Super symmetric field theories and integral modular functions

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1 Introduction

The main result of this paper, Theorem 1, is that the partition function of a 2-dimensional super symmetric quantum field theory is an integral modular function. The bulk of this work consists of incorporating a suitable geometric notion of *super symmetry* into the axiomatic description of quantum field theories going back to Atiyah and Segal. Roughly speaking, we replace the usual functors from Riemannian manifolds to (locally convex) vector spaces by functors from Riemannian super manifolds to super vector spaces and work in families parametrized by complex super manifolds. We end up with an axiomatic/geometric version of what physicists might call a super symmetric quantum field theory with minimal super symmetry.

We point out that these field theories are neither conformal (i.e. the functors depend on the Riemannian metric, not just its conformal class) nor chiral (i.e. the operators associated to the bordisms depend only smoothly, not holomorphically, on the Riemannian structures). As a consequence, it is a priori not clear why the partition function, i.e. the quantum field theory evaluated on 2-dimensional tori, should be $SL_2(\mathbb{Z})$-invariant or holomorphic. This all comes out of the right notion of super symmetry, a fact that seems to be well known in the physics community, and we thank Ed Witten for explaining it to us in the context of the super symmetric σ-model. Our own contribution is a *geometric definition* of super symmetry that implies that a certain square root exists (for an infinitesimal generator $\bar{L}_0$ of the operators associated to cylinders). It is the latter algebraic fact that's usually called super symmetry in the physics literature and it seems well known that it leads to a modular partition function.

In this introduction we will outline our definitions. In addition, we explain the relevance of this result for our program of relating 2-dimensional quantum field theories to the topological modular form spectrum TMF due to Hopkins and Miller.

A d-dimensional quantum field theory in the sense of Atiyah and Segal is a symmetric monoidal functor which associates a locally convex vector space $E(Y)$ to a closed oriented Riemannian manifold Y of dimension $(d-1)$

and a trace class operator $E(\Sigma)\colon E(Y_1) \to E(Y_2)$ to an oriented Riemannian bordism Σ from Y_1 to Y_2. If E is a 2-dimensional quantum field theory, then one obtains a complex valued function Z_E on the upper half plane $\mathfrak{h}$, by defining

$$Z_E(\tau) \overset{\text{def}}{=} E(T_\tau),$$

where $T_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ is the torus obtained by dividing the complex plane by the lattice $\mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$. This is the *partition function* of the quantum field theory E, compare definition 22.

Atiyah and Segal's only additional requirement for a quantum field theory is that the functor E is *continuous* in the usual sense, as a functor between topological categories. We strengthen this requirement to a notion of *smoothness* in the sense that will be explained in detail in section 3. It means that E can be extended to *smooth families*: for any smooth manifold S there is a functor E_S which associates to a smooth family Y of Riemannian manifolds parametrized by S (i.e. a fiber bundle $Y \to S$ equipped with a Riemannian metric along its fibers) a smooth family $E_S(Y)$ of locally convex vector spaces parametrized by S (i.e. $E_S(Y)$ is a locally trivial bundle of locally convex vector spaces over S). Similarly, E_S associates to a smooth family of Riemannian bordisms parametrized by S a smooth family of trace class operators parametrized by S. The collection of functors E_S is required to depend functorially on S. It turns out to be essential that this definition also takes into account families of *objects* in our categories, not just of morphisms, as is more common in the context of topological categories.

A *super symmetric* quantum field theory is defined completely analogously, only replacing (Riemannian) manifolds in the definition above by super (Riemannian) manifolds. More precisely, a *super symmetric quantum field theory* of dimension $d|q$ gives a symmetric monoidal functor E which associates a locally convex super vector space $E(Y)$ to a super Riemannian manifold of dimension $d-1|q$ and a trace class operator $E(\Sigma)\colon E(Y_1) \to E(Y_2)$ to a super Riemannian bordism Σ of dimension $d|q$ from Y_1 to Y_2. Here a *super Riemannian* manifold is a super manifold equipped with a super Riemannian structure, which we define in section 4 for super manifolds of dimension $d|q$ for $d = 1, 2$ and $q = 1$. Our terminology is motivated by the fact that a super Riemannian structure on a super manifold M of dimension $d|1$ induces a Riemannian metric on the reduced manifold M_{red} (an ordinary manifold of dimension d). Our definition is not the obvious generalization of a symmetric 2-tensor to super manifolds but is comes from a physicists point of view, where these structures should be useful to construct classical action functionals for certain super symmetric (classical) field theories (see remarks 57 and 65).

As for quantum field theories we require that the functor E describing a super symmetric quantum field theory is smooth, but it is essential that this smoothness condition is formulated by requiring that E does extend to families parametrized by *complex super manifolds*, see section 4 for a detailed definition. We note that ordinary manifolds give (complex) super manifolds of dimension $d|0$ and that a super symmetric quantum field theory leads to a quantum field theory as explained above.

Theorem 1. *The partition function of a super symmetric quantum field theory of dimension* $2|1$ *is a modular function with integral* q*-expansion. This continues to hold if one only has a super symmetric* flat *quantum field theory, see Remark 2.*

We recall that a modular function is a holomorphic function $f\colon \mathbb{R}^2_+ \to \mathbb{C}$ on the upper half plane, which is meromorphic at ∞ and is invariant under the usual $SL_2(\mathbb{Z})$-action on $\mathbb{R}^2_+$. The invariance implies in particular $f(\tau+1) = f(\tau)$ so that $f(\tau)$ can be expressed in the form

$$f(\tau) = \sum_{i \geq -N} a_i q^i \quad \text{where} \quad q = e^{2\pi i \tau}.$$

This is the *q-expansion* of f; it is *integral* if all coefficients a_i are integers. Our main theorem asserts the $SL_2(\mathbb{Z})$-invariance of Z_E, its holomorphicity, its meromorphicity at ∞, and the integrality of its q-expansion.

We want to point out that none of these four properties of the partition function is obvious; rather all of them are consequences of certain cancellations due to super symmetry. In fact, we believe that none of these statements holds true in general for (non-super symmetric) quantum field theories. Concerning the $SL_2(\mathbb{Z})$-invariance of Z_E, we note that for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ the torus $T_{\tau'}$ for $\tau' = \frac{a\tau+b}{c\tau+d}$ is conformally equivalent, but in general not isometric to T_τ. This implies that if E is a *conformal field theory* (for which the operators $E(\Sigma)$ are required to depend only on the conformal structure rather than the Riemannian metric on Σ), then the $SL_2(\mathbb{Z})$-invariance of the partition function is indeed obvious. A similar remark applies to the holomorphicity of the partition function under the assumption that the operators $E(\Sigma)$ depend holomorphically on Σ, i.e. if E is a *chiral* quantum field theory.

