb{Z}/2)$$

is nontrivial. Since $d_2 = (Sq^2)^*$, this is verified by the following computation (where $u \in H^1(\Sigma M(L\oplus\eta);\mathbb{Z}/2)$ denotes the Thom class for $L\oplus\eta$)

$$\begin{aligned} Sq^2(u\cdot x_1) &= Sq^2(u)\cdot x_1 + Sq^1(u)\cdot Sq^1(x_1) + u\cdot Sq^2(x_1)\\ &= w_2(L\oplus\eta)\cdot u\cdot x_1 + w_1(L\oplus\eta)\cdot u\cdot x_1^2\\ &= x_2^2\cdot x_1' u + x_1\cdot x_1^2\cdot u\\ &= x_2^2\cdot x_1\cdot u \neq 0. \end{aligned}$$

<u>2.Case:</u> $w = x_1^2$.
First note that an analogue proof as in the 1.case breaks down, because the corresponding Sq^2 is trivial. But for Q_8, the existence of an automorphism which interchanges x_1 and x_2 shows that $d_3^{Q_8} \neq 0$ in this case, too. For the higher order quaternion groups we have to analyze the situation in more detail:
Let $\{a_1, a_2\}$ respectively $\{e_1, e_2\}$ be the basis of $H_2(ML;\mathbb{Z}/2)$ respectively $H_5(ML)$ which are dual to the basis $\{x_1^2\cdot u, x_2^2\cdot u\}$ respectively $\{x_1^2\cdot p\cdot u, x_2^2\cdot p\cdot u\}$, where $u \in H^0(ML;\mathbb{Z}/2)$ is the Thom class of ML.

<u>Claim:</u> (1) $d_3(e_2) = 0$.
(2) $a_1, a_1 + a_2 \notin \mathrm{Image}(d_3)$.

Proof: (1) The cofiber sequence (II.5) gives a commutative diagram

$$\begin{array}{ccc} H_5(Mq^*L) & \xrightarrow{q^*} & H_5(ML)\\ \downarrow{\scriptstyle d_3} & & \downarrow{\scriptstyle d_3}\\ 0 = E^3_{2,2}(Mq^*L) & \xrightarrow{q^*} & E^3_{2,2}(ML) = H_2(ML;\mathbb{Z}/2) \end{array}$$

and thus it suffices to show that $e_2 \in \mathrm{Image}(q^*)$. But this follows from the computation below, where $z \in H_5(Mq^*L) \cong H_5(C)$ denotes a generator:

$$\begin{aligned} \langle p\cdot x_i\cdot u, q_*(z)\rangle &= \langle q^*(p\cdot x_i\cdot u), z\rangle\\ &= \langle q^*(p)\cdot q^*(x_i)\cdot u, z\rangle = \begin{cases} 0 & \text{if } i = 1\\ 1 & \text{if } i = 2. \end{cases} \end{aligned}$$

(2) follows from the fact that for a equal to a_1 or $a_1 + a_2$ the equation $a = d_3(e)$ would lead to the following contradiction:

$$\begin{aligned} 0 \neq \langle w, a\rangle &= \langle w, d_3(e)\rangle\\ &= \langle w^*(\iota_2), d_3(e)\rangle = \langle \iota_2, d_3(w_*(e))\rangle\\ &= 0. \end{aligned}$$

Here $0 \neq \iota_2 \in H^2(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$ and we compared with the James spectral sequence for the fibration

$$BSpin \to BSO \to K(\mathbb{Z}/2,2)$$

for which we showed in the proof of Theorem 3.3.2(2) that all relevant d_2 and d_3-differentials vanish. □

Note that the claim in particular proves the relation

$$d_3^{Q_8}(e_1[8]) = a_2[8].$$

Now let $n > 1$ and consider as in the proof of Proposition 4.2.1 the inclusion

$$i : Q_8 \hookrightarrow Q_{8n}$$

with $i^*(x_1[8n]) = x_1[8]$ and $i^*(x_2[8n]) = 0$ (implying $i^*(L[8n]) = L[8]$ and $i_*(a_2[8]) = 0$). Then the commutative diagram

$$\begin{array}{ccc} H_5(ML[8]) & \xrightarrow{i_*} & H_5(ML[8n]) \\ \downarrow{\scriptstyle d_3[8]} & & \downarrow{\scriptstyle d_3[8n]} \\ H_2(ML[8];\mathbb{Z}/2) & \xrightarrow{i_*} & H_2(ML[8n];\mathbb{Z}/2) \end{array}$$

shows that

$$d_3(e_1[8n]) = d_3(i_*(e_1[8])) = i_*(d_3(e_1[8])) = i_*(a_2[8]) = 0.$$

Together with Claim(1) this completes the proof. ■

Theorem 4.3.4. *Let $Q = Q_{8n}$ be a quaternion 2-group and $0 \neq w \in H^2(Q;\mathbb{Z}/2)$.*

(i) *If $w = x_2^2$ or $x_1^2 + x_2^2$ or $w = x_1^2, n = 1$, there is an isomorphism*

$$\sigma : \Omega_4(\xi(Q,w)) \longrightarrow 8\cdot\mathbb{Z}.$$

(ii) *If $w = x_1^2$ and $n > 1$, there is a $\mathbb{Z}/2$-valued "tertiary" bordism invariant $\mathfrak{ter}$ such that*

$$(\sigma, \mathfrak{ter}) : \Omega_4(\xi(Q,w)) \longrightarrow 8\cdot\mathbb{Z}\times\mathbb{Z}/2$$

is an isomorphism of groups.

Proof. Using the above information about the differentials in the James spectral sequence, the result follows from Theorem 3.3.2, parts(3) and (4) which say that in all cases the image of the signature on $\Omega_4(\xi(Q,w))$ is $8\cdot\mathbb{Z}$. ■

Remark . In Section 8.2 we will show that $\mathfrak{ter}$ is not a homotopy invariant whereas we show in Theorem 6.4.1 that the $\mathfrak{sec}$-invariant from the spin case is determined by the equivariant intersection form of the manifold.

4.4. Stable Classification Results for π_1 Finite with Periodic 2-Sylow Subgroups.

First recall from Section 5 that the fiber homotopy type of a 1-universal fibration $\xi : B \longrightarrow BSO$ only depends on $\Pi := \pi_1 B, \pi_2 B$ and $w_2(\xi)$ if $\pi_2 B = 0$. In Section 6.3, we denoted $\xi = \xi(\Pi, w)$ in the latter case. To have a unified language for all cases, we introduce the following notation:

Definition 4.4.1. Let $\xi : B \longrightarrow BSO$ be a 1-universal fibration with $\Pi := \pi_1 B$ and let $u : B \longrightarrow B\Pi$ be a fixed 2-equivalence.

1. The w_2-*type* of ξ is by definition ∞ if $\pi_2 B \neq 0$ and otherwise $w := (u_*)^{-1} w_2(\xi) \in H^2(\Pi;\mathbb{Z}/2)$. We will write

$$\xi = \xi(\Pi, w) \quad \text{with} \quad w \in \{\infty\} \cup H^2(\Pi;\mathbb{Z}/2).$$

2. If M is an oriented manifold with fundamental group Π then its w_2-*type* is by definition the w_2-type of the normal 1-type of M viewed as an element

$$w_2 M \in \mathcal{H}(\Pi) := \{\infty\} \cup \mathcal{H}^{\in}(\Pi;\mathbb{Z}/\in)/_{\mathrm{Out}(\Pi)}.$$

In part(2) of the definition, we have to divide out the group $\mathrm{Out}(\Pi)$ because the normal 1-type of a manifold is only well-defined up to fiber homotopy equivalence. Note that the w_2-type of an oriented manifold together with its fundamental group contains exactly the same information as its normal 1-type but is more convenient to use. Finally, recall that $w_2 M = \infty$ if and only if $w_2\widetilde{M} \neq 0$. Since this is in turn equivalent to the intersection form of $\widetilde{M}$ being odd, the w_2-type is a proper generalization of the even-odd type for 1-connected 4-manifolds.

Definition 4.4.2. Let Π be a finitely presented group and $w \in \mathcal{H}(\Pi)$. We define $\mathrm{MSt}_{2m}(\Pi)$ to be the set of all stable diffeomorphism classes of oriented $2m$-dimensional manifolds with fundamental group Π. Furthermore, we let $\mathrm{MSt}_4(\Pi, w)$ be the subset of $\mathrm{MSt}_4(\Pi)$ consisting of all manifolds with w_2-type w.

By Section 5, we know that for $m > 1$ all sets $\mathrm{MSt}_{2m}(\Pi, w)$ are non empty and since the w_2-type is obviously a stable diffeomorphism invariant and we have

$$\mathrm{MSt}_4(\Pi) = \bigcup_{w \in \mathcal{H}(\Pi)} \mathrm{MSt}_4(\Pi, w).$$

We will now describe another important stable diffeomorphism invariant:

Definition 4.4.3. Let $\xi : B \longrightarrow BSO$ be a 1-universal fibration with $\Pi := \pi_1 B$ and let $u : B \longrightarrow B\Pi$ be a fixed 2-equivalence.

1. The π_1*-fundamental class* of an n-dimensional ξ-manifold $(M, \tilde{\nu})$ is by definition the element
$$(u \circ \tilde{\nu})_*[M] \in H_n(\Pi).$$
2. If M is an oriented n-manifold with fundamental group Π then its π_1-fundamental class is by definition the element
$$v_*[M] \in H_n(\Pi)/_{\mathrm{Out}(\Pi)},$$
where $v : M \longrightarrow B\Pi$ is some 2-equivalence which can be chosen to factor over the normal 1-type of M.

Note that the π_1-fundamental class is a bordism invariant and gives a homomorphism

$$u_* : \Omega_n(\xi) \longrightarrow H_n(\Pi)$$

which is just the edge-homomorphism $\mathfrak{ed}$ from Theorem 3.2.1 but we will use the more suggestive name u_* in the following. With these invariants at hand, we can now start to describe the classification results.
Let Π be a finite group and $\Pi_{(2)}$ a 2-Sylow subgroup of Π. Following [Swan 1], we define the 2-period of Π to be the least positive integer q such that the Tate cohomology groups $\hat{H}^i(\Pi; M)$ and $\hat{H}^{i+q}(\Pi; M)$ have isomorphic 2-primary components for all $i \in \mathbb{Z}$ and all Π-modules M. The following theorem allows in certain cases a translation of the computations for $\Pi_{(2)}$ from Sections 7.1 to 7.3 to those for Π.

Theorem ([Swan 1, Thm.1])**.** *If $\Pi_{(2)}$ is cyclic, the 2-period of Π is 2. If $\Pi_{(2)}$ is a quaternion group, the 2-period of Π is 4.*

Our computations thus naturally split into two cases. Before we can state the classification theorems, we have to define the number $\underline{d(\Pi) := \dim_{\mathbb{Z}/2} H^2(\Pi; \mathbb{Z}/2)}$, which measures how many w_2-types we have to consider.

Theorem 4.4.4. *Let Π be a finite group with cyclic 2-Sylow subgroups. Then $d(\Pi) \in \{0,1\}$ and if $\xi = \xi(\Pi, w) : B \longrightarrow BSO$ is a 1-universal fibration, signature and π_1-fundamental class induce the following group isomorphisms:*

$$\begin{array}{rcll} \Omega_4(\xi) & \xrightarrow{\cong} & \mathbb{Z} \times H_4(\Pi) & \text{if } w = \infty, \\ \Omega_4(\xi) & \xrightarrow{\cong} & 16 \cdot \mathbb{Z} \times H_4(\Pi) & \text{if } w = 0, \\ \Omega_4(\xi) & \xrightarrow{\cong} & 8 \cdot \mathbb{Z} \times H_4(\Pi) & \text{if } d(\Pi) = 1 \text{ and } w \neq 0, \infty. \end{array}$$

Remark . It will soon turn out that $d(\Pi) = 0$ if and only if the group Π has odd order.

Corollary 4.4.5. *Let Π be a finite group with cyclic 2-Sylow subgroups. Then $|\mathcal{H}(\Pi)| = \lceil(\Pi)+ \in$ and signature and π_1-fundamental class induce the following 1-1 correspondences:*

$$\begin{array}{lcll} \mathrm{MSt}_4(\Pi, \infty) & \longleftrightarrow & \mathbb{Z} \times H_4(\Pi)/\mathrm{Out}(\Pi) & \\ \mathrm{MSt}_4(\Pi, 0) & \longleftrightarrow & 16 \cdot \mathbb{Z} \times H_4(\Pi)/\mathrm{Out}(\Pi) & \\ \mathrm{MSt}_4(\Pi, w) & \longleftrightarrow & 8 \cdot \mathbb{Z} \times H_4(\Pi)/\mathrm{Out}(\Pi) & \text{if } d(\Pi) = 1 \text{ and } w \neq 0, \infty. \end{array}$$

In particular, w_2-type, signature and π_1-fundamental class classify oriented 4-manifolds with fundamental group Π up to stable diffeomorphism, i.e. the mapping

$$(w_2, \sigma, u_*) : \mathrm{MSt}_4(\Pi) \longrightarrow \mathcal{H}(\Pi) \times \mathbb{Z} \times \mathcal{H}_\triangle(\Pi)/\mathrm{Out}(\Pi)$$

is injective.

Before we can proof these theorems, we need some preparations.

Lemma 4.4.6. *If C is a cyclic 2-Sylow subgroup of a finite group Π then*

1. $\hat{H}^k(\Pi; \mathbb{Z})_{(2)} \cong \begin{cases} C & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$
2. *For all $k \in \mathbb{Z}$,* $\quad \dim_{\mathbb{Z}/2} \hat{H}^k(\Pi; \mathbb{Z}/2) = \begin{cases} 1 & \text{if } |\Pi| \text{ is even,} \\ 0 & \text{if } |\Pi| \text{ is odd.} \end{cases}$
3. *The inclusion $i : C \hookrightarrow \Pi$ induces for all Π-modules M and all $k \in \mathbb{Z}$ a monomorphism*
$$i^* : \hat{H}^k(\Pi; M)_{(2)} \longrightarrow \hat{H}^k(\Pi; M).$$

Proof. (1) follows immediately from Swan's theorem because $\hat{H}^0(\Pi; \mathbb{Z})$ is cyclic of order $|\Pi|$ and $\hat{H}^1(\Pi; \mathbb{Z}) = 0$, see [Brown, p.135].
(2) is a direct consequence of (1) if one applies the long exact sequence in Tate cohomology obtained from the short exact coefficient sequence $\mathbb{Z} \rightarrowtail \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2$.
(3) If M is a Π-module, the transfer map

$$tr^* : \hat{H}^k(C; M) \longrightarrow \hat{H}^k(\Pi; M)$$

from Section 1.3 has the property $tr^* \circ i^* =$ multiplication by $|\Pi : C|$, which proves the statement. ∎

Proof (of Theorem 4.4.4 and Corollary 4.4.5). If $|\Pi|$ is odd, the proof is obvious, so let us assume that Π and thus C have even orders. The commutative diagram of fibrations

$$\begin{array}{ccccc} BSpin & \longrightarrow & B' & \longrightarrow & BC \\ \| & & \downarrow & & \downarrow{\scriptstyle Bi} \\ BSpin & \longrightarrow & B & \longrightarrow & B\Pi \end{array}$$

induces a map between the James spectral sequences for $\Omega_*(\xi(\Pi, w))$ and $\Omega_*(\xi(C, i^*w))$. With the homological information at hand, we can easily see that this map gives an exact sequence

$$0 \to \Omega_4(\xi(C, i^*w)) \to \Omega_4(\xi(\Pi, w)) \xrightarrow{u_*} H_4(\Pi) \to 0.$$

The edge-homomorphism u_* is onto, because $H_4(\Pi)$ has odd order and thus all differentials vanish on this group. In Sections 7.2 and 7.3 we showed that the signature induces isomorphisms

$$\Omega_4(\xi(C, i^*w)) \longrightarrow n \cdot \mathbb{Z}$$

where $n = 1, 16$ or 8 depending on $i^*w \in \mathcal{H}(\mathcal{C}) = \{\infty, \prime, \lfloor\}$. Since by Theorem 3.3.2(4) the image of the signature on $\Omega_4(\xi(\Pi, w))$ takes exactly the same values, the proof of Theorem 4.4.4 is finished because the signature splits the above exact sequence.
To prove the corollary, we only have to show that every element of $\mathrm{Aut}(\xi)$ respects the decomposition

$$\Omega_4(\xi) \cong \mathbb{Z} \times H_4(\Pi)$$

because the action on $H_4(\Pi)$ is given by the induced map on fundamental groups. Clearly, the torsion subgroup $H_4(\Pi)$ is preserved. Moreover, in all cases the James spectral sequence gives an exact sequence

$$0 \to n \cdot \mathbb{Z} \cong F_{2,2} \to \Omega_4(\xi) \to H_4(\Pi) \to 0$$

and the signature gives a splitting of this sequence. Since by Corollary 3.1.2, every fiber homotopy self-equivalence of ξ respects this extension (and does not change the signature of a manifold), the result follows. ■

At the end of the part on cyclic 2-Sylow subgroups, we state an interesting structure theorem for such groups, which seems to be standard for group theorists but might be unknown to some topologists.

Theorem ([Gorenstein, Ch.7,Thm.6.1]). *If Π is a finite group with cyclic 2-Sylow subgroups, it possesses a normal 2-complement, i.e.*

$$\Pi \cong N \rtimes C$$

with $|N|$ odd and C cyclic of 2-power order.

Note that this theorem implies that nonabelian simple groups cannot have cyclic 2-Sylow subgroups and that a group Π as above is always solvable.
Now let Π be a finite group with quaternion 2-Sylow subgroup $Q = Q_{8n}$. Using Swan's theorem and the same arguments as for the cyclic case (together with the observation [Brown, p.135] that $\hat{H}^{-1}(\Pi; \mathbb{Z}) = 0$ which was not necessary in the 2-periodic case), we obtain the following

Lemma 4.4.7. *Π has the following homological properties:*

1. $\hat{H}^k(\Pi; \mathbb{Z})_{(2)} \cong \begin{cases} C_{8n} & \text{if } k \equiv 0 \ (4), \\ 0 & \text{if } k \text{ is odd.} \end{cases}$
2. $\dim_{\mathbb{Z}/2} \hat{H}^k(\Pi; \mathbb{Z}/2) = 1$ *for all* $k \equiv 0, 3 \ (4)$ *and*
$$\begin{aligned} d(\Pi) &:= \dim_{\mathbb{Z}/2} H^2(\Pi; \mathbb{Z}/2) = \dim_{\mathbb{Z}/2} H^2(\Pi)_{(2)} \\ &= \dim_{\mathbb{Z}/2} \hat{H}^k(\Pi; \mathbb{Z}/2) \leq 2 \text{ for all } k \equiv 1, 2 \quad (4). \end{aligned}$$
3. *The inclusion $i : C \hookrightarrow \Pi$ induces for all Π-modules M and all $k \in \mathbb{Z}$ a monomorphism*
$$i^* : \hat{H}^k(\Pi; M)_{(2)} \longrightarrow \hat{H}^k(C; M).$$

Proof. The only thing which is not proved exactly as in the cyclic case is the equality

$$\dim_{\mathbb{Z}/2} H^1(\Pi;\mathbb{Z}/2) = \dim_{\mathbb{Z}/2} H^2(\Pi;\mathbb{Z}/2)$$

which can be shown in the following way. Because of the existence of the transfer maps, all three vertical maps i^* in the Bockstein exact sequence

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H^1(\Pi;\mathbb{Z}/2) & \xrightarrow{Sq^1_\Pi} & H^2(\Pi;\mathbb{Z}/2) & \longrightarrow & H^2(\Pi;\mathbb{Z}/4) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & \\
\cdots & \longrightarrow & H^1(Q;\mathbb{Z}/2) & \xrightarrow[\cong]{Sq^1_Q} & H^2(Q;\mathbb{Z}/2) & \longrightarrow & H^2(Q;\mathbb{Z}/4) & \longrightarrow & \cdots
\end{array}$$

are monomorphisms and thus the commutativity of the diagram shows that Sq^1_Π is an isomorphism. ■

At this point, I wanted to give examples of certain groups to show that $d(\Pi)$ can take all three possible values 0,1 and 2 and I was looking for properties of the group Π which determine $d(\Pi)$. The following result was very surprising for me. The proof uses highly nontrivial results from the theory of finite groups and can be viewed as an analogue of the structure theorem given above for finite groups with cyclic 2-Sylow subgroups.

Theorem 4.4.8. *Let Π be a finite group with quaternion 2-Sylow subgroup Q and let K be the largest normal subgroup of Π of odd order.*

a) *The factor group $\Pi/_K$ is isomorphic to either*
   (i) *a 2-Sylow subgroup Q of Π,*
   (ii) *the nontrivial extension of the alternating group A_7 by C_2, or*
   (iii) *an extension*
   $$1 \to SL_2(q) \to \Pi/_K \to C_m \to 1,$$
   where q is an odd prime power and $m \in \mathbb{N}$ is odd.

b) *$d(\Pi) = 2$ if and only if case(i) occurs in a), i.e. if*
$$\Pi \cong K \rtimes Q.$$

c) *$d(\Pi) = 1$ does not occur .*

Proof. a) is a consequence of a theorem of Brauer and Suzuki, see [Gorenstein, Chapter 12, Thm.1.1]. It states that the center of $\Pi/_K$ is of order 2. (Note that this already implies that Π cannot be simple!) It then follows that $\Pi/_K$ modulo its center has dihedral 2-Sylow subgroups. But such groups where classified by [Gorenstein, Chapter 16.3], compare also the remark in [Gorenstein, p.377].

b) Recall from Lemma 4.4.7 that

$$d(\Pi) = \dim_{\mathbb{Z}/2} H^2(\Pi;\mathbb{Z}/2) = \dim_{\mathbb{Z}/2} H^1(\Pi;\mathbb{Z}/2) = \dim_{\mathbb{Z}/2}(H_1(\Pi)\otimes\mathbb{Z}/2) =: a(\Pi).$$

Since $|K|$ is odd, we clearly have $a(\Pi) = a(\Pi/_K)$. In case(i) this directly implies that $a(\Pi) = a(Q) = 2$.

In the remaining cases(ii) and (iii), one easily checks that $a(\Pi/_K) = 0$ because the nontrivial extension of A_7 by C_2 (which is well-defined since $H^2(A_7;C_2) \cong \mathbb{Z}/2$) as well as $SL_2(q), q \geq 4$ are perfect groups and $|H_1(SL_2(3))| = 3$. This also proves part c). ■

In the following classification theorem we will not use this structure theorem except that we do not state the empty case $d(\Pi) = 1$.

Theorem 4.4.9. *Let Π be a finite group with quaternion 2-Sylow subgroups. Then $d(\Pi) \in \{0,2\}$ and if $\xi = \xi(\Pi, w) : B \longrightarrow BSO$ is a 1-universal fibration, there exist $\mathbb{Z}/2$-valued bordism invariants $\mathfrak{sec}$ (in the spin case) respectively $\mathfrak{ter}$ (in the non-spin case) which together with the signature and the π_1-fundamental class induce the following group isomorphisms: (if $d(\Pi) = 0$ only the first two cases occur !)*

$$
\begin{array}{lccl}
\Omega_4(\xi) & \xrightarrow{\cong} & \mathbb{Z} \times H_4(\Pi) & \text{if } w = \infty, \\
\Omega_4(\xi) & \xrightarrow{\cong} & 16 \cdot \mathbb{Z} \times H_4(\Pi) \times \mathbb{Z}/2 & \text{if } w = 0, \\
\Omega_4(\xi) & \xrightarrow{\cong} & 8 \cdot \mathbb{Z} \times H_4(\Pi) & \text{if } i^*w = x_2^2, x_1^2 + x_2^2 \text{ or } i^*w = x_1^2, |\Pi|_{(2)} = 8, \\
\Omega_4(\xi) & \xrightarrow{\cong} & 8 \cdot \mathbb{Z} \times H_4(\Pi) \times \mathbb{Z}/2 & \text{if } i^*w = x_1^2 \text{ and } |\Pi|_{(2)} > 8.
\end{array}
$$

Proof. As in the cyclic case we compare the James spectral sequences for $\Omega_*(\xi(\Pi, w))$ and $\Omega_*(\xi(Q, i^*w))$ to obtain an exact sequence

$$0 \longrightarrow \Omega_4(\xi(Q, i^*w)) \longrightarrow \Omega_4(\xi(\Pi, w)) \xrightarrow{u_*} H_4(\Pi) \longrightarrow 0.$$

By Theorems 4.2.2 and 4.3.4, to prove the theorem we only have to show that this sequence splits. Clearly the signature splits off a free direct summand and we are done in the cases where the invariants $\mathfrak{sec}$ respectively $\mathfrak{ter}$ do not occur . In the other cases, the kernel of the signature is an extension of $H_4(\Pi)$ by $\mathbb{Z}/2$. But since $H_4(\Pi)$ has odd order, this extension splits, too. ∎

Corollary 4.4.10. *Let Π be a finite group with quaternion 2-Sylow subgroups.*

1. *For 4-dimensional spin manifolds with fundamental group Π there exists a $\mathbb{Z}/2$-valued stable diffeomorphism invariant $\mathfrak{sec}$ which together with the signature and the π_1-fundamental class induces the following 1-1 correspondences: (if $d(\Pi) = 0$ only the first two cases occur !)*

$$\begin{array}{lcll} \mathrm{MSt}_4(\Pi, \infty) & \longleftrightarrow & \mathbb{Z} \times H_4(\Pi)/_{\mathrm{Out}(\Pi)} & \\ \mathrm{MSt}_4(\Pi, 0) & \longleftrightarrow & 16 \cdot \mathbb{Z} \times H_4(\Pi)/_{\mathrm{Out}(\Pi)} \times \mathbb{Z}/2 & \\ \mathrm{MSt}_4(\Pi, w) & \longleftrightarrow & 8 \cdot \mathbb{Z} \times H_4(\Pi)/_{\mathrm{Out}(\Pi)} & \text{if } i^*w = x_2^2, x_1^2 + x_2^2 \\ & & & \text{or } i^*w = x_1^2, |\Pi|_{(2)} = 8 \end{array}$$

2. *Let $d(\Pi) = 2, |\Pi|_{(2)} > 8$ and $i^*w = x_1^2$.*
For a 4-dimensional manifold with w_2-type (Π, w) there exists a $\mathbb{Z}/2$-valued stable diffeomorphism invariant $\mathfrak{ter}$ such that the mapping
$$(\sigma, u_*, \mathfrak{ter}) : \mathrm{MSt}_4(\Pi, w) \longrightarrow 8 \cdot \mathbb{Z} \times H_4(\Pi)/_{\mathrm{Out}(\Pi)} \times \mathbb{Z}/2$$
is injective. This mapping is onto if every automorphism of Π fixes the element
$$(i^*)^{-1}x_2 \in H^1(\Pi; \mathbb{Z}/2).$$
If not, the image is exactly
$$\{(8 \cdot s, u, t) \in 8 \cdot \mathbb{Z} \times H_4(\Pi)/_{\mathrm{Out}(\Pi)} \times \mathbb{Z}/2 \mid s \cdot t \equiv 0 \quad (2)\}.$$

Remarks:

(i) One can get a more compact form of the classification theorem by setting the invariants $\mathfrak{sec}$ and $\mathfrak{ter}$ equal to a fixed value in the cases where they are not really needed. Then we have proved that the mapping
$$(w_2, \sigma, u_*, \mathfrak{sec}, \mathfrak{ter}) : \mathrm{MSt}_4(\Pi) \longrightarrow \mathcal{H}(\Pi) \times \mathbb{Z} \times \mathcal{H}_{\triangle}(\Pi)/_{\mathrm{Out}(\Pi)} \times \mathbb{Z}/\in \times \mathbb{Z}/\in$$
is injective. (Recall that for $|\Pi|_{(2)} = 8$ we can leave out $\mathfrak{ter}$.)

(ii) This classification result will be much more useful once we have identified the invariants $\mathfrak{sec}$ and $\mathfrak{ter}$ with something better computable. This will be done in Section 9.2.

(iii) To get a complete classification for the given fundamental group, we still have to compute the set $\mathcal{H}(\Pi)$. If $d(\Pi) = 0$ then $\mathcal{H}(\Pi) = \{\infty, \prime\}$, so let us assume $d(\Pi) = 2$ in the following discussion.
Recall that $|\mathcal{H}(\mathcal{Q}_\forall)| = \ni$ whereas $|\mathcal{H}(\mathcal{Q}_{\forall\backslash})| = \triangle$ for $n > 1$ because in the latter case the element $x_1 \in H^1(Q_{8n}; \mathbb{Z}/2)$ is preserved under all automorphisms of Q_{8n}. The reason for this is that $\mathrm{Ker}(x_1)$ is the unique cyclic subgroup of index 2 in Q_{8n} whereas Q_8 has 3 such subgroups.
To control the action on $H^2(\Pi; \mathbb{Z}/2)$ for a given $\alpha \in \mathrm{Aut}(\Pi)$, we can assume that α preserves a 2-Sylow subgroup because inner automorphism act trivially on cohomology. It follows that for $|\Pi|_{(2)} > 8$ the element $(i^*)^{-1}x_1$ is preserved and thus $|\mathcal{H}(\Pi)| \geq \triangle$. Note however that $|\mathcal{H}(\Pi)| = \bigtriangledown$ is very well possible because the automorphism of Q_{8n} which interchanges x_2 and $x_1 + x_2$ must not extend to Π. This is for example the case for the semidirect product
$$\Pi := C_m \rtimes Q_{8n}$$
where m is odd and Q_{8n} acts via the map $x_2 : Q_{8n} \to \mathbb{Z}/2 \leq \mathrm{Aut}(C_m)$ on C_m.
Similarly, for $|\Pi|_{(2)} = 8$ it is easy to write down semidirect products which show that $|\mathcal{H}(\Pi)|$ can take any value between 3 and 5.

(iv) The condition in part(2) of the above corollary, namely that every automorphism of Π fixes the element
$$(i^*)^{-1}x_2 \in H^1(\Pi;\mathbb{Z}/2),$$
is equivalent to $|\mathcal{H}(\Pi)| = \bigtriangledown$ by remark(iii). For such groups the stable classification is more complicated in two aspects: There are more w_2-types and also the $\mathfrak{ter}$-invariant takes more distinct values.

Proof of Corollary 4.4.10. If the groups $\Omega_4(\xi(\Pi,w))$ in Theorem 4.4.9 do not contain 2-torsion, the proof is the same as in the cyclic case. Therefore, we have to consider only two cases:

<u>1.Case:</u> $w=0$ (and $\xi=\xi(\Pi,0)$).
We have to show that every element of $\mathrm{Aut}(\xi)$ respects the decomposition
$$\Omega_4(\xi) \cong 16\cdot\mathbb{Z}\times H_4(\Pi)\times\mathbb{Z}/2.$$
Clearly, the 2-primary torsion $\mathbb{Z}/2$ as well as the odd-primary torsion $H_4(\Pi)$ are preserved. Moreover, the same argument as in the cyclic case shows that that the free part is preserved, too. Note that we have used there that in the James filtration the term $F_{2,2}$ is free, detected by the signature. This will become wrong in the next case.

<u>2.Case:</u> $d(\Pi)=2, |\Pi|_{(2)}>8$ and $i^*w=x_1^2$ (and $\xi=\xi(\Pi,w)$).
Theorem 4.4.9 gives a decomposition
$$\Omega_4(\xi)\cong 8\cdot\mathbb{Z}\times\mathbb{Z}/2\times H_4(\Pi).$$
Again, the 2-primary respectively the odd-primary torsion parts are preserved and we have to compute the orbit of the element $(8,0,0)$ under $\mathrm{Aut}(\xi)$. The James filtration looks in this case is
$$\Omega_4^{Spin}\subseteq F_{2,2}\subseteq\Omega_4(\xi)$$
with $(8,0,0)\in F_{2,2}$ and $\Omega_4(\xi)/F_{2,2}\cong H_4(\Pi)$. Since by Corollary 3.1.2 every $\varphi\in\mathrm{Aut}(\xi)$ respects this filtration (and also the signature), we can conclude that
$$\varphi_*(8,0,0)=(8,?,0).$$
The question mark can be determined as follows: We have a commutative diagram
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_4^{Spin} & \longrightarrow & F_{2,2} & \xrightarrow{p} & H_2(\Pi;\mathbb{Z}/2) & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle\varphi_*} & & \downarrow{\scriptstyle\pi_1(\varphi)_*} & & \\
0 & \longrightarrow & \Omega_4^{Spin} & \longrightarrow & F_{2,2} & \xrightarrow{p} & H_2(\Pi;\mathbb{Z}/2) & \longrightarrow & 0.
\end{array}$$
If $\{a_1,a_2\}$ is a basis of $H_2(\Pi;\mathbb{Z}/2)$ which is dual to the basis $\{(i^*)^{-1}x_1^2,(i^*)^{-1}x_2^2\}$ of $H^2(\Pi;\mathbb{Z}/2)$, we have seen in Remark(iii) above that $(i^*)^{-1}x_1^2$ and thus a_2 is preserved under $\pi_1(\varphi)_*$.
Identifying elements of $F_{2,2}$ with tupels in $8\cdot\mathbb{Z}\times\mathbb{Z}/2$, we know that
$$\varphi_*(0,1)=(0,1)\quad\text{and}\quad p(0,1)=a_2.$$
The surjectivity of p then shows that $p(1,0)=a_1$ or a_1+a_2, but in both cases we have
$$\begin{aligned}
\varphi_*(8,0)=(8,0) &\Longleftrightarrow \pi_1(\varphi)_*a_1=a_1\\
&\Longleftrightarrow \pi_1(\varphi)^*((i^*)^{-1}x_2^2)=(i^*)^{-1}x_2^2\\
&\Longleftrightarrow \pi_1(\varphi)^*((i^*)^{-1}x_2)=(i^*)^{-1}x_2
\end{aligned}$$

and otherwise $\varphi_*(8,0) = (8,1)$. Therefore, φ respects the direct decomposition of $\Omega_4(\xi)$ if and only if $\pi_1(\varphi)$ fixes $(i^*)^{-1}x_2$. Since by Theorem 2.2.6 the map

$$\pi_1 : \mathrm{Aut}(\xi) \longrightarrow \mathrm{Aut}(\Pi)_w = \mathrm{Aut}(\Pi)$$

is surjective, we proved the assertion using the following remark: if $\varphi_*(8,0) = (8,1)$ for some $\varphi \in \mathrm{Aut}(\xi)$ then the $\mathfrak{ter}$-invariant is determined by the signature if and only if $\sigma \equiv 8 \ (16)$. ■

5. Topological 4-Manifolds

In this Section we want to move from the category of differentiable manifolds to the category of topological manifolds. The changes which are necessary to achieve a stable homeomorphism classification of 4-dimensional topological manifold are described, with the result being that the difference is measured by the Kirby-Siebenmann invariant only. Further, in Section 8.2 topological surgery in dimension 4 is used to show that the $\mathfrak{ter}$-invariant from Section 7.3 can be changed without changing the homotopy type of the manifold. This will, in particular, produce two oriented differentiable manifolds which are homotopy equivalent but not stably diffeomorphic, see Example 5.2.4.