Theorem 1 above is an important step in our approach to understand the infinite loop space TMF, proven to exist by Hopkins-Miller and recently constructed by Lurie. This space gives the universal elliptic cohomology theory and is a topological version of modular forms. So by construction, there is a map to the ring MF of integral modular forms:

$$\pi_0 TMF \to \mathrm{MF}$$

which induces a rational isomorphism (and where the kernel and cokernel were completely calculated by Hopkins, Mahowald and Miller). We now believe that instead of studying conformal field theories as in [ST], one should define an infinite loop space $FQFT$ of super symmetric *flat* quantum field theories, together with a continuous map $F : FQFT \to TMF$ which leads to a commutative diagram

$$\begin{tikzcd} \pi_0 FQFT \arrow[rr, "F_*", "\cong"'] \arrow[dr, "Z"'] & & \pi_0 TMF \arrow[dl] \\ & \mathrm{MF} & \end{tikzcd}.$$

This would give a computation of the connected components of the space $FQFT$, otherwise a seemingly impossible task.

Remark 2. By the Gauss-Bonnet Theorem the only closed orientable 2-manifold with a flat Riemannian structure is the surface of genus one. Moreover, the set of isomorphism classes of such structures agrees with the moduli space of elliptic curves over $\mathbb{C}$ (up to rescaling, i.e. a factor $\mathbb{R}_+$). This is our reason to restrict to flat surface when trying to understand the space TMF.

One aspect that is not addressed in this paper are modular forms of weight $w \neq 0$. There is a notion of quantum field theories of *degree* n, $n \in \mathbb{Z}$, very similar to that in [ST]. We expect that a super symmetric version of these will make Theorem 1 hold true, where the degree and weight are related by the equation $n = 2w$. Moreover, there should be spaces $FQFT_n$ of degree n super symmetric flat quantum field theories that are deloopings of the space $FQFT$, and the above diagram should continue to hold on the level of $\pi_n FQFT \cong \pi_0 FQFT_{-n}$.

A final aspect of quantum field theories that needs to be included in order to define the space $FQFT$ above is the *space-time locality*. In [ST] we gave an approach to this aspect via 2-functors from the 2-category of $0-, 1-$ and

2-dimensional conformal manifolds to a 2-category algebras, bimodules and intertwiners. We believe that this continues to work in the super symmetric Riemannian context.

We also expect that examples of such fully fletched (super symmetric, local) quantum field theories are given by the super symmetric σ-models for Riemannian string manifolds as target. In this regard, it is an advantage not to have to check conformal invariance since it only seems to hold for Ricci flat targets. We expect that these σ-models will lead to a continuous map on the infinite loop space that represents string cobordism

$$\Omega^\infty MString \longrightarrow FQFT$$

which is our candidate for the family Witten genus. It should commute with the recently constructed Witten genus of Lurie (with values in TMF) via the map F used in the above diagram.

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2 Field theories a la Segal

As mentioned in the introduction, Graeme Segal has proposed an axiomatic definition of conformal field theories (CFT's) and quantum field theories (QFT's) as functors from a suitable bordism category to a category of locally convex (topological) vector spaces satisfying certain axioms. This proposal is described in his paper 'The definition of conformal field theory', which was circulated widely as a preprint for many years before it was published in the proceedings of the conference in honor of his sixtieth birthday [Se2]. We have been very much influenced by this important paper; refining his ideas we define a d-dimensional QFT as a functor

$$E\colon \mathrm{RB}_d \longrightarrow \mathrm{TV}$$

from the category of d-dimensional Riemannian spin bordisms (defined in subsection 2.3) to the category TV of locally convex (topological) vector spaces (defined in subsection 2.1). This functor is required to be compatible with the symmetric monoidal structure (given by disjoint union in the domain category and tensor product in the range category) and to be *smooth* in the sense to be explained in section 3.

The details of these definitions are new; in particular, the notion of a 'smooth functor' explained in section 3. We want to stress that unlike Segal's definition of a field theory in [Se2] and our definition in our earlier paper [ST], there are *no additional axioms for the functor E* besides the smoothness and the compatibility with the symmetric monoidal structure. We compare these definitions in more detail in remark **??**.

In the following two subsections we will describe the categories TV and RB_d. We provide more detail than might be necessary at this point; however, we try to phrase things in such a way that the generalization of these categories to their family versions (discussed in section 3) and their super versions (section 4) is straightforward.

There are many possible variants of the definition of the categories RB_d and TV; e.g., different choices for the geometric structure on the bordisms involved lead to variants of the bordism category RB_d. While the focus of [Se2] and [ST] is on *conformal field theories* (where the bordisms are equipped with conformal structures), in this paper we are interested in *quantum field theories* which corresponds to requiring Riemannian metrics on bordisms. We also require our bordisms to be equipped with a *spin structure* (rather than just an orientation, which is more usual), and we will build our category TV using $\mathbb{Z}/2$-graded vector spaces. The resulting kind of field theories are more closely related to super symmetric field theories.

2.1 The category TV

Definition 3. (The category TV**)**

objects are $\mathbb{Z}/2$-graded locally convex vector spaces;

morphisms are grading preserving continuous linear maps.

Definition 4. (The projective tensor product) If V, W are locally convex vector spaces, their algebraic tensor product $V \otimes_{\mathrm{alg}} W$ is a vector space which carries again a locally convex topology known as the *projective topology* which is characterized by the property that it is the finest locally convex topology such that the canonical bilinear map

$$V \times W \longrightarrow V \otimes_{\mathrm{alg}} W$$

is continuous (see [Koe, §41.2]). We denote by $V \otimes W$ the *projective tensor product* which is defined to be the completion of $V \otimes_{\mathrm{alg}} W$ w.r.t. the projective topology. A $\mathbb{Z}/2$-grading on V, W induces the usual $\mathbb{Z}/2$-grading on $V \otimes W$. The projective tensor product gives TV the structure of a symmetric monoidal category. As usual for graded vector spaces, the symmetry isomorphism

$$V \otimes W \cong W \otimes V \qquad \text{is given by} \qquad v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$$

for homogeneous elements $v \in V$, $w \in W$ of degree $|v|$ and $|w|$.

2.2 Preliminaries on spin structures

We recall that a *spin structure* on a Riemannian d-manifold M consists of an orientation of M together with a double covering $Spin(M) \to SO(M)$ of the oriented frame bundle

$$SO(M) \stackrel{\mathrm{def}}{=} \{(x, f) \mid x \in M,\ f\colon \mathbb{R}^d \to T_x M \text{ isometry}\} \xrightarrow{p} M,$$

such that the restriction to each fiber $p^{-1}(x) \subset SO(M)$ is a non-trivial double covering for $d \geq 2$.