5.1. Necessary Modifications in the Stable Classification Program.

In this Section we will consider the stable homeomorphism classification of closed oriented topological 4-manifolds. Such a manifold has a stable normal Gauß map $\nu : M \longrightarrow BTOP$, where

$$TOP = \bigcup_{n\geq 0} TOP(n)$$

and $TOP(n)$ is the topological group of all base point preserving self-homeomorphisms of $\mathbb{R}^{\ltimes}$. Then $BTOP$ is the classifying space of stable fiber bundles with fiber $\mathbb{R}^{\ltimes}$ and specified zero-section. There are the obvious inclusion maps $O(n) \longrightarrow TOP(n)$ which induce a fibration $BO \longrightarrow BTOP$ with fiber TOP/O.
Similarly, if BPL is the classifying space for stable piecewise linear bundles then there is a fibration $BPL \longrightarrow BTOP$ with fiber TOP/PL. The fundamental result of [Kirby-Siebenmann] says that this fibration is a principal fibration, induced by an H-map

$$\mathfrak{ks} : BTOP \longrightarrow K(\mathbb{Z}/2, 4)$$

which the authors call the triangulation obstruction and which is today known as the *Kirby-Siebenmann invariant*. Up to dimension 6, the spaces BPL and BO are equal and thus the Kirby-Siebenmann invariant gives in the 4-dimensional case a unique $\mathbb{Z}/2$-valued obstruction for the existence of a lift of the topological stable normal Gauß map $M^4 \longrightarrow BTOP$ over BO. It was not until the striking results of M.Freedman that one could prove the nontriviality of this obstruction. Freedman showed (see [Freedman] or the book of [Freedman-Quinn, Ch.11.3]) that the topological surgery sequence

$$L_5^s(\pi_1 X, w_1 X) \to \mathcal{S}^{\mathcal{TOP}}_{\triangle}(\mathcal{X},\mathcal{N}) \to \mathcal{N}^{\mathcal{TOP}}_{\triangle}(\mathcal{X},\mathcal{N}) \to \mathcal{L}^{\int}_{\triangle}(\pi_\infty \mathcal{X}, \sqsupseteq_\infty \mathcal{X}) \tag{II.6}$$

is exact for any 4-dimensional Poincaré pair (X, N) where N^3 is a manifold and $\pi_1 X$ is a *good* group (e.g. finite or cyclic). Now if N is a homology 3-sphere then the cone over N gives a Poincaré pair $(C(N), N)$. But since one already knew that

$$L_5^s(1) = \{0\} \quad \text{and} \quad \mathcal{N}^{\mathcal{TOP}}_{\triangle}(\mathcal{C}(\mathcal{N}),\mathcal{N}) \xrightarrow{\cong} \mathcal{L}^{\int}_{\triangle}(\infty) \quad (\cong \mathbb{Z}),$$

Freedman concluded that there is a unique contractible 4-manifold whose boundary is N. By taking N to be the boundary of the plumbing construction on 8 tangent disc bundles of S^2 with respect to the graph E_8, and closing this off with a contractible manifold as above, he obtained a topological 4-manifold with even definite intersection form of rank 8. Since for spin manifolds [Kirby-Siebenmann, p.325,Thm.13.1] had proven the formula

$$\mathfrak{ks}(M) \equiv \frac{\sigma(M)}{8} \pmod 2, \tag{II.7}$$

Freedman had obtained a manifold with nontrivial $\mathfrak{ks}$-invariant which he called $|E_8|$. Note that from the surgery sequence (II.6), the topological s-cobordism theorem and Milnor's homotopy classification of simply-connected Poincaré complexes (compare Section 1.6), Freedman's classification of simply-connected 4-manifolds (see [Freedman-Quinn, Ch.10.1]) directly follows from an observation of [Wall 1, Ch.16] about the realization of the torsion classes in

$$\mathcal{N}^{\mathcal{TOP}}_{\triangle}(\mathcal{M}) \cong [\mathcal{M}, \mathcal{G}/\mathcal{TOP}] \cong \mathcal{H}^{\in}(\mathcal{M};\mathbb{Z}/\in) \times \mathcal{H}^{\triangle}(\mathcal{M};\mathbb{Z})$$

by homotopy self-equivalences of the manifold M (there is a corrected version of Wall's construction in [Cochran-Habegger]).

We now want to turn to the computation of topological bordism groups. Using the manifold $|E_8|$, it follows directly from the work of [Kirby-Siebenmann, p.322-325] that for $G = O, SO$ or $Spin$ the natural maps

$$\Omega_i^G \longrightarrow \Omega_i^{GTOP}$$

are isomorphisms for $i \leq 3$ and injective with cokernel $\mathbb{Z}/2$ for $i = 4$. Moreover, the signature divided by 8 gives an isomorphism of $\Omega_4^{SpinTOP}$ onto $\mathbb{Z}$, whereas $\Omega_4^{STOP} \cong \mathbb{Z} \times \mathbb{Z}/2$ via $(\sigma, \mathfrak{ks})$. Our aim is to extend these results to the bordism groups $\Omega_4^{TOP}(\xi)$ for a 1-universal fibration $\xi : B \longrightarrow BTOP$. The constructions in Section 5 can be repeated word by word to obtain all such fibrations and their fiber homotopy classification, just by replacing BO by $BTOP$ at every step. The reason why this works well follows from the fact that the natural map $BO \longrightarrow BTOP$ is a 3-equivalence. In particular, we obtain the linear 1-universal fibrations by pulling back the topological ones via this map.

Theorem 5.1.1. *Let $\xi' : B' \longrightarrow BSTOP$ be an oriented topological normal 1-type and let $\xi : B \longrightarrow BSO$ be the corresponding differentiable normal 1-type. Then the pullback diagram*

$$\begin{tikzcd} B \arrow[r] \arrow[d, "\xi"] & B' \arrow[d, "\xi'"] \\ BSO \arrow[r] & BSTOP \end{tikzcd}$$

induces an exact sequence

$$0 \longrightarrow \Omega_4(\xi) \longrightarrow \Omega_4^{TOP}(\xi') \xrightarrow{\mathfrak{ks}} \mathbb{Z}/2 \longrightarrow 0 \tag{II.8}$$

which splits if and only if $w_2(\xi) \neq 0$.

Proof. The exactness of the sequence follows directly from the above information about Ω_i^{STOP} respectively $\Omega_i^{SpinTOP}, i \leq 4$, by comparing the James spectral sequences of the fibrations ξ and ξ'. Note that it is easy to generalize the James spectral sequence from vector bundles to fiber bundles with fiber $\mathbb{R}^{\Bbbk}$ and zero-section. We only have to replace the relative fibration (disc bundle, sphere bundle) by the relative fibration (total space, total space $\setminus$ zero-section) to which we can also apply the relative Serre spectral sequence.
However, we also have to use topological transversality in dimension 4 (see [Scharlemann] or [Freedman-Quinn, Ch.9]), to be able to use the Pontrjagin-Thom isomorphism

$$\Omega_4^{TOP}(\xi') \cong \pi_4(M\xi').$$

It is clear that a splitting of the exact sequence(II.8) is the same as the choice of an element of order 2 in $\Omega_4^{TOP}(\xi')$ with nontrivial $\mathfrak{ks}$-invariant. If $\pi_2 B \neq 0$ then $\Omega_4^{STOP} \subseteq \Omega_4^{TOP}(\xi')$ and the sequence splits because Ω_4^{STOP} contains such an element: one can take e.g. $[|E_8|] - 8 \cdot [\mathbb{CP}^{\nvDash}]$.
If $w_2(\xi) = 0$ then the relation(II.7) holds in $\Omega_4^{TOP}(\xi')$. This implies that a ξ'-manifold with nontrivial $\mathfrak{ks}$-invariant has nontrivial signature and thus cannot have finite order. In particular, the exact sequence(II.8) does not split.

Finally, let $\pi_2 B = 0$ and $w_2(\xi) \neq 0$. The James spectral sequence for ξ and ξ' gives a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \Omega_4^{Spin} \arrow[r] \arrow[d, "\cdot 2"] & F_{2,2}(\xi) \arrow[r, "p"] \arrow[d] & E^\infty_{2,2}(\xi) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega_4^{SpinTOP} \arrow[r] \arrow[d, "\mathfrak{ks}"] & F_{2,2}(\xi') \arrow[r, "p^{TOP}"] \arrow[d, "\mathfrak{ks}"] & E^\infty_{2,2}(\xi') \arrow[r] & 0 \\
 & \mathbb{Z}/2 \arrow[r, equal] & \mathbb{Z}/2. & &
\end{tikzcd}$$

Since $E^\infty_{2,2} = H_2(\pi_1 B; \mathbb{Z}/2)/\operatorname{Image}(d_i)$ is 2-torsion and the first vertical map is multiplication by 2, the middle horizontal sequence splits out of the obvious algebraic reason. This implies that the image of the signature on $F_{2,2}(\xi')$ is $8 \cdot \mathbb{Z}$ and therefore the map

$$(\frac{\sigma}{8}, p^{TOP}) : F_{2,2}(\xi') \longrightarrow \mathbb{Z} \times E^\infty_{2,2}$$

is a well-defined isomorphism. Recall from Theorem 3.3.2(3) that there exists a (differentiable) ξ-manifold $[M] \in F_{2,2}(\xi)$ with signature 8. It follows directly from the above isomorphism that

$$[M] - [|E_8|] \in F_{2,2}(\xi') \subseteq \Omega_4^{TOP}(\xi')$$

is an element of order two which clearly has nontrivial $\mathfrak{ks}$-invariant. ■

Corollary 5.1.2. *If $w_2(\xi) \neq 0$, the image of the signature on $\Omega_4(\xi)$ equals the image of the signature on $\Omega_4^{TOP}(\xi')$.*

It is now clear how to compute the groups $\Omega_4^{TOP}(\xi')$, assuming that $\Omega_4(\xi)$ is known. In the nonspin case there is the additional $\mathbb{Z}/2$-valued $\mathfrak{ks}$-invariant whereas in the spin case just the image of the signature changes from $16 \cdot \mathbb{Z}$ to $8 \cdot \mathbb{Z}$.
There also is a topological version of the stable classification result of M.Kreck (see the Main Theorem in Section 4), which states that the natural map

$$\mathrm{NSt}_4^{TOP}(\xi) \longrightarrow \Omega_4^{TOP}(\xi)$$

is a 1-1 correspondence under the same assumptions as in the differentiable case. If we use this result in both categories $DIFF$ and TOP, we obtain a corollary which shows that with respect to stable questions, these two categories only differ by the $\mathfrak{ks}$-invariant.

Corollary 5.1.3 (compare [Kreck 1, Thm.4.1,4.12])**.**

a) *If M is a closed orientable 4-manifold with trivial Kirby-Siebenmann invariant then there is a natural number r such that $M \# r \cdot (S^2 \times S^2)$ has a differentiable structure.*
b) *If M and N are two closed oriented differentiable 4-manifolds which are stably homeomorphic then they are stably diffeomorphic.*

5.2. **Extensions of the $*$-Operation.**

In this Section we want to prove that for the fundamental groups under consideration, i.e. finite groups with periodic 2-Sylow subgroups, the invariants $\mathfrak{ks}$ and $\mathfrak{ter}$ from Theorems 4.4.4 respectively 4.4.9 can be varied without changing the homotopy type of the manifold. The most famous example of such a construction is the $*$-operator of [Freedman-Quinn, Ch.10.4]. They prove that for a 4-manifold M with good fundamental group and $w_2\widetilde{M} \neq 0$ (i.e. w_2-type ∞ in our notation), there always exists a manifold $*M$ with the properties:

$$*M \simeq M \quad \text{and} \quad \mathfrak{ks}(*M) \neq \mathfrak{ks}(M).$$

Without taking uniqueness into account, we will call any manifold with the above two properties a representative of $*M$. Note that this convention differs from the original definition of the $*$-operation in [Freedman-Quinn] who set $*M := M$ if $w_2\widetilde{M} = 0$.

Remark 5.2.1. Although the normal 1-types of M and $*M$ are the same (because they are determined by $\pi_1 M, w_2 M, w_2\widetilde{M}$ which are all homotopy invariants), the homotopy equivalence $*M \simeq M$ cannot be a normal map with respect to ν_M and ν_{*M}. This follows from the definition of $\mathfrak{ks}(M) \in H^4(M;\mathbb{Z}/2) \cong \mathbb{Z}/2$ as the composition

$$M \xrightarrow{\nu_M} BTOP \xrightarrow{\mathfrak{ks}} K(\mathbb{Z}/2, 4)$$

which shows that any odd degree map $f : X \longrightarrow M$ with $\nu_X = \nu_M \circ f$ preserves the $\mathfrak{ks}$-invariant. Recall that for a normal map $f : X \longrightarrow M$ by definition the bundle over M can be any vector bundle (respectively TOP-bundle) reduction of the Spivak normal bundle. This is the reason why we are careful about specifying the bundle in question.

It is clear that $*M$ cannot exist if M is spin, because the relation(II.7) shows that the $\mathfrak{ks}$-invariant is a homotopy invariant for spin manifolds. This leaves the question about the existence of $*M$ for all w_2-types $\neq 0, \infty$.

Proposition 5.2.2. *Let M be a closed manifold with finite fundamental group whose 2-Sylow subgroups are periodic. If M is not spin then $*M$ exists.*

Proof. Since for the fundamental group $\pi_1 M$ in question the surgery obstruction for closed 4-manifold problems is detected by the signature (see [Hambleton et al.]), the surgery sequence(II.6) shows that it is enough to construct a degree 1 normal map $X \longrightarrow M$ with

$$\mathfrak{ks}(X) \neq \mathfrak{ks}(M) \quad \text{and} \quad \sigma(X) = \sigma(M).$$

To this end we have to study the set of degree 1 normal maps

$$\mathcal{N}_{\triangle}^{\mathcal{TOP}}(\mathcal{M}) \cong [\mathcal{M}, \mathcal{G}/\mathcal{TOP}].$$

By [Kirby-Siebenmann, p.329], the 5-skeleton of G/TOP equals the 5-skeleton of $K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}, 4)$ and therefore

$$[M, G/TOP] \cong H^2(M;\mathbb{Z}/2) \times H^4(M).$$

Moreover, in the same reference it is proven that the difference of the $\mathfrak{ks}$-invariants, as a map from $H^2(M;\mathbb{Z}/2) \times H^4(M)$ to $\mathbb{Z}/2$ is given by the correspondence

$$(x, y) \longmapsto \langle x^2 + r_2(y), [M] \rangle.$$

Finally, by [Madson-Milgram, pp.97,98], a nontrivial multiple of the difference of the signatures is given by the projection onto the second factor of $H^2(M;\mathbb{Z}/2) \times H^4(M)$.

We finish the proof by observing that $w_2(M) \neq 0$ implies the existence of an element $x \in H^2(M;\mathbb{Z}/2)$ with $x^2 \neq 0$. Thus under the above isomorphisms we can take as the degree 1 normal map the element $(x, 0) \in H^2(M;\mathbb{Z}/2) \times H^4(M)$. It obviously satisfies the desired properties. ∎

If M has normal 1-type ξ and the tertiary bordism invariant $\mathfrak{ter}$ is trivial on $\Omega_4^{TOP}(\xi)$ then $*M$ represents a well-defined element in this bordism group because all other bordism invariants are in fact homotopy invariants (for the $\mathfrak{sec}$-invariant compare Theorem 6.4.1). But if the $\mathfrak{ter}$-invariant is nontrivial then we did not control $\mathfrak{ter}(*M)$ in Proposition 5.2.2. Therefore, we have to be more careful in the next construction.

Proposition 5.2.3. *Let $\xi : B \longrightarrow BSTOP$ be 1-universal with $\Pi := \pi_1 B$ finite with periodic 2-Sylow subgroups such that in (the topological analogue of) Theorem 4.4.9 the following map ϕ is an isomorphism*

$$\phi : (\sigma, u_*, \mathfrak{ter}, \mathfrak{ks}) : \Omega_4^{TOP}(\xi) \longrightarrow 8 \cdot \mathbb{Z} \times H_4(\Pi) \times \mathbb{Z}/2 \times \mathbb{Z}/2.$$

Given a pair $(t, k) \in \mathbb{Z}/2 \times \mathbb{Z}/2$ and a normal 1-smoothing (M, μ) in ξ which is zero-bordant in $\Omega_4^{TOP}(\xi)$, there exists a normal 1-smoothing $(\bullet M, \bullet\mu)$ in ξ such that

$$\bullet M \simeq M \quad \text{and} \quad \phi(\bullet M, \bullet\mu) = (0, 0, t, k).$$

Proof. We will first construct a manifold $\bullet M$ for the pair $(t, k) = (1, 0)$.

The ξ-structure $M \longrightarrow B$ induces a homomorphism

$$\mu_* : \Omega_4(\nu) = \Omega_4(\xi \circ \mu) \longrightarrow \Omega_4(\xi)$$

where $\nu = \xi \circ \mu : M \longrightarrow BSTOP$ is the topological stable normal Gauß map for M. For the same reasons as in the proof of the preceding proposition, it is enough to construct a degree 1 normal map $f : X \longrightarrow M$ such that

$$\nu_X = \nu \circ f \quad \text{and} \quad \mu_*(M, f) = \phi^{-1}(0, 0, 1, 0).$$

This time we will work entirely inside the subset of $\mathcal{N}_{\triangle}^{\mathcal{TOP}}(\mathcal{M})$, consisting of the affine subspace of degree 1 normal maps in $\Omega_4^{TOP}(\nu)$ because we have to control also the normal 1-smoothings of the manifolds in question. We start with the most obvious element of degree 1, namely $(M, \mathrm{id}_M) \in \Omega_4^{TOP}(\nu)$ (mapping to 0 under μ_*), and we want to add an element $T \in \Omega_4^{TOP}(\nu)$ such that

$$\deg(T) = 0 \quad \text{and} \quad \mu_*(T) = \phi^{-1}(0, 1, 0, 0).$$

Then the sum $(M, \mathrm{id}_M) + T$ will give the desired bordism class of (X, f) above. To prove the existence of an element T with the specified properties, we compare the Atiyah-Hirzebruch spectral sequences for

$$\Omega_*^{TOP}(\nu) \quad \text{and} \quad \Omega_*^{TOP}(\xi)$$

with E^2-terms $H_*(M\nu; \pi_*(S^0))$ respectively $H_*(M\xi; \pi_*(S^0))$. We obtain the following maps on the induced filtrations:

$$\begin{array}{ccccccc} E_{1,3}^\infty(\nu) = F_{1,3}(\nu) & \subseteq & F_{2,2}(\nu) & \subseteq & F_{3,1}(\nu) & \subseteq & \pi_4(M\nu) \cong \Omega_4^{TOP}(\nu) \\ \downarrow & & \downarrow \mu_{2,2} & & \downarrow & & \downarrow \mu_* \\ E_{1,3}^\infty(\xi) = F_{1,3}(\xi) & \subseteq & F_{2,2}(\xi) & \subseteq & F_{3,1}(\xi) & \subseteq & \pi_4(M\xi) \cong \Omega_4^{TOP}(\xi). \end{array}$$

First note that all elements in $F_{3,1}(\nu)$ have degree 0 because

$$E_{4,0}^\infty(\nu) = H_4(M\nu) \cong H_4(X) \cong \mathbb{Z}$$

and the edge-homomorphism is just the image of the fundamental class (compare Proposition 3.2.1). Secondly, by Theorem 4.3.4 we know that

$$E_{3,1}^3(\xi) = H_3(M\xi; \mathbb{Z}/2) / \operatorname{Image}(d_2) = 0$$

and therefore $\phi^{-1}(0, 0, 1, 0) \in F_{3,1}(\xi) = F_{2,2}(\xi)$. It thus suffices to show that $\mu_{2,2}$ is an epimorphism which in turn is guaranteed if the maps

$$\mu_{i,4-i}^\infty : E_{i,4-i}^\infty(\nu) \longrightarrow E_{i,4-i}^\infty(\xi)$$

are surjective for $i = 1, 2$.
We first recall that since $\mu : M \longrightarrow B$ is a normal 1-smoothing, it is a 2-equivalence and thus by the Thom isomorphism

$$\mu_* : H_i(M\nu; \mathbb{Z}/2) \longrightarrow H_i(M\xi; \mathbb{Z}/2)$$

is an isomorphism for $i = 1$ and an epimorphism for $i = 2$. The commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Image}(d_i(\nu)) \arrow[r] \arrow[d] & H_1(M\nu; \mathbb{Z}/2) \arrow[r] \arrow[d, "\cong"] & E^\infty_{1,3}(\nu) \arrow[r] \arrow[d, "\mu^\infty_{3,1}"] & 0 \\
0 \arrow[r] & \mathrm{Image}(d_i(\xi)) \arrow[r] & H_1(M\xi; \mathbb{Z}/2) \arrow[r] & E^\infty_{1,3}(\xi) \arrow[r] & 0
\end{tikzcd}$$

then proves the surjectivity of $\mu^\infty_{1,3}$.
For $\mu^\infty_{2,2}$, remember that $H_5(M\nu) \cong H_5(M) = 0$ and thus $E^\infty_{2,2}(\nu) = E^3_{2,2}(\nu)$. Moreover, the differential

$$d_2(\nu) : E^2_{4,1}(\nu) \longrightarrow E^2_{2,2}(\nu)$$

is trivial because it is dual to $Sq^2_{M\nu}$ and we have for all $a \in H^2(M; \mathbb{Z}/2)$ the relation

$$\begin{aligned}
Sq^2(x \cdot u) &= Sq^2(x) \cdot u + Sq^1(x) \cdot Sq^1(u) + x \cdot Sq^2(u) \\
&= x^2 \cdot u + x \cdot w_2(u) \\
&= x^2 \cdot u + x \cdot w_2(\nu) \cdot u \\
&= x^2 \cdot u + x^2 \cdot u = 0
\end{aligned}$$

where $u \in H^0(M\nu; \mathbb{Z}/2)$ is the Thom class of ν.
Thus there is a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & E^\infty_{2,2}(\nu) \arrow[r] \arrow[d] & H_2(M\nu; \mathbb{Z}/2) \arrow[r, "d_2(\nu)"] \arrow[d, "\mu_*"] & \mathbb{Z}/24 \arrow[d, equal] \\
& E^3_{2,2}(\xi) \arrow[r] & H_2(M\xi; \mathbb{Z}/2) \arrow[r, "d_2(\xi)"] & \mathbb{Z}/24
\end{tikzcd}$$

which proves the surjectivity of $\mu^\infty_{2,2}$ because $E^\infty_{2,2}(\xi) = E^3_{2,2}(\xi)/\,\mathrm{Image}(d_3(\xi))$ and the middle vertical map μ_* is an epimorphism.
To finish the proof of the proposition, the changes that are necessary in order to realize an arbitrary pair $(t, k) \in \mathbb{Z}/2 \times \mathbb{Z}/2$ instead of the special pair $(1, 0)$ have to be outlined. To get a nontrivial $\mathfrak{ks}$-invariant, we start with a manifold $*M$ instead of M. Then for any normal 1-smoothing $*\mu$ of $*M$, we have

$$\phi(*M, *\mu) = (0, 0, ?, 1).$$

We want to show that we can vary $? = \mathfrak{ter}(*M)$ arbitrarily without changing the homotopy type and the $\mathfrak{ks}$-invariant. This can be done by the same strategy as above, namely by surgery on a degree 1 normal map $f : X \longrightarrow *M$ (with $\nu_X = \nu_{*M} \circ f$) obtained from the bordism class of

$$(*M, \nu_{*M}) + T \in \Omega^{TOP}_4(\nu_{*M}).$$

The result is a normal homotopy equivalence $\bullet M \simeq *M$ over ν_{*M} and thus by Remark 5.2.1 the $\mathfrak{ks}$-invariants of $\bullet M$ and $*M$ agree. ■

Example 5.2.4. If Π is a finite group with quaternion 2-Sylow subgroups then Theorem 4.3.4 together with the above proposition proves the existence of two oriented differentiable 4-manifolds with fundamental groups Π which are homotopy equivalent but not stably diffeomorphic. Remember that a 4-manifold with trivial $\mathfrak{ks}$-invariant has stably a differentiable structure and that the $\mathfrak{ter}$-invariant is nontrivial on a suitable bordism group corresponding to this fundamental group. Note that we are comparing the zero element in the group $\Omega_4(\xi)$ with a nontrivial element, and thus the (linear) action of $\mathrm{Aut}(\xi)$ cannot move one into the other.

Had we only wanted to obtain the above example, we need not have switched to the topological category and use M.Freedman's deep result. The reason for this is that [Cappell-Shaneson] proved that in the differentiable setting the surgery sequence is exact in the stable category. But for the reasons described in Section 9.1, we will really need the unstable result of Proposition 5.2.3 once we try to classify up to homeomorphism.

Part 2. Homeomorphism Classification of 4-Dimensional Manifolds

6. Cancellation of $S^2 \times S^2$-Summands

This Section is centered around the question whether two stably homeomorphic 4-manifolds are homeomorphic. It is clear that the number of $S^2 \times S^2$-summands on both sides have to be equal and that this is controlled by the Euler characteristic χ. Thus our aim will be to prove results in the following direction:

$$\text{If } \chi(M) = \chi(N) \text{ and } M \underset{\text{st.}}{\approx} N \text{ then } M \approx N.$$

Together with the results of Part(II), we will obtain a homeomorphism classification for special fundamental groups under rather mild restrictions on the intersection form on $H_2(.;\mathbb{Z})$. This will be carried out in detail in Section 9.4 for the following 3 classes of finite fundamental groups.

(i) Groups with cyclic Sylow subgroups.
(ii) $SL_2(p), p$ a prime with $p \equiv 3, 5\ (8)$.
(iii) Groups with 4-periodic cohomology.

Before we start this Section, we remind the reader that without mentioning we always consider closed oriented topological 4-manifolds.

6.1. **The Cancellation Theorem.**

Theorem ([Hambleton-Kreck 3]). *Let M and N be two 4-manifolds with finite fundamental group. Suppose that the connected sum $M\#r\cdot(S^2\times S^2)$ is homeomorphic to $N\#r\cdot(S^2\times S^2)$. If $N\approx N_0\#(S^2\times S^2)$ then M is homeomorphic to N.*

A brief outline of the proof of the cancellation theorem can be given as follows: First note that if the homeomorphism

$$f: M\#r\cdot(S^2\times S^2)\longrightarrow N\#r\cdot(S^2\times S^2)$$

carries the hyperbolic form $H(\Lambda^r), \Lambda := \mathbb{Z}\pi_1 M$, coming from the $S^2\times S^2$-summands, identically from the left to the right hand side then one can check (or see [Kreck 1, Lem.6.1]) that the cobordism

$$(M\times I)\#r\cdot(S^2\times D^3)\underset{f}{\cup}(N\times I)\#r\cdot(D^3\times S^2)$$

is an s-cobordism between M and N. By the topological s-cobordism theorem it follows that M and N are homeomorphic since finite fundamental groups are *good* in the sense of [Freedman]. (At this point the corresponding proof breaks down in the differentiable category.)
We now face the problem that we do not have given the homeomorphism f explicitly and thus we cannot control what happens to the hyperbolic summands. We just know that f_* maps any pair of hyperbolic basis vectors $(e,f)\in H(\Lambda)$ to a *hyperbolic pair* inside $H_2(N;\Lambda)\oplus H(\Lambda^r)$. To be more precise, we should mention that we are talking about the quadratic forms on the universal coverings given by intersections and self-intersections (see Section 9.2) defined only on

$$K := \mathrm{Ker}(w_2 : H_2(N\#r\cdot(S^2\times S^2);\Lambda)\longrightarrow\mathbb{Z}/2).$$

Now I.Hambleton and M.Kreck improve a theorem of H.Bass who showed that a certain group of automorphisms of the quadratic form on K acts transitively on such hyperbolic pairs. The main problem they have to solve is that Bass assumes the existence of two additional hyperbolic summands (which would correspond to $N\approx N_0\#2\cdot(S^2\times S^2)$). Roughly, what they use is that $H_2(N_0;\Lambda)$ itself contains a form which is almost as good as a hyperbolic summand.
The last step in the proof of the cancellation theorem is to realize the necessary algebraic automorphisms of K by self-automorphisms of $N\#r\cdot(S^2\times S^2)$ and hereby changing f to a homeomorphism which maps the hyperbolic summands identically.
The existence of such self-homeomorphisms goes back to a beautiful idea of [Wall 2] and was carried out in [Cappell-Shaneson]. □
Let me now discuss, how a stable homeomorphism classification together with the cancellation theorem leads to a homeomorphism classification under mild restrictions.

Corollary 6.1.1. *Let Π be a finite group with $H_2(\Pi)_{(2)}=0$ and let M and N be 4-manifolds with fundamental group Π and indefinite intersection form on $H_2(.;\mathbb{Z})$. If M and N are stably homeomorphic, $\chi(M)=\chi(N)$ and if every element of signature 0 in*

$$\mathrm{MSt}_4^{TOP}(\Pi, w_2M)$$

is represented by a rational homology 4-sphere then M and N are homeomorphic.

Remark . By M.Freedman's classification of simply-connected 4-manifolds, the assumption on the indefiniteness of the intersection form is already necessary in the simply-connected case.

Proof. Recall from Theorem 3.3.2(4) that the assumption $H_2(\Pi)_{(2)} = 0$ implies that

$$\sigma(\Pi, w) = \begin{cases} 1 & \text{if } w = \infty, \\ 16 & \text{if } w = 0, \\ 8 & \text{if } w \neq \infty, 0. \end{cases}$$

It follows that if we define

$$Z := \begin{cases} \sigma(M) \cdot \mathbb{CP}^2 & \text{if } w_2 M = \infty \\ \frac{\sigma(M)}{8} \cdot |E_8| & \text{if } w_2 M \neq \infty \end{cases}$$

then Z is a simply-connected 4-manifold with definite intersection form and $M\#(-Z)$ has signature 0. By assumption, there exists a rational homology 4-sphere Σ which is stably homeomorphic to this connected sum. Now observe that M is stably homeomorphic to $M\#(-Z)\#Z$ because

$$\mathrm{MSt}_4(\Pi, w_2 M) \cong \Omega_4(\xi(\Pi, w_2 M))/_{\mathrm{Aut}(\xi)}$$

and $(-Z)$ is the inverse of Z in the bordism group. Maybe it is worth mentioning a different proof of this fact: In the case $w_2 M \neq \infty$ it follows from the classification of 1-connected 4-manifolds which implies that $(-Z)\#Z$ is homeomorphic to $\sigma(M) \cdot (S^2 \times S^2)$. If $w_2 M = \infty$, one can use a trick from [Wall 2, Rem.5.2] who shows that under this condition

$$M\#\mathbb{CP}^2\#(-\mathbb{CP}^2) \approx M\#(S^2 \times S^2).$$

Coming back to the main line of the proof, we obtain a homeomorphism

$$M\#r \cdot (S^2 \times S^2) \approx \Sigma\#Z\#s \cdot (S^2 \times S^2),$$

where $\Sigma\#Z$ has a definite intersection form. But since by assumption M has an indefinite intersection form, we may conclude that $s > r$. Thus the cancellation theorem implies that

$$M \approx \Sigma\#Z\#(s-r) \cdot (S^2 \times S^2).$$

Since exactly the same arguments apply for the manifold N, the conclusion follows. ■

Remark 6.1.2. If Π is an arbitrary group, there does not exist a rational homology 4-sphere in $\mathrm{MSt}_4(\Pi, \infty)$. This follows from the observation that the intersection form on $H_2(M; \mathbb{Z})$ is even if and only if $w_2 M$ vanishes on all integral homology classes. Thus if Σ is a rational homology 4-sphere and $x \in H_2(\widetilde{\Sigma}; \mathbb{Z})$ then

$$\langle w_2(\widetilde{\Sigma}), x \rangle = \langle p^*(w_2 \Sigma), x \rangle = \langle w_2 \Sigma, p_*(x) \rangle = 0$$

because the intersection form on $H_2(\Sigma; \mathbb{Z})$ is identically zero and in particular even. Since on $\widetilde{\Sigma}$ any 2-dimensional $\mathbb{Z}/2$-homology class is a reduction of an integer class, we conclude that $w_2(\widetilde{\Sigma}) = 0$.

This shows that Corollary 6.1.1 is never applicable for the w_2-type ∞. However, if there exists any rational homology 4-sphere Σ in $\mathrm{MSt}_4(\Pi)$ then the manifold

$$\Sigma\#\mathbb{CP}^2\#(-\mathbb{CP}^2)$$

has signature 0 and lies in $\mathrm{MSt}_4(\Pi, \infty)$. Thus the assertion of the corollary still holds if we assume that

$$\chi(M) > 4 \quad \text{if} \quad w_2 M = \infty, \sigma(M) = 0.$$

Note that we do not need this assumption if $w_2 M = \infty$ and $\sigma(M) \neq 0$ because in this case we added a $\mathbb{CP}^2$-summand to Σ anyway.
I hope that the reader is now strongly motivated to read a digression on the construction of

rational homology 4-spheres. But first I have to insert a chapter on the algebra of intersection forms.