We want to stress that spin structures on a manifold are best viewed as a *groupoid*, whose objects are spin structures as defined above, and whose morphisms are isomorphisms of double coverings. The set of isomorphism classes of spin structures on M is a torsor for $H^0(M; \mathbb{Z}/2) \oplus H^1(M; \mathbb{Z}/2)$ (i.e., this group acts freely and transitively on the set of isomorphism classes; the first summand acts by changing the orientation, the second by tensoring with double coverings pulled back from M). We want to alert the reader that in the following a spin structure is an *object* in this groupoid, not an isomorphism class of objects (the latter usage is quite common in the literature).

Example 5. (Examples of spin structures.) There is a *standard spin structure* on $M = \mathbb{R}^d$ given by the double covering

$$M \times Spin(d) \longrightarrow M \times SO(d) \cong SO(M);$$

here $SO(M)$ is the oriented frame bundle w.r.t. the standard orientation of $M = \mathbb{R}^d$; it is isomorphic to the trivial bundle $M \times SO(d)$ via the standard trivialization of the tangent bundle of $\mathbb{R}^d$.

We note that the translation action is compatible with this trivialization and hence the translation action on $SO(M)$ lifts to a translation action on $M \times Spin(d)$ (which acts trivially on the second summand). In particular if G is a discrete subgroup of $\mathbb{R}^d$, then the spin structure on $\mathbb{R}^d$ induces a spin structure on $\mathbb{R}^d/G$.

We note that up to isomorphism there are *two* spin structures on the circle $S^1 = \mathbb{R}/\mathbb{Z}$ equipped with its standard orientation; the spin structure induced by the standard spin structure on $\mathbb{R}$ is often called the *periodic spin structure* (since sections of the associated spinor bundle can be interpreted as periodic functions on $\mathbb{R}$). The circle equipped with this spin structure represents the non-trivial element in the spin bordism group $\Omega_1^{spin} \cong \mathbb{Z}/2$.

Similarly, up to isomorphism there are four spin structures on the torus $T = \mathbb{R}^2/\mathbb{Z}^2$ equipped with its induced orientation; the spin structure induced from $\mathbb{R}^2$ is often called the periodic-periodic spin structure and equipped with this spin structure the torus represents the non-trivial element in the spin bordism group $\Omega_2^{spin} \cong \mathbb{Z}/2$.

Definition 6. Let M, N be two Riemannian spin manifolds of dimension d. An *isometric spin embedding* from M to N is a pair $(f, \widehat{f_*})$, consisting of an isometric embedding $f\colon M \to N$ and a map $\widehat{f_*}\colon Spin(M) \to Spin(N)$ which covers the $SO(d)$-equivariant map $f_*\colon SO(M) \to SO(N)$ induced by the differential of f.

Example 7. The groups of isometries of $\mathbb{R}^d$ is the semi-direct product $\mathbb{R}^d \rtimes SO(d)$, where $\mathbb{R}^d$ acts by translations and $SO(d)$ by rotations. The group of spin isometries of $\mathbb{R}^d$ (i.e., invertible isometric spin embeddings $\mathbb{R}^d \to \mathbb{R}^d$) is the semi-direct product $\mathbb{R}^d \rtimes Spin(d)$. Here $\mathbb{R}^d$ acts on $Spin(\mathbb{R}^d) = \mathbb{R}^d \times Spin(d)$ by translations on the first factor and trivially on the second, while $Spin(d)$ acts on $\mathbb{R}^d$ by rotations (via the double cover $Spin(d) \to SO(d)$) and on $Spin(d)$ by left-multiplication.

(Spin structures on d-manifolds for $d = 1, 2$.) Spin structures on Riemannian manifolds M of dimension $d = 1, 2$ have the following alternative description that will be useful for us when relating them to super manifolds of dimension $d|1$ equipped with a super Riemannian structure (see **??**). For $d = 1$ (resp. $d = 2$), a spin structure on M can be described as a pair (L, α)

consisting of a real (resp. complex) line bundle $L \to M$ and a vector bundle isomorphism $\alpha\colon L^{\otimes 2} \xrightarrow{\cong} TM$ (this is an isomorphism of real vector bundles for $d = 2$). To see that a pair (L, α) determines a spin structure in the sense of definition **??** consider the map

$$q\colon L \to TM \qquad \text{given by} \qquad \ell \mapsto \alpha(\ell \otimes \ell).$$

Via the isomorphism α, the Riemannian metric on TM induces a metric on $L^{\otimes 2}$ (which is hermitian for $d = 2$), and hence on L. Restricting q to the sphere bundle $S(L)$ gives a double covering map

$$q\colon S(L) \longrightarrow q(S(L)) \subset S(TM).$$

As discussed below, the image $q(S(L))$ can be identified in both cases with the oriented orthonormal frame bundle $SO(M) \to M$ and hence the double covering above gives a spin structure. For $d = 1$, the image $q(S(L_x))$ consists of one of the two unit tangent vectors in T_xM; in particular, α determines an orientation of T_xM, and $q(S(L_x))$ can be thought of as the oriented orthonormal frame. For $d = 2$, the image $q(S(L))$ is the unit sphere bundle $S(TM)$, which in turn can be identified with the oriented orthonormal frame bundle $SO(M)$ by sending a unit tangent vector v to the frame (v, iv), where the complex structure and the orientation of TM is induced by that of $L^{\otimes 2}$ via the isomorphism α.

2.3 The Riemannian bordism category RB_d

Remark 8. Before defining the bordism category RB_d we want to motivate a special feature of it that distinguishes it from the usual bordism categories, namely its *asymmetry*. Suppose that Σ is an oriented bordism of dimension d from Y_1 to Y_2. Then Σ can also be interpreted as a bordism from Y_2^* to Y_1^*, where Y_i^* is the manifold Y_i equipped with the opposite orientation. In other words, on the oriented bordism category B_d (whose objects are closed $(d-1)$-manifolds and whose morphisms are oriented d-bordisms) there is an anti-involution ${}^*\colon \mathrm{B}_d \to \mathrm{B}_d$. Our bordism category RB_d is *asymmetric* in the sense that there is no such anti-involution on RB_d (unlike e.g. the bordism category $\mathcal{B}^d$ in our survey paper, which is equipped with the anti-involution * [ST, Def. 2.1.1]).

This is motivated by a similar asymmetry of the category TV. We note that sending a locally convex space V to its continuous dual V^* is not strictly speaking an anti-involution of the category TV; even after picking one of the many possibly topologies (weak, strong,etc.) on V^*, the double dual $(V^*)^*$ is at best naturally isomorphic to V. After restricting to reflexive spaces, and after picking a topology for the dual, we obtain a weak anti-involution. However, this anti-involution will not be compatible with tensor products. E.g., the evaluation map provides us with a natural continuous map

$$ev\colon V^* \otimes V \longrightarrow \mathbb{C};$$

however, there is no natural continuous map $\mathbb{C} \to V \otimes V^*$ (except if V is finite-dimensional; in which case it is given by $1 \mapsto \sum_i e_i \otimes e_i^*$, where $\{e_i\}$ is a basis of V and $\{e_i^*\}$ is the dual basis of V^*). The map $ev^*\colon (V^* \otimes V)^* \to \mathbb{C}^* = \mathbb{C}$ dual to ev is *not* such a map since $(V^* \otimes V)^*$ is typically not isomorphic to $V \otimes V^*$ (with some assumptions on the topological vector spaces V, W the dual of the *projective* tensor product $V \otimes W$ is isomorphic to the *injective* tensor product of V^* and W^*).

The following terminology will be convenient for our definition of the bordism category RB_d.

(Ends of a space and metric completions.) We recall that for a topological space Y the set $e(Y)$ of *ends of* Y is the inverse limit

$$e(Y) \stackrel{\text{def}}{=} \lim_{\overleftarrow{K}} \pi_0(Y \setminus K),$$

taken over all compact subspaces $K \subset Y$. There is a natural topology on the set

$$bY \stackrel{\text{def}}{=} Y \amalg e(Y)$$

which makes bY a compact space, called the *Freundenthal compactification of* Y.

If Y is a metric space, its *metric completion* $\bar{Y}$ is a complete metric space, whose points are equivalence classes of Cauchy sequences in Y. There is a continuous map

$$p\colon \bar{Y} \longrightarrow bY \tag{9}$$

which sends a Cauchy sequence in $Y \subset bY$ to its limit in bY (every sequence in Y has an accumulation point in the compact space bY; for a Cauchy sequence, there is only one accumulation point which is its limit).

In general, there is little relationship between the Freundenthal compactification bY which depends only on the *topology* of Y, and $\bar{Y}$ which depends on the metric involved; e.g., $Y = \mathbb{R}$ with its usual metric is complete

Definition 10. (The category Riem_d **of Riemannian spin** d**-manifolds.)** An object of Riem_d is a smooth Riemannian spin d-manifold Y without boundary, but not necessarily compact, which comes equipped with a decomposition of its set of ends as a disjoint union

$$e(Y) = e^l(Y) \amalg e^r(Y)$$

into *left ends* $e^l(Y)$ and *right ends* $e^r(Y)$. Here the words 'left' and 'right' just refer to on which side we draw the ends in our pictures of objects of Riem_d. We define subsets of $\delta Y = Y \setminus Y$ by

$$\delta^l(Y) \stackrel{\text{def}}{=} p^{-1}(e^l(Y)) \subset \delta Y \qquad \delta^r(Y) \stackrel{\text{def}}{=} p^{-1}(e^r(Y)) \subset \delta Y$$

in other words, $\delta^l(Y)$ (resp. $\delta^r(Y)$) is the subspace of the metric completion given by Cauchy sequences in Y which converge to points in $e^l(Y)$ (resp. $e^r(Y)$). We set $\bar{Y}^l \stackrel{\text{def}}{=} Y \cup \delta^l Y$ and $\bar{Y}^r \stackrel{\text{def}}{=} Y \cup \delta^r Y$.

So far, the roles of the 'left ends' and the 'right ends' have been completely symmetric; now we remove that symmetry by the following
Requirement. $\bar{Y}^l$ is a topological manifold with interior Y and boundary $\delta^l Y \stackrel{\text{def}}{=} p^{-1}(e^l(Y))$. The map $p_{\delta^l Y}\colon \delta^l Y \to e^l(Y)$ induces a bijection between $\pi_0 \delta^l Y$ and $e^l(Y)$.

Here is a picture of an object of Riem_2.

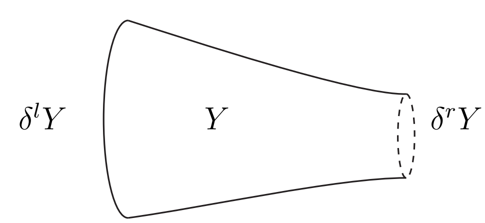


Figure 1: **An object of** Riem_2

In this example, the above requirement for left ends of Y also holds for the right ends of Y. An explicit example of an object of Riem_2 where that isn't the case is given by $Y = S^1 \times (0, \infty)$ with its standard metric; Y has two ends, which can be identified with 0 and ∞ in an obvious way, and we declare 0 to be a 'left end' and ∞ to be a 'right end' (which corresponds to the usual way of drawing the interval $(0,\infty)$). The metric completion $\bar{Y}$ can be identified with $S^1 \times [0,\infty)$ and we have $\delta^l Y = S^1 \times \{0\}$ and $\delta^r Y = \emptyset$. In particular, Y satisfies the requirement on left end above, but not the analogous requirement for right ends.

Here is another example of an object of Riem_2 for which $\delta^l Y$ is not a smooth $(d-1)$-manifold, and $\delta^r Y$ isn't a $(d-1)$-manifold at all. Let $\Delta^d \subset \mathbb{R}^d$ be a d-dimensional simplex whose vertices are $d+1$ points in general position such that the origin is in the interior of Δ^d. Let Y be the interior of Δ^d without the origin. Then the metric completion $\bar{Y}$ consists of the union of Y and $\delta Y = \partial\Delta^d \cup \{0\}$. There are two ends and $p_{|\delta Y}\colon \delta Y \to e(Y)$ a bijective correspondence between $\pi_0(\delta Y)$ and $e(Y)$. We note that neither component of δY is a smooth $(d-1)$-manifold, but $\partial\Delta$ is a topological $(d-1)$-manifold, while $\{0\}$ isn't. In particular, we obtain an object of Riem_2 if we declare the end given by $\partial\Delta^d$ to be a left end, and the other end a right end.

If Y_0 and Y_1 are objects of Riem_d, the *morphisms* from Y_0 to Y_1 are Riemannian spin embeddings $f\colon Y_0 \hookrightarrow Y_1$.

We want to emphasize that we *do not* require any compatibility of this map with the decomposition of the ends of Y_i into 'left' and 'right'. In general, a Riemannian embedding $f\colon Y_0 \to Y_1$ does not induce a map $bY_0 \to bY_1$ between their Freudenthal compactifications; in particular, there is no 'induced map' from the ends of Y_0 to the ends of Y_1 (consider e.g. the inclusion of intervals $(-1,1) \hookrightarrow (-2,2)$ equipped with the standard metrics). If f is *proper*, then it extends to a unique continuous map $f\colon bY_0 \to bY_1$.

We call a morphism $f\colon Y_0 \to Y_1$ *left-proper* (resp. *right-proper*) if f extends to a continuous map $b^l Y_0 \to b^l Y_1$ (resp. $b^r Y_0 \to b^r Y_1$) which induces a bijection on left ends (resp. right ends).