6.2. **Sesquilinear and Metabolic Forms.**

We first recall the relevant definitions from [Bass, § 1-3]: Let Λ be a ring with involution, i.e. we have given an antiautomorphism $a \mapsto \bar{a}$ of order ≤ 2. Using this involution, one can view a right Λ-module M as a left Λ-module by setting

$$a \cdot m := m \cdot \bar{a} \quad \forall a \in \Lambda, m \in M$$

and vice versa. As a matter of education at the University of Mainz, I prefer to work with left modules. So let M be a left Λ-module. Then $\mathrm{Hom}_\Lambda(M, \Lambda)$ is naturally a right Λ-module via

$$(\varphi \cdot a)(m) := \varphi(m) \cdot a \quad \forall a \in \Lambda, m \in M, \varphi \in \mathrm{Hom}_\Lambda(M, \Lambda).$$

We shall write $\overline{M}$ for the corresponding left Λ-module.

Example 6.2.1. Let Π be a finite group and $\Lambda := \mathbb{Z}\Pi$ be the group ring with the involution

$$\overline{\sum_{g \in \Pi} a_g g} := \sum_{g \in \Pi} a_g w_1(g) g^{-1}$$

for a given homomorphism $w_1 : \Pi \longrightarrow \mathbb{Z}/2$. Let M be a left Λ-module and M^* be the abelian group $\mathrm{Hom}_\mathbb{Z}(M, \mathbb{Z})$ with the left Λ-action

$$(a \cdot u)(m) := u(\bar{a} \cdot m) \quad \forall a \in \Lambda, m \in M, u \in M^*.$$

From this definition it follows that $\mathrm{Hom}_\Lambda(M, \mathbb{Z}) = (M^*)^{(\Pi, w_1)}$, where for a Λ-module V we define (in analogy to the fixed-point set for $w_1 \equiv 0$)

$$V^{(\Pi, w_1)} := \{v \in V \mid g \cdot v = w_1(g) v \quad \forall g \in \Pi\}.$$

There is a natural isomorphism of left Λ-modules (compare [Brown, Ch.VI,Prop.3.4])

$$\psi : M^* \longrightarrow \overline{M}$$

given by

$$\psi(u)(m) := \sum_{g \in \Pi} u(g^{-1} \cdot m) g \quad \forall m \in M, u \in M^*.$$

If $\epsilon_1 : \Lambda \longrightarrow \mathbb{Z}$ is defined by $\epsilon_1(\sum_{g \in \Pi} a_g g) := a_1$ then the inverse of ψ is given by $\varphi \mapsto \epsilon_1 \circ \varphi$.

Definition 6.2.2. A *sesquilinear form* on a Λ-module M is a biadditive function

$$h : M \times M \longrightarrow \Lambda$$

satisfying $\quad h(a \cdot m, b \cdot n) = a \cdot h(m, n) \cdot \bar{b} \quad \forall a, b \in \Lambda, m, n \in M.$
The set of all such forms is denoted by $\mathrm{Sesq}(M)$. The *dual* of h is the sesquilinear form $\bar{h}$ defined by

$$\bar{h}(m, n) := \overline{h(n, m)} \quad \forall m, n \in M.$$

h is called *hermitian* if $h = \bar{h}$ and the pair (M, h) is called in this case a *hermitian module.*

Remark . There is a canonical isomorphism of abelian groups

$$d : \mathrm{Sesq}(M) \longrightarrow \mathrm{Hom}_\Lambda(M, \overline{M})$$

given by $d(h)(m)(n) := h(n, m) \quad \forall h \in \mathrm{Sesq}(M), m, n \in M$. One calls the form h *nonsingular* if $d(h)$ is an isomorphism.

Example 6.2.3. If Π is a finite group, $\Lambda = \mathbb{Z}\Pi$ the group ring with involution as in Example 6.2.1 and M is a Λ-module, we have natural isomorphisms

$$\mathrm{Hom}_\Lambda(M, \overline{M}) \cong \mathrm{Hom}_\Lambda(M, M^*) \cong \mathrm{Hom}_\Lambda(M \otimes_\mathbb{Z} M, \mathbb{Z}) = \mathrm{Hom}_\mathbb{Z}(M \otimes_\mathbb{Z} M, \mathbb{Z})^{(\Pi, w_1)}.$$

To make the second isomorphism well-defined, we have to give $M \otimes_\mathbb{Z} M$ the Λ-module structure defined by

$$g \cdot (m \otimes n) := w_1(g)(g \cdot m \otimes g \cdot n) \quad \forall g \in \Pi, m, n \in M.$$

It follows that elements $s \in \mathrm{Hom}_\mathbb{Z}(M \otimes_\mathbb{Z} M, \mathbb{Z})^{(\Pi, w_1)}$ are characterized by the property

$$s(a \cdot m, n) = s(m, \bar{a} \cdot n) \quad \forall a \in \Lambda, m \in M.$$

This leads to an isomorphism of abelian groups

$$\Psi : \mathrm{Hom}_\mathbb{Z}(M \otimes_\mathbb{Z} M, \mathbb{Z})^{(\Pi, w_1)} \longrightarrow \mathrm{Sesq}(M)$$

given by

$$\Psi(s)(m, n) := \sum_{g \in \Pi} s(g^{-1} \cdot m, n) g.$$

As in Example 6.2.1, the inverse of Ψ is simply given by $h \mapsto \epsilon_1 \circ h$. It follows that taking the dual of a sesquilinear form h corresponds to the involution $s \mapsto s^*$ on $\mathrm{Hom}_\mathbb{Z}(M \otimes_\mathbb{Z} M, \mathbb{Z})$ where

$$s^*(m, n) := s(n, m).$$

Such hermitian forms arise in geometry for example as follows: Let X be a 4-dimensional Poincaré complex with finite fundamental group Π. Then the cup-product plus evaluation on the fundamental class $[\widetilde{X}]$ (in other words Poincaré duality) gives a unimodular form (compare Part(I) of this thesis)

$$S_X \in \mathrm{Hom}_\mathbb{Z}(H^2(\widetilde{X}) \otimes_\mathbb{Z} H^2(\widetilde{X}), \mathbb{Z})^{(\Pi, w_1 X)}.$$

Via the correspondence in Example 6.2.3 we obtain a nonsingular hermitian form on $H^2(\widetilde{X})$ (with values in the group ring with its usual involution). But this is *not* the intersection form usually considered in the literature, because if one tries to perform concrete computations, one always uses the intersection numbers of submanifolds representing certain homology or homotopy classes. Algebraically, this is reflected by the fact that the universal coefficient theorem together with the Hurewicz homomorphism give a natural isomorphism

$$H^2(\widetilde{X}) \cong \mathrm{Hom}_\mathbb{Z}(H_2(\widetilde{X}), \mathbb{Z}) \cong H_2(\widetilde{X})^* \cong \pi_2(\widetilde{X})^* \cong \pi_2(X)^*.$$

Via the natural isomorphisms ($\pi_2 := \pi_2 X$)

$$\mathrm{Hom}_\mathbb{Z}(\pi_2^* \otimes \pi_2^*, \mathbb{Z}) \cong \mathrm{Hom}_\mathbb{Z}(\pi_2^*, \pi_2^{**}) \cong \mathrm{Hom}_\mathbb{Z}(\pi_2^*, \pi_2)$$

S_X can be therefore viewed as an element in $\mathrm{Hom}_\mathbb{Z}(\pi_2^*, \pi_2)$ and the unimodularity of S_X means that this element is an isomorphism. Now what people usually understand under the equivariant intersection form of the Poincaré complex X, is the nonsingular hermitian form on π_2 with values in Λ which corresponds to

$$S_X^{-1} \in \mathrm{Hom}_\mathbb{Z}(\pi_2, \pi_2^*) \cong \mathrm{Sesq}(\pi_2).$$

It can be computed by counting intersection numbers in $\widetilde{X}$ as described in [Wall 1, Thm.5.2]. Although in Part(I) of my thesis we rigorously worked with S_X, we will now switch to its inverse. Moreover, following the notation in [Wall 1], we will denote the hermitian form S_X^{-1} by

$$\lambda_X : \pi_2 X \times \pi_2 X \longrightarrow \Lambda.$$

Remark 6.2.4. If $\Pi = \pi_1 X$ is an infinite group, one has to be more careful since on the one hand, the algebraic isomorphisms ψ respectively Ψ cannot be defined and on the other hand, $\widetilde{X}$ is not compact and thus Poincaré duality is an isomorphism

$$S_X : H^2_{comp}(\widetilde{X}) \longrightarrow H_2(\widetilde{X})$$

where H^i_{comp} denotes cohomology with compact support. But these two problems *cancel* in the following sense: On the one hand , the homomorphism Ψ from Example 6.2.3 can still be defined on the subgroup of all $s \in \mathrm{Hom}_{\mathbb{Z}}(M \otimes_{\mathbb{Z}} M, \mathbb{Z})^{(\Pi, w_1)}$ such that for all $m, n \in M$

$$s(g^{-1} \cdot m, n) = 0 \text{ for all but finitely many } g \in \Pi.$$

On the other hand, the isomorphisms

$$H_2(\widetilde{X}) \cong \pi_2(\widetilde{X}) \cong \pi_2 X \quad \text{ and } \quad H^2(\widetilde{X}) \cong (\pi_2 X)^*$$

are still valid and there is a forgetful map

$$f : H^2_{comp}(\widetilde{X}) \longrightarrow H^2(\widetilde{X}).$$

Therefore, we obtain a homomorphism

$$\lambda_X : \pi_2 X \xrightarrow[\cong]{S_X^{-1}} H^2_{comp}(\widetilde{X}) \xrightarrow{f} H^2(\widetilde{X}) \xrightarrow[\cong]{} (\pi_2 X)^*$$

on which the above homomorphism Ψ is well-defined and gives an element $\lambda_X \in \mathrm{Sesq}(\pi_2 X)$. This follows from the fact that one counts intersections between two (compact) 2-spheres so that only finitely many deck-transformations keep a given sphere in an area where it can intersect a second given sphere.

Note that for infinite fundamental groups we are forced to work with $\lambda_X \in \mathrm{Sesq}(\pi_2 X)$ since in general there is no intersection form on $(\pi_2 X)^*$. We shall now describe a further condition which is satisfied for the intersection form λ of a 4-dimensional Poincaré complex: From Lemma **??** it follows that the form $\epsilon_1 \circ \lambda$ satisfies the Bredon-conditions

$$\epsilon_1 \circ \lambda(\tau(m), m) \equiv 0 \quad (2) \quad \forall m \in \pi_2 \text{ and } \tau \in \Pi \text{ of order 2.}$$

This is clearly equivalent to

$$\epsilon_\tau \circ \lambda(m, m) \equiv 0 \quad (2) \quad \forall m \in \pi_2 \text{ and } \tau \in \Pi \text{ of order 2.}$$

If we assume $w_1(\tau) = 0 \quad \forall \tau \in \Pi$ with $\tau^2 = 1$, an elementary computation shows that the following two conditions are equivalent for a given element $m \in \pi_2$:

(i) $\epsilon_\tau \circ \lambda(m, m) \equiv 0 \ (2) \quad \forall \tau \in \Pi$ with $\tau^2 = 1$.
(ii) $\lambda(m, m) \in \{a + \bar{a} \mid a \in \Lambda\}$.

The oriented case $w_1 \equiv 0$ is particularly interesting because then one easily shows that for $a, b \in \Lambda$ the equation $a + \bar{a} = b + \bar{b}$ implies that there exists an element $l \in \Lambda$ such that $a - b = l - \bar{l}$. In particular, the element $m \in \pi_2$ determines via the equation

$$\lambda(m, m) = a + \bar{a}$$

the element $a \in \Lambda$ modulo an indeterminacy of the form $l - \bar{l}$. This leads to the following

Definition 6.2.5. Let (M, λ) be a hermitian Λ-module. If $I := \{l - \bar{l} \mid l \in \Lambda\}$, a function

$$\mu : M \longrightarrow \Lambda/I$$

is called a *quadratic refinement* of λ if it satisfies the following properties:

1. $\lambda(m, m) = \mu(m) + \overline{\mu(m)} \in \Lambda \quad \forall m \in M$,
2. $\mu(m + n) = \mu(m) + \mu(n) + [\lambda(m, n)] \quad \forall m, n \in M$,

3. $\mu(a \cdot m) = a \cdot \mu(m) \cdot \bar{a} \quad \forall a \in \Lambda, m \in M.$

Definition 6.2.6. In [Wall 1], a *quadratic form* with coefficients in Λ is a triple (M, λ, μ), where (M, λ) is a hermitian Λ-module with quadratic refinement μ.

From the discussion preceding the definitions it follows that an oriented Poincaré complex X has a quadratic refinement of the intersection form λ restricted to the module

$$K(X) := \operatorname{Ker}(w_2(\widetilde{X}) : \pi_2 X \longrightarrow \mathbb{Z}/2) = \{m \in \pi_2 X \mid \epsilon_1 \circ \lambda(m, m) \equiv 0 \quad (2)\}.$$

In Section 2, we used this additional geometric restriction on λ to prove the Main Theorem. From the viewpoint of manifolds, the above approach to a quadratic refinement is very artificial because one obtains μ without an assumption on the orientation character directly from the geometry of manifolds as follows: If N is a 4-dimensional manifold, an element in the kernel of $w_2(\widetilde{N}) : \pi_2 N \longrightarrow \mathbb{Z}/2$ can be uniquely (up to regular homotopy) represented by a framed immersion $S^2 \looparrowright N$, compare [Kreck 1, Prop.6.6]. Then one can apply the description of [Wall 1, Thm.5.2] for counting the self-intersections of this immersion to obtain a function

$$\mu : K(N) \longrightarrow \Lambda/I$$

with the algebraic properties of a quadratic refinement for the intersection form $\lambda|K(N)$.

Remark 6.2.7. The inclusion of groups $\{1\} \hookrightarrow \Pi$ induces an inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}\Pi = \Lambda$ of rings with involution and since $\bar{1} = 1$ also an inclusion

$$\mathbb{Z} \cdot 1 \hookrightarrow \Lambda/I$$

which splits the well-defined map $\tilde{\epsilon}_1 : \Lambda/I \longrightarrow \mathbb{Z}$ induced by ϵ_1. We can therefore write

$$\Lambda/I = \mathbb{Z} \cdot 1 \oplus R$$

as abelian groups. Now let $x \in K(N)$ be given. We represent x by a (not necessarily framed) immersion $f : S^2 \looparrowright N$. Then $\mu(f)$ is defined by [Wall 1, Thm.5.2] and we want to compare

$$\mu(x) = \mu(x)_1 + \mu(x)_R \quad \text{with} \quad \mu(f) = \mu(f)_1 + \mu(f)_R \quad \in \mathbb{Z} \cdot 1 \oplus R.$$

One can change the immersion f to a framed immersion by introducing so called twistings which are certain explicit nonregular homotopies of f, see [Freedman-Quinn, p.14,22]. Since these changes take place inside a (contractible) coordinate neighborhood, one has the equation

$$\mu(x)_R = \mu(f)_R.$$

For the 1-component of μ one can use the following formula (iii) of [Wall 1, Thm.5.2]:

$$\lambda(f, f) = \mu(f) + \overline{\mu(f)} + \chi(\nu(f)) \cdot 1$$

to conclude from the homotopy invariance of λ that

$$\mu(x)_1 = \mu(f)_1 + \tfrac{1}{2}\chi(\nu(f)).$$

Note that since $x \in K(N)$, the Euler number $\chi(\nu(f))$ of the normal bundle of f is even. These formulas show that $\mu(x)$ is well-defined by setting

$$\mu(x) := \mu(f) + \tfrac{1}{2}\chi(\nu(f)) \cdot 1 \quad \in \Lambda/I$$

for any immersion f representing $x \in K(N)$.

We have seen that the existence of μ is an additional condition on λ but that in the orientable case μ is completely determined by λ. However, in the nonoriented case the quadratic refinement is an extra structure which is important for doing surgery, see Section 9.2. In this Section, we want to concentrate on the oriented case and can therefore forget about the function μ if we just keep the condition for its existence in mind:

Definition 6.2.8. A hermitian form $\lambda \in \mathrm{Sesq}(M)$ is called *weakly even* if for all $m \in M$

$$\lambda(m,m) \in \{a + \bar{a} \mid a \in \Lambda\}.$$

Recall that the intersection form of an oriented Poincaré complex with w_2-type $\neq \infty$ is always weakly even. Further examples of weakly even hermitian forms are obviously those of the form $q + \bar{q}$ with $q \in \mathrm{Sesq}(M)$.

Definition 6.2.9. A hermitian form $\lambda \in \mathrm{Sesq}(M)$ is called *even* if there exists a $q \in \mathrm{Sesq}(M)$ such that

$$\lambda = q + \bar{q}.$$

Remark . In the definition of [Bass, (4.4)], a *quadratic module* is a pair $(M, [q])$ where $q \in \mathrm{Sesq}(M)$ and $[q]$ is its class modulo $\{s - \bar{s} \mid s \in \mathrm{Sesq}(M)\}$. Then the associated hermitian form $\lambda := q + \bar{q}$ is well-defined and has the quadratic refinement $\mu(m) := [q(m,m)]$. In particular, a quadratic module gives a quadratic form in the sense of Definition 6.2.6.
Since in this Section we are not interested in the additional structures given by q or μ but only in the hermitian form λ and its properties, we have chosen the above language.