We note that the continuity of f implies that it extends to a continuous map $\bar{f}\colon \bar{Y}_0 \to \bar{Y}_1$; this map in general does not map $\delta Y_0 \to \delta Y_1$; an example is provided by the inclusion $(-1,1) \hookrightarrow (-2,2)$. We also observe that $\bar{f}$ isn't necessarily injective anymore as the inclusion $(0,1) \amalg (1,2) \hookrightarrow (0,2)$ shows.

Here is a typical picture of an object Y of RB^2. The manifold Y is non-compact; the dotted circle on the right hand side is not part of Y; rather, Y can be compactified by adding that circle. We think of ∂Y as the primary datum; the role of Y is mostly to provide a neighborhood of ∂Y equipped with the necessary geometric data to make gluing possible. We will see that if $U \subset Y$ is an open neighborhood of $\partial Y \subset Y$, then U and Y are isomorphic as objects of RB_d; in other words, what matters is the germ of the neighborhood of ∂Y provided by Y.

Definition 11. (The Riemannian spin bordism category RB_d.) The objects of RB_d are just the objects of the category Riem_d defined above. The morphisms from Y_0 to Y_1 are equivalence classes of Riemannian spin bordisms from Y_0 to Y_1; here a *Riemannian spin bordism* is a diagram

$$U_1 \overset{\iota_1}{\hookrightarrow} \Sigma \overset{\iota_0}{\hookleftarrow} U_0$$

in the category Riem_d, where $U_i \subset Y_i \subset \bar{Y}_i$ is an open neighborhood of $\delta_{\mathrm{out}} Y_i$ whose metric completion $\bar{U}_i$

Σ is an object of Riem_d;

$\iota_i \colon U_i \hookrightarrow \Sigma$ is an isometric spin embedding (i.e., a morphism of Riem^d) of some open neighborhood $U_i \subset Y_i$ of the boundary $\partial Y_i \subset Y_i$.

We require that

- $\iota_0 \colon U_0 \to \Sigma$ is *proper* and it induces a bijection between the ends of U_0 and Σ;
- $\iota_1 \colon U_1 \to \Sigma$ restricts to a homeomorphism $\partial U_1 \xrightarrow{\cong} \partial \Sigma$.

Here is a picture of a Riemannian bordism; we usually draw the domain of the bordism to the right of its range, since we want to read compositions of bordisms, like compositions of maps, from right to left.

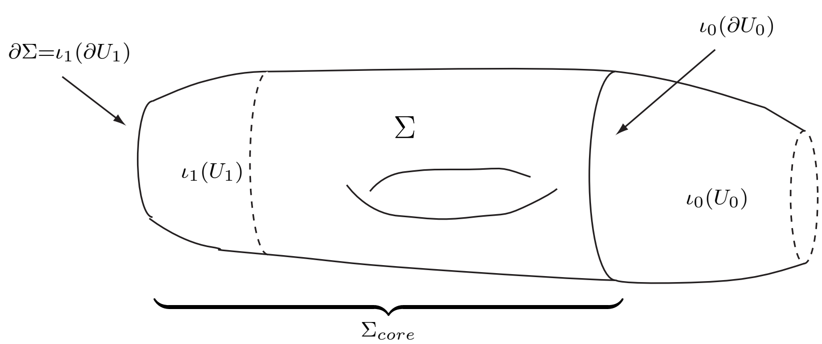


Figure 2: **A Riemannian bordism**

We will call $\Sigma_{core} \overset{\text{def}}{=} \Sigma \setminus \iota_0(\mathrm{int}(U_0))$ the *core* of Σ, where $\mathrm{int}(U_0)$ is the interior of U_0. We note that the second condition above implies that Σ_{core} is compact. In particular, if $\iota_0(\partial U_0)$ does not intersect $\iota_1(\partial U_1)$, then Σ_{core} is a bordism between $\partial Y_0 = \partial U_0$ and $\partial Y_1 = \partial U_1$ in the usual sense. We will refer to $\iota_0(\mathrm{int}(U_0))$ as the *incoming neighborhood* (associated to the domain of the morphism) and to $\iota_1(\mathrm{int}(U_1))$ as the *outgoing neighborhood* (associated to the range). We note the asymmetry between domain and range of this bordism (which is a desired feature as explained in remark 8): the incoming neighborhood is *outside* the core bordism; we think of it as a 'tab' which will be essential for composing bordisms by gluing then.

Now suppose that $(\Sigma, \iota_0, \iota_1)$ and $(\Sigma', \iota_0', \iota_1')$ are two Riemannian spin bordisms from Y_0 to Y_1 with $U_0 \subset U_0'$, $U_1 \subset U_1'$ and that there is Riemannian spin embedding $F\colon \Sigma \hookrightarrow \Sigma'$ (i.e., a morphism in Riem^d) which makes the following diagram commutative

$$\begin{tikzcd}
U_1 \arrow[r, hook, "\iota_2"] \arrow[d, hook] & \Sigma \arrow[d, hook, "F"'] & U_0 \arrow[l, hook', "\iota_0"'] \arrow[d, hook] \\
U_1' \arrow[r, hook, "\iota_1'"] & \Sigma' & U_0' \arrow[l, hook', "\iota_0'"']
\end{tikzcd}$$

Then we declare $(\Sigma, \iota_0, \iota_1)$ to be equivalent to $(\Sigma', \iota_0', \iota_1')$. A *morphism* from Y_0 to Y_1 is an equivalence class of Riemannian spin bordisms with respect to the equivalence relation generated by the relation just described.

composition of Riemannian spin bordisms is given by gluing as shown in the picture below.

More precisely, let $(\Sigma', \iota_0', \iota_1')$ be a Riemannian spin bordism from Y_0 to Y_1 and $(\Sigma, \iota_1, \iota_2)$ a Riemannian spin bordism from Y_1 to Y_2. Without loss of generality, we can assume that the domains of the spin isometries ι_1' and ι_1 agree (both of which are open neighborhoods of $\partial Y_1 \subset Y_1$); suppose $U_1 \subset Y_1$ is this common domain. Then identifying $\iota_1'(U_1) \subset \Sigma'$ with $\iota_1(U_1) \subset \Sigma$ via the isometry $\iota_1 \circ (\iota_1')^{-1}$ gives the Riemannian spin manifold

$$\Sigma'' \overset{\text{def}}{=} \Sigma \cup_{U_1} \Sigma'.$$

Together with the isometric spin embeddings

$$\iota_2''\colon U_2 \xrightarrow{\iota_2} \Sigma \subset \Sigma'' \qquad \text{and} \qquad \iota_0''\colon U_0' \xrightarrow{\iota_0'} \Sigma' \subset \Sigma''$$

this is a Riemannian bordism from Y_0 to Y_2.

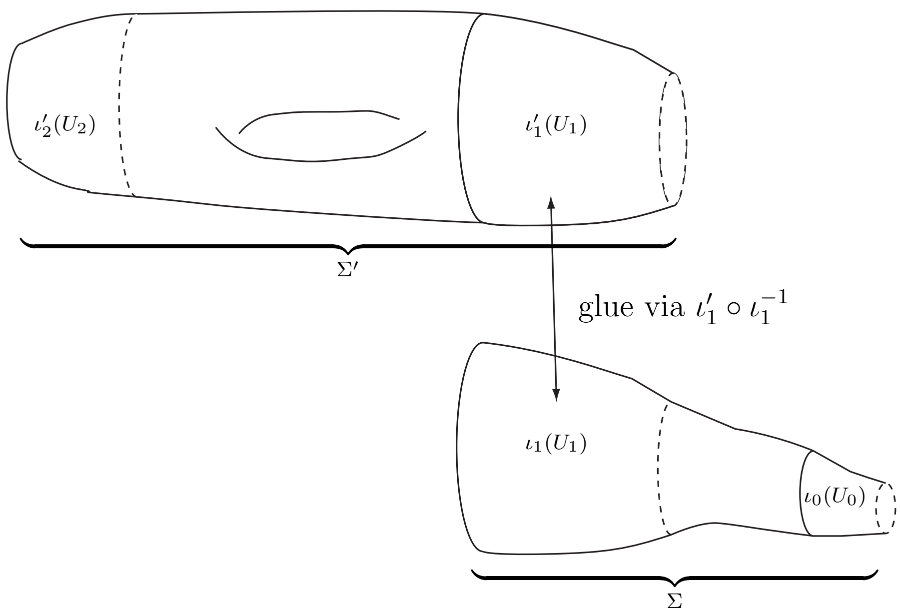


Figure 3: **The composition** $\Sigma'' = \Sigma \circ \Sigma'$ **in** RB^2

Example 12. Examples of morphisms in the category RB_d**.**

Identity morphism. If Y is an object of RB_d, then

$$Y \xhookrightarrow{\mathrm{id}} \Sigma = Y \xhookleftarrow{\mathrm{id}} Y$$

is a Riemannian bordism which represents the identity endomorphism of U.

Isomorphism. Let Y_0, Y_1 be Riemannian spin d-manifolds (i.e., objects of Riem^d) and suppose that there are open neighborhoods U_i of $\partial Y_i \subset Y_i$ for $i = 0, 1$ and an isomorphism $f\colon U_0 \hookrightarrow U_1$ in the category Riem^d. Then

$$U_1 \xhookrightarrow{\mathrm{id}} \Sigma = U_1 \xhookleftarrow{f} U_0$$

represents a morphism from Y_0 to Y_1 in RB_d which obviously is an *isomorphism*.

Two-sided neighborhood. Let Σ, Y_1, Y_2 be objects of Riem^d, and let $\iota_i\colon Y_i \hookrightarrow \Sigma$ be morphisms such that $\iota_1(\partial Y_1) = \iota_2(\partial Y_2)$ is a hypersurface in Σ (not necessarily smooth) such that

$$\iota_1(Y_1) \cup \iota_2(Y_2) = \Sigma \qquad \iota_1(\mathrm{int}(Y_1)) \cap \iota_2(\mathrm{int}(Y_2)) = \emptyset$$

as shown in the figure below. We think of Σ is a two-sided neighborhood

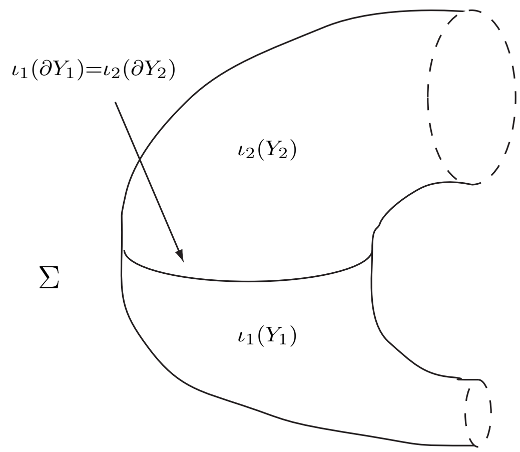


Figure 4: **A 2-sided neighborhood as morphism in** RB^2

of $\iota_1(\partial Y_1) = \iota_2(\partial Y_2)$; it gives rise to the following morphism from $Y_1 \amalg Y_2$ to $\emptyset$:

$$\emptyset \hookrightarrow \Sigma \xleftarrow{\iota_1 \amalg \iota_2} Y_1 \amalg Y_2\,.$$

2.4 Examples of objects and morphisms of RB_d

In this subsection we introduce some objects and morphisms of RB_d for $d = 1, 2$. Moreover, we show some relations between these morphisms. Analogous relations in the super bordism category SRB_d (see Example 81 and Lemmas 82 and 83) will play a central role in the proof of our main theorem.

Example 13. (Examples of objects and morphisms of RB^1.)

the points $\mathrm{pt}, \bar{\mathrm{pt}} \in \mathrm{RB}^1$**.** We define objects $\mathrm{pt}, \bar{\mathrm{pt}} \in \mathrm{RB}^1$ as follows:

$$\mathrm{pt} \stackrel{\mathrm{def}}{=} [0, \infty) \qquad \bar{\mathrm{pt}} \stackrel{\mathrm{def}}{=} (-\infty, 0],$$

where the real line $\mathbb{R}$ is equipped with its standard Riemannian metric and its standard spin structure. We call these objects 'points' since we think of them as consisting of $\{0\}$ equipped with a one-sided collar. We note that pt, $\bar{\text{pt}}$ are not isomorphic as objects of RB^1 (the isometry $[0,\infty) \to (-\infty,0]$, $s \mapsto -s$ is not spin structure preserving).

the circle $S^1_\ell \in \text{RB}^1(\emptyset,\emptyset)$. For $\ell \in \mathbb{R}_+$ let

$$S^1_\ell \stackrel{\text{def}}{=} \mathbb{R}/\ell\mathbb{Z}$$

be the *circle of length* ℓ, equipped with the Riemannian metric (resp. spin structure) induced by the standard Riemannian metric (resp. spin structure) on $\mathbb{R}$. This is a Riemannian bordism from $\emptyset$ to $\emptyset$ and represents the nontrivial element of $\Omega_1^{spin} \cong \mathbb{Z}/2$.

the interval $I^1_\ell \in \text{RB}^1(\emptyset, \text{pt} \amalg \bar{\text{pt}})$. For $\ell \in (0,\infty)$ let R^1_ℓ be the morphism