Lemma 6.2.10 ([Bass, Prop.3.4])**.** *If (M, λ) is a finitely generated projective hermitian module then λ is even if and only if it is weakly even.*

It is very easy to algebraically construct examples of weakly even hermitian forms which are not even. But the interesting question is, whether such examples also appear as intersection forms of 4-manifolds. In Proposition 6.3.4, we will see that for certain groups Π the following example does in fact occur in the geometry of 4-manifolds.

Theorem 6.2.11. *Let Π be a finite group and $\Lambda = \mathbb{Z}\Pi$ endowed with the involution from Example 6.2.1 with $w_1 \equiv 0$. Let $\mathfrak{I}$ be the augmentation ideal of Λ and $h \in \mathrm{Sesq}(\mathfrak{I})$ be the restriction of the "identity form"*

$$\begin{array}{rccc} (1): & \Lambda \times \Lambda & \longrightarrow & \Lambda \\ & (a,b) & \longmapsto & a \cdot \bar{b} \end{array} .$$

Then h is a weakly even hermitian form, which is even if and only if Π has cyclic 2-Sylow subgroups. We call h the <u>canonical form</u> *on $\mathfrak{I}$.*

Proof. Since the (1)-form is hermitian, so is h. However, the (1)-form is not weakly even. But if we take the $\mathbb{Z}$-basis $\{g - 1 \mid g \in \Pi \setminus 1\}$ of $\mathfrak{I}$ then for any $g \in \Pi$

$$\begin{aligned} h(g-1, g-1) &= (g-1) \cdot (g^{-1} - 1) \\ &= (1-g) + \overline{(1-g)} \end{aligned}$$

and thus h is weakly even.
If Π is a cyclic group generated by g then $g - 1$ generates $\mathfrak{I}$ as a Λ-module and has the defining relation $N \cdot (g-1)$, where $N \in \Lambda$ is the sum of all group elements. (We just put the isomorphism $\mathfrak{I} \cong \Lambda/N$ in words!) Thus a sesquilinear form q on $\mathfrak{I}$ is determined by its value $q(g-1, g-1)$ and to any given $l \in \Lambda$ with $N \cdot l = 0$ there exists a sesquilinear form q with $q(g-1, g-1) = l$. In particular, there exists a $q_0 \in \mathrm{Sesq}(\mathfrak{I})$ with $q_0(g-1, g-1) = (1-g)$ and it follows that $h = q_0 + \bar{q}_0$ is even.
Let me now translate the hermitian form h via the correspondence Ψ of Example 6.2.3 into the $\mathbb{Z}$-valued, Π-equivariant symmetric form $\epsilon_1 \circ h$. Since we will not switch back to the Λ-valued form, let me omit the ϵ_1-symbol and recall that

$$h : \mathfrak{I} \times \mathfrak{I} \longrightarrow \mathbb{Z}$$

is then given by $h(g-1, g'-1) = 1 + \delta_{g,g'}$. The evenness translates into the question whether

$$h \in \mathrm{Hom}_\Lambda(\mathfrak{I} \otimes \mathfrak{I}, \mathbb{Z}) = \mathrm{Hom}_\mathbb{Z}(\mathfrak{I} \otimes \mathfrak{I}, \mathbb{Z})^\Pi$$

can be written as $h = q + q^*$ with $q \in \mathrm{Hom}_\Lambda(\mathfrak{I} \otimes \mathfrak{I}, \mathbb{Z})$.

Lemma 6.2.12. *Let $h = h_\Pi$ be the form above.*

1. *If h_Π is even and $U \leq \Pi$ is some subgroup then the canonical form h_U on $\mathfrak{I}U$ is even, too.*
2. *If $U \leq \Pi$ is a 2-Sylow subgroup and h_U is even then h_Π is even, too.*

Proof: (1) In the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{I}U & \longrightarrow & \mathbb{Z}U & \xrightarrow{\epsilon} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathfrak{I}\Pi & \longrightarrow & \mathbb{Z}\Pi & \xrightarrow{\epsilon} & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

where the vertical arrows are the natural inclusions, one knows that their common cokernel F is a free U-module, see [Brown, p.14]. Therefore, F splits back to give an isomorphism of U-modules

$$\mathfrak{I}U \oplus F \cong \mathfrak{I}\Pi.$$

With respect to this decomposition, h_Π is given by the matrix of U-homomorphisms

$$\begin{pmatrix} h_U & A \\ A^* & B \end{pmatrix}.$$

It now follows that if $h_\Pi = q + q^*$ with

$$q \in \mathrm{Hom}_\mathbb{Z}(\mathfrak{I}\Pi \otimes \mathfrak{I}\Pi, \mathbb{Z})^\Pi \subseteq \mathrm{Hom}_\mathbb{Z}(\mathfrak{I}\Pi \otimes \mathfrak{I}\Pi, \mathbb{Z})^U$$

then h_U is even, too.
(2) In the terminology of part(1), we know that the form B is weakly even because h_Π has this property. But since $B \in \mathrm{Sesq}(F)$ and F is a free U-module, it follows from Lemma 6.2.10 that B is in fact even. Since by assumption also h_U is even, there exists a $q \in \mathrm{Hom}_\mathbb{Z}(\mathfrak{I}\Pi \otimes \mathfrak{I}\Pi, \mathbb{Z})^U$ such that

$$h_\Pi = q + q^*.$$

We have to show that we can also choose $q \in \mathrm{Hom}_\mathbb{Z}(\mathfrak{I}\Pi \otimes \mathfrak{I}\Pi, \mathbb{Z})^\Pi$.
Set $\mathcal{H} := \mathrm{Hom}_\mathbb{Z}(\mathfrak{I}\Pi \otimes \mathfrak{I}\Pi, \mathbb{Z})$. Then Π acts on $\mathcal{H}$ as described in Example 6.2.3 and the correspondence $f \mapsto f^*$ induces an additional $\mathbb{Z}/2$-action on $\mathcal{H}$ which commutes with the Π-action. Using Tate-cohomology groups with respect to this $\mathbb{Z}/2$-action, the problem is whether

$$0 = [h_\Pi] \in \hat{H}^0(\mathbb{Z}/2; \mathcal{H}^\Pi),$$

knowing that h_Π represents the zero element in $\hat{H}^0(\mathbb{Z}/2; \mathcal{H}^\mathcal{U})$. But now the transfer map described in Section 1.3 is a $\mathbb{Z}/2$-equivariant homomorphism tr such that the composition

$$\mathcal{H}^\Pi \overset{\rangle}{\hookrightarrow} \mathcal{H}^\mathcal{U} \overset{\sqcup\nabla}{\to} \mathcal{H}^\Pi$$

is multiplication by $|\Pi : U|$. Since this is an odd number and the Tate-cohomology groups have exponent 2, we conclude that

$$i_* : \hat{H}^0(\mathbb{Z}/2; \mathcal{H}^\Pi) \longrightarrow \hat{\mathcal{H}}'(\mathbb{Z}/\in; \mathcal{H}^\mathcal{U})$$

is injective, which proves assertion(2). $\square$
With this lemma at hand, we can now restrict to the case where Π is a noncyclic 2-group. For the smallest such group $\Pi = \mathbb{Z}/2 \times \mathbb{Z}/2$, I did a very explicit computation by writing down all necessary equations in terms of certain (3×3)-matrices. It turned out that h_Π is not even in

this case. But since this is far away from being exciting or interesting and this form does not occur in the geometry of manifolds, we skip the details. But note that it follows from the above lemma that h_Π is not even for Π a dihedral 2-group because such a group contains $\mathbb{Z}/2 \times \mathbb{Z}/2$ as a subgroup.
The next case which I also computed by hand is $\Pi = Q_8$, the ordinary quaternion group of order 8. Here one has to solve equations for (7×7)-matrices, but with quite a bit of patience I could show that h_Π is not even is this case, neither. It follows that for Π an arbitrary quaternion 2-group, h_Π is not even. This is the geometrically interesting case but the method of proof is by far not satisfactory. We will now present a proof which originated from a very fruitful discussion with Dr.W.Willems, University of Mainz, who in particular pointed out to me the usefulness of P.Webb's theorem below.

The idea is to project from the ring $\mathbb{Z}$ to the field $\mathbb{F}_2$ and to show that the form h is not even over this field. Therefore, let $\mathfrak{I}_2$ be the augmentation ideal in $\mathbb{F}_2\Pi$ and let h_2 be the $\mathbb{F}_2$-valued, Π-equivariant symmetric bilinear form given by $h_2 \equiv h \mod 2$.

Claim: *h_2 is not even if Π is a 2-group which is neither cyclic nor dihedral.*

Since this claim will finish the proof of the theorem, we do not have to return to the $\mathbb{Z}$-valued form and can therefore suppress the subscript 2. To prove the claim, we first note that norm element $N \in \mathbb{F}_2\Pi$ generates the sum of all minimal submodules of $\mathfrak{I}$ and is thus contained in an arbitrary (nontrivial) submodule of $\mathfrak{I}$. If we show that the radical of h is 1-dimensional then it follows that it is generated by N. For this we write h with respect to the $\mathbb{F}_2$-basis $\{g-1 \mid g \in \Pi \setminus 1\}$ of $\mathfrak{I}$ as the matrix

$$h = \begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} =: \mathbb{1} + A$$

and observe that A has the $(|\Pi|-2)$-fold eigenvalue 0 and the simple eigenvalue

$$\operatorname{trace}(A) = \operatorname{rank}(\mathfrak{I}) = |\Pi| - 1 = 1 \in \mathbb{F}_2.$$

Therefore, h has the $(|\Pi|-2)$-fold eigenvalue 1 and the simple eigenvalue 0 proving that the radical of h is 1-dimensional. Setting $H := \mathfrak{I}/_{\langle N \rangle}$ (this the so called "heart" of the module $\mathbb{F}_2\Pi$), we conclude that h induces an isomorphism

$$h|H : H \xrightarrow{\cong} H^*. \tag{III.1}$$

We now suppose that h is even and will derive a contradiction to the isomorphism(III.1) above. Let

$$q \in \operatorname{Hom}_{\mathbb{F}_2\Pi}(\mathfrak{I} \otimes \mathfrak{I}, \mathbb{F}_2) \quad \text{fulfill the relation } h = q + q^*.$$

Then $\langle N \rangle \subseteq \operatorname{Rad}(q), \operatorname{Rad}(q^*)$ because otherwise q (or q^*) would give a Π-equivariant isomorphism $\mathfrak{I} \cong \mathfrak{I}^*$. This is impossible since it would follow that

$$\mathbb{Z}/2 = \hat{H}_{-1}(\Pi; \mathbb{Z}/2) \cong \hat{H}_0(\Pi; \mathfrak{I}^*) \cong \hat{H}_0(\Pi; \mathfrak{I}) \cong \hat{H}_1(\Pi; \mathbb{Z}/2) \cong \Pi^{ab} \otimes \mathbb{Z}/2.$$

But since Π is a 2-group, $\Pi^{ab} \otimes \mathbb{Z}/2 \cong \Pi/\phi(\Pi)$ and this group can only be cyclic if Π is cyclic itself. (Here $\phi(\Pi)$ is the Frattini subgroup of Π, see [Huppert I, p.268 ff].)
Since we excluded cyclic groups in the claim, we can conclude that the forms q and q^* induce forms on the heart H of $\mathbb{F}_2\Pi$. The main ingredient in the proof is the following

Theorem 6.2.13 ([Webb]). *If Π is a 2-group which is not dihedral then H is absolutely indecomposible.*

Here the term absolutely indecomposible means that H stays indecomposible under all finite field extensions of $\mathbb{F}_2$. One knows that a module H is indecomposible if and only if the endomorphism ring $\mathrm{End}(H) = \mathrm{End}_{\mathbb{F}_2\Pi}(H)$ is a local ring, i.e. the set of all nonunits forms the unique maximal ideal or Jacobson radical

$$J := J(\mathrm{End}(H)),$$

compare [Reiner, p.82,88]. But if H is in addition absolutely indecomposible then it follows by [Huppert II, p.77] that

$$\mathrm{End}(H)/_{J} \cong \mathbb{F}_2. \tag{III.2}$$

Regarding the forms h, q, q^* as elements of $\mathrm{Hom}_{\mathbb{F}_2\Pi}(H, H^*)$, both of the following two cases lead to a contradiction (note that the equality $\det(q) = \det(q^*)$ shows that there are only these two cases to consider).

1. q and q^* are singular: Then it follows that $h^{-1} \circ q, h^{-1} \circ q^* \in J$ and thus
$$\mathrm{id}_H = h^{-1} \circ h = h^{-1} \circ (q + q^*) = h^{-1} \circ q + h^{-1} \circ q^* \in J.$$
2. q and q^* are nonsingular: Then there exists a unit $u \in \mathrm{End}(H)$ such that $q^* = q \circ u$. Applying the isomorphism(III.2), we know that $u = \mathrm{id}_H + j$ for some $j \in J$ and thus
$$h = q + q^* = q + q \circ (\mathrm{id}_H + j) = 2 \cdot q + q \circ j = q \circ j$$
is a singular form on H which contradicts the isomorphism(III.1). ■

Remark . The above proof of the innocent looking Theorem 6.2.11 was quite involved. To underline the nontriviality of this theorem, I want to remark that in the paper [Plotnick], the author fails to decide whether for quaternion Π the canonical form h is even.

We will now define a special kind of hermitian modules which will turn up as the intersection forms of various 4-dimensional manifolds in the next Section:

Definition 6.2.14 ([Bass, (3.5)])**.** Let (M, h) be a hermitian Λ-module.

1. The associated *metabolic module* is
$$H(M, h) := (\overline{M} \oplus M, \lambda_{(M,h)})$$
where $\lambda_{(M,h)}$ is defined by
$$\lambda_{(M,h)}((\varphi, m), (\varphi', m')) := \overline{\varphi(m')} + \varphi'(m) + h(m, m').$$
In matrix notation one may write
$$H(M, h) = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & h \end{pmatrix}.$$
2. If $h \equiv 0$ then $H(M, 0) =: H(M)$ is called the *hyperbolic module* on M.

It follows directly from the definition that $\lambda_{(M,h)}$ is (weakly) even if and only if h is (weakly) even.

Lemma 6.2.15 ([Bass, Prop.3.6])**.** *If (M, h) is an even hermitian Λ-module then*

$$H(M, h) \cong H(M).$$

Proof. If $h = q + \bar{q}$ then the equation

$$\begin{pmatrix} \mathbb{1} & 0 \\ q & \mathbb{1} \end{pmatrix} \cdot \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbb{1} & \bar{q} \\ 0 & \mathbb{1} \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & q + \bar{q} \end{pmatrix}$$

proves the assertion. ■

The following lemma from [Bass, Prop.3.6,3.7,3.10] is often very useful:

Lemma 6.2.16. *Let (M, h) be a hermitian Λ-module.*

(a) *The metabolic module $H(M, h)$ is nonsingular if and only if the natural map $M \longrightarrow \overline{\overline{M}}$ is an isomorphism.*
(b) *If h is nonsingular then there is an orthogonal decomposition*

$$H(M, h) \cong (M, h) \perp (M, -h).$$

(c) *If h is nonsingular and $M \cong M_0 \oplus M_1$ with $M_0 = M_0^{\perp}$ then there is an isomorphism of hermitian modules*

$$(M, h) \cong H(M_1, h|M_1).$$

We finish this Section by observing that the condition in part(a) of the above lemma is always fulfilled if Λ is the group ring of a finite group and M is a finitely generated free abelian group. This follows from the correspondence ψ from Example 6.2.1, since the natural map $M \longrightarrow M^{**}$ is an isomorphism.

6.3. **Construction Methods for Rational Homology 4-Spheres.**

The first thing I have to say is that there is no overall method for constructing rational homology 4-spheres. It is even possible that for a given fundamental group there exists no rational homology 4-sphere at all. For example, if Π is a finite abelian group then I proved in my Diplom Thesis that Π is the fundamental group of a rational homology 4-sphere if and only if it can be generated by 3 elements. But there is one well known construction which can be applied in some special cases (for abelian groups it just gives the cyclic case).

Proposition 6.3.1. *Given a finite 2-complex K with fundamental group Π and a cohomology class $w \in H^2(\Pi; \mathbb{Z}/2)$, there exists a differentiable compact, oriented 5-manifold $N(K)$ with boundary such that the following diagram commutes up to homotopy:*

$$\begin{array}{ccc} K & \xrightarrow{f} & B \\ {\scriptstyle j}\big\downarrow{\scriptstyle \simeq} & & \big\downarrow{\scriptstyle \xi(\Pi,w)} \\ N(K) & \xrightarrow[\nu_{N(K)}]{} & BSO \end{array}$$

Here f is a 2-equivalence and j a homotopy equivalence, in particular $N(K)$ admits a normal 1-smoothing in $\xi(\Pi, w)$. Moreover, the inclusion of the boundary $M(K)$ into $N(K)$ is a 2-equivalence inducing a normal 1-smoothing of $M(K)$ in $\xi(\Pi, w)$.
Finally, the intersection form on $M(K)$ is a metabolic form $H(\pi_2 K, s)$ for some hermitian form $s \in \mathrm{Sesq}(\pi_2 K)$.