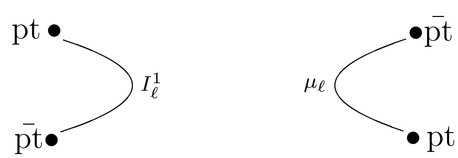
$$[0,\ell/2) \amalg (-\ell/2,0] \xhookrightarrow{\mathrm{id}\amalg\ell} [0,\ell] \hookleftarrow \emptyset\,,$$

where abusing notation we write $\ell\colon \mathbb{R} \to \mathbb{R}$ for the translation by ℓ (i.e., $s \mapsto s+\ell$).

the pairing $\mu_\ell \in \text{RB}^1(\bar{\text{pt}} \amalg \text{pt}, \emptyset)$. For $\ell \in [0,\infty)$ let μ^1_ℓ be the following morphism in RB^1 from $\bar{\text{pt}} \amalg \text{pt}$ to $\emptyset$.

$$\emptyset \hookrightarrow \mathbb{R} \xhookleftarrow{\mathrm{id}\amalg\ell} (-\infty,0] \amalg [0,\infty)\,.$$

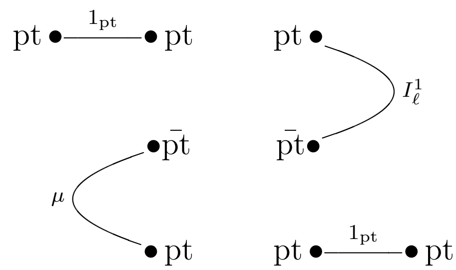
Remark 14. Pictorially we represent the morphisms I^1_ℓ and μ_ℓ by the following pictures (to be read from right to left):



We note that there is an important difference between the interval I^1_ℓ and the pairing μ_ℓ which reflects the asymmetry in our bordism category RB_d: the pairing μ_ℓ is defined for $\ell = 0$, while the interval I^1_ℓ is not. The reader might wonder why we chose to consider the interval of length ℓ as morphism $I^1_\ell\colon \emptyset \to \text{pt} \amalg \bar{\text{pt}}$ rather than as morphism $\widehat{I}^1_\ell\colon \text{pt} \to \text{pt}$. The answer is that I^1_ℓ is more *basic* then $\widehat{I}^1_\ell$ in the sense that $\widehat{I}^1_\ell$ can be expressed in terms of I^1_ℓ, the pairing $\mu \stackrel{\text{def}}{=} \mu_0$ and the monoidal structure (but not vice versa!). It is straightforward to show that $\widehat{I}^1_\ell$ is equal to the following composition:

$$\mathrm{pt} = \emptyset \amalg \mathrm{pt} \xrightarrow{I^1_\ell \amalg 1_{\mathrm{pt}}} \mathrm{pt} \amalg \bar{\mathrm{pt}} \amalg \mathrm{pt} \xrightarrow{1_{\mathrm{pt}} \amalg \mu} \mathrm{pt} \amalg \emptyset = \mathrm{pt}$$

In pictures, this composition looks as follows:



Similarly, we can express μ_ℓ in terms of μ and I^1_ℓ, as well as the endomorphism of $\bar{\mathrm{pt}}$ given by an interval of length ℓ.

Next we want to state some relations between the morphisms S^1_ℓ, I^1_ℓ and μ in the category RB^1. To do so, it will be convenient to use the following construction.

Definition 15. Let $\mathcal{C}$ be a symmetric monoidal category and let $\mu\colon Z \amalg Y \to \emptyset$ be a morphism in $\mathcal{C}$. Here we write $\amalg$ for the monoidal structure in $\mathcal{C}$, and $\emptyset$ for the unit object. Motivated by the case where $\mathcal{C}$ is the category of vector spaces, we think of μ as a *pairing* between Z and Y in the category $\mathcal{C}$. Then the morphism set $\mathcal{C}(\emptyset, Y \amalg Z)$ is a monoid with multiplication

$$\circ_\mu \colon \mathcal{C}(\emptyset, Y \amalg Z) \times \mathcal{C}(\emptyset, Y \amalg Z) \longrightarrow \mathcal{C}(\emptyset, Y \amalg Z)$$

given by defining $f \circ_\mu g$ for $f, g \in \mathcal{C}(\emptyset, Y \amalg Z)$ to be the composition

$$\emptyset = \emptyset \amalg \emptyset \xrightarrow{f \amalg g} Y \amalg Z \amalg Y \amalg Z \xrightarrow{\mathrm{id} \amalg \mu \amalg \mathrm{id}} Y \amalg Z$$

Lemma 16. *For $\ell, \ell' \in \mathbb{R}_+$ the following relations hold in the category* RB^1*:*

1. $I^1_\ell \circ_\mu I^1_{\ell'} = I^1_{\ell+\ell'} \in \mathrm{RB}^1(\emptyset, \mathrm{pt} \amalg \bar{\mathrm{pt}})$*;*

2. S^1_ℓ is equal to the composition

$$\emptyset \xrightarrow{I^1_\ell} \mathrm{pt} \amalg \bar{\mathrm{pt}} \xrightarrow{\tau} \bar{\mathrm{pt}} \amalg \mathrm{pt} \xrightarrow{\mu} \emptyset,$$

where τ is the symmetry isomorphism.

Example 17. (Examples of objects and morphisms of RB^2.) The circle S^1_ℓ does double duty in the bordism categories RB_d as endomorphism of $\emptyset \in \mathrm{RB}^1$ as well as object in RB^2; we will use the same notation in both situations, hoping that the context will make the meaning clear.

the circles $S^1_\ell, \bar{S}^1_\ell \in \mathrm{RB}^2$. Let $\mathbb{R}^2_\geq = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$ (resp. $\mathbb{R}^2_\leq = \{(x,y) \in \mathbb{R}^2 \mid y \leq 0\}$ be the closed upper (resp. lower) half plane equipped with the standard Riemannian metric and spin structure. These are not objects in RB^2 (since $\partial\mathbb{R}^2_\geq = \partial\mathbb{R}^2_\leq = \mathbb{R}$ is not compact), but we obtain an object by taking the quotient with respect to the translation action of $\mathbb{Z}\ell \subset \mathbb{R}$ for some $\ell \in \mathbb{R}_+$. We note that $\partial\mathbb{R}^2_\geq/\ell\mathbb{Z} = \partial\mathbb{R}^2_\leq/\ell\mathbb{Z} = \mathbb{R}/\mathbb{Z}\ell$ is a Riemannian circle of length ℓ. This motivates the notation

$$S^1_\ell \stackrel{\mathrm{def}}{=} \mathbb{R}^2_\geq/\mathbb{Z}\ell \qquad \bar{S}^1_\ell \stackrel{\mathrm{def}}{=} \mathbb{R}^2_\leq/\mathbb{Z}\ell$$

the torus $T^2_{\ell,\tau} \in \mathrm{RB}^2(\emptyset,\emptyset)$. For $\ell \in \mathbb{R}_+$ and $\tau \in \mathbb{R}^2_+$, consider the closed 2-manifold

$$T_{\ell,\tau} \stackrel{\mathrm{def}}{=} \mathbb{R}^2/\ell(\mathbb{Z}\tau + \mathbb{Z}1).$$