Proof. The existence of $N(K)$ was proved in [Mazur]. He replaces the cells of K by handles, choosing the framings so that the above diagram commutes. Note that we have to assume that the w_2-type is $\neq \infty$, since otherwise the map f is in general not onto in π_2. For example, it is very well possible that the boundary $M(K)$ is a rational homology 4-sphere and we have seen in Remark 6.1.2 that then necessarily $w_2 M(K) \neq \infty$.

To compute the equivariant intersection form of $M(K)$, we consider the commutative diagram of cohomology groups with coefficients in $\mathbb{Z}\Pi$ given by Lefschetz duality on the universal coverings: (We write $N := N(K), M := M(K)$)

$$
\begin{array}{ccccccccc}
H^2(N,M) & \longrightarrow & H^2(N) & \longrightarrow & H^2(M) & \longrightarrow & H^3(N,M) & \longrightarrow & H^3(N) \\
\cong \big\downarrow .\cap[N] & & \cong \big\downarrow .\cap[N] & & \cong \big\downarrow .\cap[M] & & \cong \big\downarrow .\cap[N] & & \cong \big\downarrow .\cap[N] \\
H_3(N) & \longrightarrow & H_3(N,M) & \longrightarrow & H_2(M) & \longrightarrow & H_2(N) & \longrightarrow & H_2(N,M)
\end{array}
$$

Since N is homotopy equivalence to a 2-complex, these exact sequences are in fact short exact. They split because on the cells of K one can define a homotopy splitting of the composition

$$M \overset{i}{\hookrightarrow} N \xrightarrow[\simeq]{j^{-1}} K$$

since i is a 2-equivalence, compare [Baues, Thm.4.3.1]. Finally, the commutativity of the above diagram shows that

$$V := H_3(N,M) \cong H^2(N) \cong H_2(N)^* \cong (\pi_2 K)^*$$

is direct summand of $\pi_2 M$ satisfying $V \subseteq V^\perp$. But since the intersection form on $\pi_2 M$ is nonsingular and V is a direct summand of half rank, it follows that $V = V^\perp$ and thus by Lemma 6.2.16(c) the intersection form of M is metabolic. ■

We will call a manifold $M(K)$ constructed as above a *thickening of* K (with respect to w) although a priori the manifold $N(K)$ deserves this name. I do not think that this will lead to confusion. It is surprisingly an open question whether the intersection form of a thickening is always hyperbolic.

Remark 6.3.2. By construction, the Euler characteristic of a thickening is given by

$$\chi(M(K)) = 2 \cdot \chi(N(K)) = 2 \cdot \chi(K).$$

If the 2-complex K is modelled on a presentation

$$(g_1, \ldots, g_n \mid r_1, \ldots, r_m) \quad \text{of } \underline{\text{deficiency}} \quad m - n,$$

then it follows that

$$\chi(K) = 1 - n + m.$$

If one defines the *deficiency* of a finitely presentable group Π to be the minimum over all deficiencies of finite presentations for Π, it follows that for a group of deficiency zero there exists a thickening with Euler characteristic 2. If $H_1(\Pi;\mathbb{Q}) = 0$, this thickening is a rational homology 4-sphere.

Recall that the cellular chain complex of the universal covering of the finite 2-complex K gives an exact sequence

$$0 \longrightarrow \pi_2 K \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

in which the C_i are finitely generated free Λ-modules. Generalizing the notions from Section 2.2, we call a Λ-module L a *representative of* $\Omega^3\mathbb{Z}$ if it fits into an exact sequence

$$0 \longrightarrow L \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

with finitely generated projective Λ-modules P_i. We will now generalize the thickening construction in the topological category:

Proposition 6.3.3. *Let Π be a finite group and $w \in H^2(\Pi;\mathbb{Z}/2)$. If the Λ-module L is a representative of $\Omega^3\mathbb{Z}$, there exists a 4-dimensional topological normal 1-smoothing $(M,\tilde{\nu})$ in $\xi(\Pi,w)$ which bounds in $\Omega_4^{TOP}(\xi(\Pi,w))$. The intersection form of M is a metabolic form*

$H(L', s')$ for some Λ-module L' in the same genus as L and some hermitian form $s' \in \mathrm{Sesq}(L')$. In particular we have

$$\chi(M) = 2 \cdot \frac{1 + \mathrm{rank}_{\mathbb{Z}} L}{|\Pi|} \quad .$$

Proof. Let K be a finite 2-complex with fundamental group Π. Then by Shanuel's Lemma ([Brown, p.192]), there exists a finitely generated projective Λ-module P and a natural number r such that

$$\pi_2 K \oplus \Lambda^r \cong L \oplus P.$$

Replacing K by $K \vee \bigvee_r S^2$ we can therefore assume that $\pi_2 K \cong L \oplus P$.
Now [Hambleton-Kreck 2, Cor.1.19] prove a generalization of Roiter's Replacement Lemma, from which it follows that there exists a Λ-module L' in the same genus as L such that

$$L \oplus P \cong L' \oplus F$$

for some finitely generated free Λ-module F.
Let $M(K)$ be a thickening of K with respect to w. It has a metabolic intersection form

$$\lambda = H(\pi_2 K, s) \cong H(L' \oplus F, s).$$

Since $w \neq \infty$, λ is weakly even and thus the restriction $\lambda|(\overline{F} \oplus F)$ is a weakly even, nonsingular metabolic form on a free module and thus by Lemmas 6.2.10 and 6.2.15 this restriction is hyperbolic. We obtain an orthogonal decomposition

$$\lambda = \lambda|(\overline{F} \oplus F) \perp (\lambda|(\overline{F} \oplus F))^{\perp} \cong H(F) \perp H(L', s').$$

It follows from M.Freedman's main theorem ([Freedman]) that this algebraic decomposition of the intersection form is induced by a geometric decomposition ($n := \mathrm{rank}_\Lambda(F)$)

$$M(K) \approx M \# n \cdot (S^2 \times S^2)$$

for some topological 4-manifold M with intersection form $H(L', s')$. Clearly M and $M(K)$ have the same fundamental groups and w_2-types and by the formula

$$|\Pi| \cdot \chi(K) = \chi(\widetilde{K}) = 1 + \mathrm{rank}_{\mathbb{Z}} H_2(\widetilde{K}) = 1 + \mathrm{rank}_{\mathbb{Z}} \pi_2 K$$

the Euler characteristic condition follows from Remark 6.3.2. ∎

Let me now describe yet another method for obtaining rational homology 4-spheres: Let Y^3 be a 3-manifold with fundamental group Π and $\dot{D}^3$ an open coordinate 3-disc inside Y. Define

$$\Sigma_\theta := (Y^3 \setminus \dot{D}^3) \times S^1 \underset{\theta}{\cup} S^2 \times D^2,$$

where $\theta : S^2 \times S^1 \longrightarrow S^2 \times S^1$ is derived from an element in $\pi_1(SO(3)) \cong \mathbb{Z}/2$. Then Σ_θ is a 4-manifold with fundamental group Π and it is a rational homology 4-sphere if and only if $H_1(\Pi; \mathbb{Q}) = 0$. Note that Σ_θ is obtained by surgery on the circle $* \times S^1$ in $Y \times S^1$ where θ corresponds to the choice of the framing of the normal bundle. If we do spin structure preserving surgery, $\theta = 0$ corresponds to the zero bordant spin structure on S^1 whereas $\theta \neq 0$ means that we take the nontrivial spin structure denoted by $\hat{S}^1$. Thus by construction $\Sigma := \Sigma_{\theta=0}$ bounds in $\Omega_4^{Spin}(B\Pi)$ but a priori we do not know the bordism class of $\widehat{\Sigma} := \Sigma_{\theta \neq 0}$. A straightforward computation proves the following

Proposition 6.3.4 ([Plotnick]). *If $\Pi = \pi_1(Y^3)$ is a finite group and $\mathfrak{I}$ denotes the augmentation ideal of the group ring Λ then the intersection form of Σ is the hyperbolic form $H(\mathfrak{I})$ whereas the intersection form of $\widehat{\Sigma}$ is the metabolic form $H(\mathfrak{I}, h)$ with $h \in \mathrm{Sesq}(\mathfrak{I})$ the canonical form on $\mathfrak{I}$ as in Theorem 6.2.11.*

To compare this proposition with the computation for thickenings, note that $Y \setminus \dot{D}^3$ is homotopy equivalent to a 2-complex and for finite Π the universal covering $\widetilde{Y}$ is a homotopy 3-sphere and thus

$$\pi_2(Y \setminus \dot{D}^3) \cong \pi_2(\widetilde{Y} \setminus |\Pi| \cdot \dot{D}^3) \cong \overline{\mathfrak{I}}.$$

Moreover, Σ can also be obtained as the double of $(Y \setminus \dot{D}^3) \times I$ which shows that Σ is the thickening of $Y \setminus \dot{D}^3$ with respect to $w = 0$. In particular, for finite fundamental groups of 3-manifolds, the thickenings with respect to $w = 0$ have hyperbolic intersection forms.
On the contrary, we have shown in Theorem 6.2.11 that the metabolic intersection form $H(\mathfrak{I}, h)$ of $\widehat{\Sigma}$ is hyperbolic ($\Leftrightarrow h$ is even) if and only if $\Pi = \pi_1 \widehat{\Sigma}$ has cyclic 2-Sylow subgroups. It follows that in the other possible case, where Π has quaternion 2-Sylow subgroups, the intersection forms of Σ and $\widehat{\Sigma}$ are not isomorphic and thus these two rational homology 4-spheres are not homotopy equivalent.

Remark . The question whether Σ and $\widehat{\Sigma}$ are homotopy equivalent was raised by [Plotnick] but he vainly tried to solve the algebraic problem whether the sesquilinear form h is even. In [Hambleton-Kreck 1, Rem.4.3], the authors answered the question by comparing the Thom spectra of $B\Pi \wedge BSpin$ and $B\Pi \wedge BSpinG$ and concluded that Σ and $\widehat{\Sigma}$ must have distinct quadratic 2-types. Our computation shows that it is in fact the evenness of the intersection form which distinguishes these two rational homology 4-spheres.

There are various possibilities to generalize the construction leading to Σ and $\widehat{\Sigma}$. We will not make use of the first one, so we describe it only briefly:
Let $\Pi = \pi_1(Y^3)$ be finite and let $w \in H^2(\Pi; \mathbb{Z}/2)$. Then there exists a unique oriented S^1-bundle L over Y with $w_2 L = w$. Since L restricted to a 3-disc in Y is trivial, we can form

$$\Sigma^w_\theta := L|(Y \setminus \dot{D}^3) \underset{\theta}{\cup} S^2 \times D^2, \quad \theta \text{ as before.}$$

One checks that one choice of θ leads to a $\xi(\Pi, w)$-manifold which is zero-bordant in $\Omega_4(\xi(\Pi, w))$. Moreover, a similar computation as in Proposition 6.3.4 shows that the intersection form in this case is again the hyperbolic form $H(\mathfrak{I})$. In particular, a thickening of $Y \setminus \dot{D}^3$ with respect to w has also hyperbolic intersection form. The other choice of θ leads to a spin manifold with metabolic intersection form, stably diffeomorphic to $\widehat{\Sigma}$ as one can show by computing the invariant $\mathfrak{sec}$ from Corollary 4.4.10.

Remark 6.3.5. For all $w \in H^2(\Pi; \mathbb{Z}/2)$ we constructed rational homology 4-spheres Σ^w_θ with w_2-type w and isomorphic (hyperbolic) intersection forms on the universal covering. But since by the Wu formula the w_2-type is a homotopy invariant and for the fundamental groups in question the homotopy type is detected by the quadratic 2-type (see [Hambleton-Kreck 1, Thm.A]), the k-invariant must distinguish these manifolds. This gives a nice extension of an example in [Hambleton-Kreck 1, Rem.4.5] where the authors consider the case $\Pi = \mathbb{Z}/2$.

The second generalization is concerned with a wider class of possible fundamental groups and was introduced in [Hambleton-Kreck 1]. If Π is a finite group with 4-periodic cohomology, there exists a 3-dimensional Swan-complex X for Π. This means that Π acts freely on the CW-complex $X \simeq S^3$. The finiteness obstruction for X as an element in $K_0(X) = K_0(\mathbb{Z}\Pi)$ is represented by a projective module P_X and can be very well nontrivial, compare[Milgram]. By doing surgery on a degree 1 normal map

$$Y^3 \times S^1 \longrightarrow X \times S^1$$

[Hambleton-Kreck 1, Thm.4.2] obtain two 4-dimensional spin manifolds M and $\widehat{M}$ with fundamental group Π and

$$\pi_2 M \cong \overline{\mathfrak{I}} \oplus \mathfrak{I} \oplus \overline{P} \oplus P \cong \pi_2 \widehat{M}$$

form some projective module P in the same projective class as P_X. Moreover, the authors show that the intersection form on M is the hyperbolic form $H(\mathfrak{I} \oplus P)$ and the intersection form on $\widehat{M}$ is the metabolic form $H(\mathfrak{I} \oplus P, h \oplus 0)$ where $h \in \mathrm{Sesq}(\mathfrak{I})$ is the canonical form from Theorem 6.2.11.
By doing exactly the same algebraic trick as in the proof of Theorem 6.3.3, we can now conclude that there exists a Λ-module $\mathfrak{I}'$ in the same genus as $\mathfrak{I}$ and a natural number n such that

$$H(\mathfrak{I} \oplus P) \cong H(\mathfrak{I}' \oplus \Lambda^n) \cong H(\mathfrak{I}') \perp H(\Lambda^n)$$

and similarly

$$H(\mathfrak{I} \oplus P, h \oplus 0) \cong H(\mathfrak{I}' \oplus \Lambda^n, s) \cong H(\mathfrak{I}', s') \perp H(\Lambda^n)$$

for some $s' \in \mathrm{Sesq}(\mathfrak{I}')$. Note that s' is even if and only if h is even. Since these decomposition can be realized geometrically, we obtain from Theorem 6.2.11 the following

Proposition 6.3.6. *Let Π be a finite group with 4-periodic cohomology. Then there exist two topological spin rational homology 4-spheres Σ and $\widehat{\Sigma}$ with fundamental group Π and intersection forms $H(\mathfrak{I}')$ respectively $H(\mathfrak{I}', s')$. The metabolic intersection form of $\widehat{\Sigma}$ is hyperbolic if and only if Π has cyclic 2-Sylow subgroups.*

In the last part of this Section, I would like to survey a construction method which I employed in my Diplom thesis. It is also a generalization of the above construction: If Y^3 is again a 3-manifold then one can perform surgery (with either framing) on an arbitrary element $g \in \pi_1(Y \times S^1)$, not just on the circle $* \times S^1$. Then it might be possible that even though $\pi_1 Y$ is infinite, for some choice of g the resulting 4-manifold has finite fundamental group and is thus a rational homology 4-sphere. By taking Y to be one of the 3-dimensional Brieskorn spheres

$$\{(z_1, z_2, z_3) \in S^5 \subseteq \mathbb{C}^{[illegible]} \mid [illegible]\},$$

I proved the following result.

Proposition 6.3.7. *Given a prime number p, there exist two differentiable homology 4-spheres Σ and $\widehat{\Sigma}$ with fundamental group $SL_2(p)$. Moreover, $\mathfrak{sec}(\Sigma) = \mathfrak{o}$ and*

$$\mathfrak{sec}(\widehat{\Sigma}) = \begin{cases} 1 & \text{if } p \equiv 3, 5 \quad (8) \\ 0 & \text{if } p \equiv 1, 7 \quad (8) \end{cases}.$$

We will show in Theorem 6.4.1 that the above invariant $\mathfrak{sec}$ measures whether the intersection form of the manifold is even or not. Out of this reason and because the cohomological period of the group $SL_2(p)$ is 4 if and only if $p = 3, 5$, this theorem is an extension of Theorem 6.3.6.
I must admit that I could not settle the question whether there exist homology 4-spheres with nontrivial $\mathfrak{sec}$-invariant for $p \equiv 1, 7$ (8). For the smallest values $p = 7, 17$, I constructed such manifolds using very special presentations for $SL_2(7)$ respectively $SL_2(17)$ which I could not generalize to larger primes.
However, for an arbitrary prime p, it is not hard to construct a 4-manifold with fundamental group $SL_2(p)$, nontrivial $\mathfrak{sec}$-invariant and Euler characteristic 4. But since we decided only to apply the cancellation theorem in case a rational homology 4-sphere exists, we will not make use of these manifolds. Nevertheless, it is clear how to obtain a classification theorem under a stronger hypothesis in these cases, too.

6.4. **Classification Results for Special Fundamental Groups.**

The fundamental groups for which we will give a classification result will be certain finite groups with periodic cohomology so that we can apply the stable classification results from Section 7.4. Since in addition the fourth homology group of a finite group with periodic cohomology vanishes, the stable diffeomorphism invariants are exactly

$$w_2,\ \text{signature}\ ,\mathfrak{sec}, \mathfrak{ter}\ \text{and}\ \mathfrak{ks}.$$

Here the $\mathfrak{sec}$-invariant is defined for spin manifolds, whereas the $\mathfrak{ter}$-invariant is defined in one special w_2-type $\neq \infty, 0$. We abbreviate this w_2-type by $w_{\mathfrak{ter}}$. In Section 8.2 we showed that the $\mathfrak{ter}$-invariant is not a homotopy invariant, whereas the following theorem proves that the $\mathfrak{sec}$-invariant is determined by the stable equivariant intersection form.

Theorem 6.4.1. *Let M be a 4-dimensional spin manifold with finite fundamental group whose 2-Sylow subgroups are quaternion. Then*

$$\mathfrak{sec}(\mathfrak{M}) = \mathfrak{o} \quad \textit{if and only if the equivariant intersection form of } M \textit{ is even.}$$

Proof. By the usual transfer argument (compare Lemma 6.2.12), we can assume that $\Pi := \pi_1 M$ is a quaternion 2-group. By Theorem 6.2.11 and Proposition 6.3.4 there exist two rational homology 4-spheres Σ and $\widehat{\Sigma}$ with fundamental group Π such that the intersection form λ_Σ is even but $\lambda_{\widehat{\Sigma}}$ is not even. Therefore, Σ and $\widehat{\Sigma}$ can not be stably diffeomorphic. But by Corollary 4.4.10, signature and $\mathfrak{sec}$-invariant detect the stable diffeomorphism class implying that

$$\mathfrak{sec}(\Sigma) \neq \mathfrak{sec}(\widehat{\Sigma}).$$

Since by construction Σ was the manifold which bounds in $\Omega_4^{Spin}(B\Pi)$, it follows that $\mathfrak{sec}(\Sigma) = \mathfrak{o}$ and the proof is complete for the special examples Σ and $\widehat{\Sigma}$. This implies the general case, since the manifold

$$M\#(-\tfrac{\sigma(M)}{8} \cdot |E_8|)$$

has an even intersection form if and only if M does and also their $\mathfrak{sec}$-invariants agree. Since this connected sum is stably homeomorphic to either Σ or $\widehat{\Sigma}$, the result follows. ∎

Let me now state our main classification result.

Theorem 6.4.2. *Let M and N be two 4-manifolds with finite fundamental group Π and same w_2-type, signature, $\mathfrak{ks}$-invariant and Euler characteristic. Assume that the intersection form on $H_2(M;\mathbb{Z})$ is indefinite (and that $\chi(M) > 4$ if $w_2M = \infty$ and $\sigma(M) = 0$). Then M and N are homeomorphic if either of the following conditions is fulfilled:*

(i) *All Sylow subgroups of Π are cyclic.*

(ii) $\Pi \cong SL_2(p), p$ *a prime with* $p \equiv 3, 5,\ (8)$ *and*

$$\mathfrak{sec}(\mathfrak{M}) \quad = \quad \mathfrak{sec}(\mathfrak{N}) \qquad \textit{if} \qquad w_2M = 0.$$

(iii) Π *has 4-periodic cohomology and*

$$\begin{array}{lclcll} \mathfrak{sec}(\mathfrak{M}) & = & \mathfrak{sec}(\mathfrak{N}) & \textit{if} & w_2M = 0 & \textit{respectively} \\ \mathfrak{ter}(\mathfrak{M}) & = & \mathfrak{ter}(\mathfrak{N}) & \textit{if} & w_2M = w_{\mathfrak{ter}}. & \end{array}$$

(Recall from Corollary 4.4.10 that in some cases $\mathfrak{ter}(\mathfrak{M})$ is determined by the signature.)

Proof. To apply Corollary 6.1.1, we have to construct a rational homology 4-sphere in every stable homeomorphism class of signature 0 manifolds with the given fundamental group and w_2-type. For this we go through the single cases:
(i) It is well known that a group whose Sylow subgroups are cyclic is a metacyclic group with vanishing second homology. By [Beyl-Trappe, Ch.IV], Π has deficiency zero and so a suitable thickening gives a rational homology 4-sphere Σ, see Remark 6.3.2. By the stable classification result (Corollary 4.4.5), we need just one more rational homology 4-sphere, namely one with w_2-type $\neq \infty, 0$ and nontrivial $\mathfrak{ks}$-invariant. This is provided by Proposition 5.2.2, where we showed that $*\Sigma$ exists in this case.
(ii) The necessary homology 4-spheres are provided by Propositions 6.3.7 and 5.2.2 as above.
(iii) If Π has 4-periodic cohomology, there exists an exact sequence (see [Swan 2])

$$0 \longrightarrow \overline{\mathfrak{I}} \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

where P_i are finitely generated projective Π-modules and $\overline{\mathfrak{I}}$ is the dual of the augmentation ideal $\mathfrak{I}$.
Therefore, Proposition 6.3.3 gives for every w_2-type $w \neq \infty$ a rational homology 4-sphere which bounds in $\Omega_4^{TOP}(\xi(\Pi, w))$. In the spin case, Proposition 6.3.6 also gives a rational homology 4-sphere with nontrivial $\mathfrak{sec}$-invariant. In the w_2-types $w \neq \infty, 0$ take the zero-bordant rational homology 4-sphere and apply Proposition 5.2.2 (respectively Proposition 5.2.3 if $w = w_{\mathfrak{ter}}$) which shows that without changing the homotopy type, one can prescribe the $\mathfrak{ks}$-invariant (respectively $\mathfrak{ks}$- and $\mathfrak{ter}$-invariant if $w = w_{\mathfrak{ter}}$) arbitrarily. Since by the stable classification result (Corollary 4.4.10), these rational homology 4-spheres cover all elements of signature 0 in $\mathrm{MSt}_4^{TOP}(\Pi, w)$, the result follows. ■

Remark . For finite cyclic groups, the above result was improved in [Hambleton-Kreck 4] because the authors could show that in this case one can cancel *all the way down.* They prove that two 4-manifolds with the same finite cyclic fundamental group are homeomorphic if and only if they have the same w_2-type, $\mathfrak{ks}$-invariant and isomorphic intersection forms on $H_2(.;\mathbb{Z})$.

6.5. **Some Conjectures.**

We want to conclude the discussion of a homeomorphism classification by the method of stabilization and then cancellation by stating a number of conjectures for arbitrary fundamental groups which we partially proved for (say) quaternion groups. We again consider connected closed oriented topological 4-manifolds.

Conjecture A: *Two 4-manifolds M and N have the same π_1-fundamental class if and only if $\pi_2 M$ and $\pi_2 N$ are stably isomorphic in the sense that there exist $r, s \in \mathbb{N}$ such that*

$$\pi_2 M \oplus \Lambda^r \cong \pi_2 N \oplus \Lambda^s.$$

The *only if* part of Conjecture A was proven in [Hambleton-Kreck 1, Prop.2.4], where the authors show that for a fixed 2-equivalence $u : M \longrightarrow B\Pi$, the π_1-fundamental class

$$u_*[M] \in H_4(\Pi) \cong \mathrm{Ext}^1_\Lambda((\Omega^3\mathbb{Z})^*, \Omega^3\mathbb{Z})$$

determines by the above isomorphism an extension

$$0 \longrightarrow \Omega^3\mathbb{Z} \xrightarrow{i} \pi_2 M \oplus \Lambda^r \xrightarrow{p} (\Omega^3\mathbb{Z})^* \longrightarrow 0.$$

To prove the *if* part of Conjecture A, one has to find a natural way in which the stable isomorphism type of $\pi_2 M$ determines one of the maps i or p above.

Let me now assume that the π_1-fundamental class of M vanishes and that the w_2-type w of M is $\neq \infty$. By adding copies of $S^2 \times S^2$ to M, we can assume that

$$\pi_2 M \cong \overline{\pi_2 K} \oplus \pi_2 K$$

for some finite 2-complex K. Furthermore, the $\mathfrak{sec}$-invariant of M is defined as an element

$$\mathfrak{sec}(\mathfrak{M}) \in \left(\mathfrak{H}_3(\Pi;\mathbb{Z}/2)/\operatorname{Image}((\mathfrak{Sq}^2_{\mathfrak{w}})^* \circ \mathfrak{r}_2)\right) /_{\operatorname{Out}(\Pi)}.$$

Conjecture B. *The stable isometry class of the equivariant intersection form λ_M determines $\mathfrak{sec}(\mathfrak{M})$, the trivial $\mathfrak{sec}$-invariant corresponding to an even hermitian form (compare Conjecture C). Moreover, given $\mathfrak{s} \in \mathfrak{H}_3(\Pi;\mathbb{Z}/2)$, there exists a 4-dimensional spin manifold $M_{\mathfrak{s}}$ with fundamental group Π and $\mathfrak{sec}$-invariant $[\mathfrak{s}]$ such that for some hermitian form $h \in \operatorname{Sesq}(\overline{\pi_2 K})$*

$$\lambda_{M_{\mathfrak{s}}} \cong H(\overline{\pi_2 K}, h).$$

Remark . The *moreover* part is only stated for spin manifolds because otherwise there might occur problems coming from the weaker divisibility conditions for the signature. For example, at the end of Section 6.3 we constructed a 4-manifold with fundamental group $\mathbb{Z}/2 \times \mathbb{Z}/2$, vanishing π_1-fundamental class, signature 4 and universal covering spin. This manifold must necessarily have a nontrivial $\mathfrak{sec}$-invariant but its intersection form cannot be metabolic since it has signature 4. Even more is true: for the normal 1-type ξ in question, the $\mathfrak{sec}$-invariant is a homomorphism

$$\mathfrak{sec} : \mathfrak{F}_{3,1}(\xi) \longrightarrow 4\mathbb{Z}/8\mathbb{Z} \cong \mathbb{Z}/2$$

and is induced by the signature (which implies the first part of Conjecture B for this normal 1-type).

Let me discuss the following evidence for the *moreover* part of Conjecture B. From the exact sequence

$$0 \longrightarrow H_3(\Pi;\mathbb{Z}/2) \xrightarrow{\partial} \pi_2 K \otimes_\Lambda \mathbb{Z}/2 \longrightarrow H_2(K;\mathbb{Z}/2)$$

we see that there exists an element $\alpha \in \pi_2 K$ with $\partial(\mathfrak{s}) = \alpha \otimes 1$. This element α determines a hermitian form h_α on $\overline{\pi_2 K}$ by the equation

$$h_\alpha(\phi,\psi) := \overline{\phi(\alpha)} \cdot \psi(\alpha) \ \in \Lambda \quad \forall \phi,\psi \in \overline{\pi_2 K}.$$

Furthermore, it is easy to see that if

$$\alpha \otimes 1 = \beta \otimes 1 \in \pi_2 K \otimes_\Lambda \mathbb{Z}/2 \qquad \text{for some } \beta \in \pi_2 K$$

then there exists a $q \in \operatorname{Sesq}(\overline{\pi_2 K})$ with $h_\beta = h_\alpha + q + \bar{q}$. The same proof as the one for Proposition 6.2.15 then shows that

$$H(\overline{\pi_2 K}, h_\alpha) \cong H(\overline{\pi_2 K}, h_\beta).$$

We conclude that algebraically there is a well-defined map

$$H_3(\Pi;\mathbb{Z}/2) \longrightarrow \{\text{ Isomorphism classes of metabolic modules } H(\overline{\pi_2 K}, h)\}.$$

But what about the realization of these metabolic forms as intersection forms of 4-dimensional manifolds? The only necessary condition is that the metabolic form must be weakly even because the w_2-type is $\neq \infty$. But for the metabolic forms defined above, this can be verified using the fact that

$$\alpha \otimes 1 \in \operatorname{Image}(\partial) \subseteq \pi_2 K \otimes_\Lambda \mathbb{Z}/2.$$

Let me now describe a sufficient condition for the realizability of the metabolic forms above: Assume that there exists a thickening $M(K)$ (with respect to $w = 0$) and a homotopy retraction $r : K \longrightarrow M(K)$ of the composition

$$M(K) \subseteq N(K) \xrightarrow{\simeq} K$$

such that the element

$$(r_*(\alpha), 1, 1) \in \pi_2 M(K) \oplus \Lambda \oplus \Lambda \cong \pi_2(M(K)\#\mathbb{CP}^2\#(-\mathbb{CP}^2))$$

is represented by an embedding $\varphi : S^2 \times D^2 \longrightarrow M(K)\#\mathbb{CP}^2\#(-\mathbb{CP}^2)$.
Then we can do surgery on φ to obtain a spin manifold $M_{\mathfrak{s}}$ with fundamental group Π and intersection form

$$\lambda_{M_{\mathfrak{s}}} \cong H(\overline{\pi_2 K}, h_\alpha).$$

Moreover, using Proposition 3.2.3, it is easy to show that $\mathfrak{sec}(\mathfrak{M}_{\mathfrak{s}}) = [\mathfrak{s}]$. By construction, we have a homeomorphism

$$M_{\mathfrak{s}}\#\mathbb{CP}^2\#(-\mathbb{CP}^2) \approx M(K)\#\mathbb{CP}^2\#(-\mathbb{CP}^2)$$

and we remark that from the stable point of view this is not surprising. In fact, the connected sum with $\mathbb{CP}^2\#(-\mathbb{CP}^2)$ changes the w_2-type to be ∞ and thus kills the $\mathfrak{sec}$-invariant.
Note however, that even if Π is a good group the existence of the embedding φ as above *cannot* be concluded from the embedding theorem in [Freedman-Quinn, Thm10.3]. In fact, w_2 is neither trivial on $\pi_2(M(K)\#\mathbb{CP}^2\#(-\mathbb{CP}^2))$ nor is it nontrivial on the orthogonal complement of the two vectors

$$(r_*(\alpha), 1, 1), (0, 1, 0) \in \pi_2 M(K) \oplus \Lambda \oplus \Lambda$$

and thus non of the possible assumptions in Theorem 10.3 of [Freedman-Quinn] are satisfied. Note that an obvious necessary condition for the existence of the embedding φ is the vanishing of $\lambda_{M(K)}(r_*(\alpha), r_*(\alpha))$. This leads to

Conjecture C. *Given a finite 2-complex K, there exists a pair $(M(K), r)$ as above such that*

$$\lambda_{M(K)}|r_*(\pi_2 K) \equiv 0.$$

My idea for the proof of this conjecture would be to construct a 4-dimensional manifold W with boundary ∂W such that

$$W \simeq K \quad \text{and} \quad \pi_2 \partial W \xrightarrow{i_*} \pi_2 W \ \text{ is surjective.}$$

Then the double of W would be a thickening as described in Conjecture C. This approach certainly needs techniques from 3-dimensional topology and the reason why I believe it to work is that if

$$K \simeq Y^3 \setminus \overset{\circ}{D}{}^3$$

for some closed 3-manifold Y then $W := (Y^3 \setminus \overset{\circ}{D}{}^3) \times I$ satisfies the desired properties since $\partial W = Y\#Y$.

Conjecture D. *Let M and N be two 4-manifolds with finite fundamental group. Suppose that the connected sum $M\#r \cdot (S^2 \times S^2)$ is homeomorphic to $N\#r \cdot (S^2 \times S^2)$. Then M is homeomorphic to N if there exists a decomposition $N \approx N_0\#\mathbb{CP}^2\#(-\mathbb{CP}^2)$.*

Note that this conjecture is exactly the cancellation theorem from [Hambleton-Kreck 3] (compare Section 9.1), except that the assumption $N \approx N_0\#(S^2 \times S^2)$ is replaced by the assumption

$N \approx N_0 \# \mathbb{CP}^2 \# (-\mathbb{CP}^2)$. This hypothesis implies that $w_2 N = \infty$ and thus the stable classification is easy but there cannot exist a rational homology 4-sphere in the stable homeomorphism class of N, see Remark 6.1.2. Conjecture D would imply that the extra assumption

$$\chi(M) > 4 \quad \text{if} \quad w_2 M = \infty, \sigma(M) = 0$$

could be omitted in our classification theorems. The reason why I believe Conjecture D to be true is that the intersection form of $\mathbb{CP}^2 \# (-\mathbb{CP}^2)$ contains a hyperbolic form on a finite index submodule and the methods of [Hambleton-Kreck 3] are designed to work in such a situation. We finish this Section with a conjecture whose proof would have to make use of the verification of all previous conjectures.

Conjecture E. *Let M and N be two 4-manifolds with the same finite fundamental group of deficiency zero. Assume that the intersection form on $H_2(M;\mathbb{Z})$ is indefinite. Then M and N are homeomorphic if and only if they are stably homeomorphic and have the same Euler characteristic.*

Note that if one adds the hypothesis

$$\chi(M) > 4 \quad \text{if} \quad w_2 M = \infty, \sigma(M) = 0$$

or proves Conjecture D, the above conjecture is proven in [Hambleton-Kreck 4] for odd order groups. This assumption on the group order makes the situation considerably easier because the w_2-type is either ∞ or 0 and there are no $\mathfrak{sec}$- respectively $\mathfrak{ter}$-invariants.

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