This torus has a Riemannian metric (resp. spin structure) induced by the standard Riemannian metric (resp. spin structure) on $\mathbb{R}^2$, and hence it is a Riemannian spin bordism from $\emptyset$ to $\emptyset$; it represents the non-trivial element of $\Omega_2^{spin} \cong \mathbb{Z}/2$.

the cylinder $C^2_{\ell,\tau} \in \mathrm{RB}^2(\emptyset, S^1_\ell \amalg \bar{S}^1_\ell)$. For $\ell \in \mathbb{R}_+$ and $\tau = (\tau_1,\tau_2) \in \mathbb{R}^2_+$ the morphism $C^2_{\ell,\tau}$ is given by the Riemannian bordism

$$(\mathbb{R} \times [0, \ell\tau_2/2))/\mathbb{Z}\ell \amalg (\mathbb{R} \times (-\ell\tau_2/2, 0])/\mathbb{Z}\ell$$
$$\xhookrightarrow{\mathrm{id} \amalg \ell\tau} (\mathbb{R} \times [0, \ell\tau_2])/\mathbb{Z}\ell \hookleftarrow \emptyset,$$

where $\ell\tau \colon \mathbb{R}^2 \to \mathbb{R}^2$ is translation by $\ell\tau \in \mathbb{R}^2$.

the pairing $\mu \in \mathrm{RB}^2(S^1_\ell \amalg \bar{S}^1_\ell, \emptyset)$ is represented by the Riemannian bordism

$$\emptyset \hookrightarrow \mathbb{R}^2/\mathbb{Z}\ell \xleftarrow{\mathrm{id} \amalg \mathrm{id}} \mathbb{R}^2_{\geq}/\mathbb{Z}\ell \amalg \mathbb{R}^2_{\leq}/\mathbb{Z}\ell \ ,$$

Lemma 18. *For $\ell \in \mathbb{R}_+$, $\tau, \tau' \in \mathbb{R}^2_+$ the following relations hold in the category RB^2:*

1. $C^2_{\ell,\tau} \circ_\mu C^2_{\ell,\tau'} = C^2_{\ell,\tau+\tau'} \in \mathrm{RB}^2(\emptyset, S^1_\ell \amalg \bar{S}^1_\ell)$;
2. $C^2_{\ell,\tau+1} = C^2_{\ell,\tau} \in \mathrm{RB}^2(S^1_\ell, S^1_\ell)$;
3. $T^2_{\ell,\tau} \in \mathrm{RB}^2(\emptyset, \emptyset)$ *is equal to the composition*

$$\emptyset \xrightarrow{C^2_{\ell,\tau}} S^1_\ell \amalg \bar{S}^1_\ell \xrightarrow{\tau} \bar{S}^1_\ell \amalg S^1_\ell \xrightarrow{\mu} \emptyset,$$

4. $T^2_{g(\ell,\tau)} = T^2_{\ell,\tau} \in \mathrm{RB}^2(\emptyset, \emptyset)$ *for every* $g \in SL_2(\mathbb{Z})$;

Here $g(\ell, \tau) \in \mathbb{R}^2_+$ in part 4 is defined by

$$g(\ell, \tau) \overset{\text{def}}{=} (\ell|c\tau + d|, \frac{a\tau + b}{c\tau + d}) \qquad \text{for} \qquad g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right). \tag{19}$$

This describes an action of $SL_2(\mathbb{Z})$ on $\mathbb{R}_+ \times \mathbb{R}^2_+$. The space

$$\mathbb{R}_+ \times \mathbb{R}^2_+ \cong GL_2^+(\mathbb{R})/SO(2)$$

is the moduli space of flat Riemannian structures on the torus with a basis of its first homology group via $(\ell, \tau) \mapsto T^2_{\ell,\tau}$. Points in the same orbit lead to isometric tori; the converse holds as well and hence the quotient $SL_2(\mathbb{Z})\backslash(\mathbb{R}_+ \times \mathbb{R}^2_+)$ can be interpreted geometrically as the moduli space (stack) of flat Riemannian tori.

The proof of these relations is straightforward. We will prove a family version of these relations (Lemmas 47 and 48) below.

2.5 Quantum field theories, preliminary definition

Definition 20. (Preliminary.) A *quantum field theory of dimension d* is a symmetric monoidal functor

$$E\colon \mathrm{RB}_d \longrightarrow \mathrm{TV}\,.$$

Remark 21. The above definition of a QFT is simpler than the one we suggested in our survey paper [ST], where the functor was required to be compatible with additional (anti-) involutions and a natural transformation we called 'adjunction transformation'. The asymmetric definition of the bordism category RB_d allows us to interpret a 2-sided neighborhood of a closed $d-1$-manifold as a 'pairing' μ (see Example 12 (two-sided neighborhood), 13 (the pairing μ), and 17(the pairing μ)). Our old 'adjunction transformation' in the bordism category can then be a expressed in terms of these pairings and similarly, the adjunction transformation in the algebraic range category is obtained from the pairing $E(\mu)$; then the compatibility of the functor with the adjoint transformation is simply a consequence of functoriality.

The above definition is quite similar to Graeme Segal's definition of a QFT in the 'postscript' to his paper [**?**], where he comments on how his point of view has changed since that paper was originally written. Our definition is more compact since our notion of bordism incorporates both, usual bordisms and well as diffeomorphism which Segal treats separately. This seems necessary when we require that a QFT is a 'smooth' functor in section 3, since our family bordism category should contain families that for example contain annuli as well as diffeomorphism (which are infinitesimally thin annuli).

Definition 22. Let $E\colon\ \mathrm{RB}_d \to \mathrm{TV}^{\pm}$ be a QFT of dimension $d = 1$ or $d = 2$, and let $E^+\colon\ \mathrm{RB}_d \to \mathrm{TV}$ be the composition of E with the forgetful functor $\mathrm{TV}^{\pm} \to \mathrm{TV}$ (see Remark **??**). For $d = 1$, the *partition function* of E is the function

$$Z_E\colon \mathbb{R}_+ \longrightarrow \mathbb{C} \quad \text{defined by} \quad \ell \mapsto E^+(S^1_\ell),$$

where $S^1_\ell \in \mathrm{RB}^1(\emptyset,\emptyset)$ is the circle of length ℓ (see example 13), and we used that $E^+(\emptyset) = \mathbb{C} \in \mathrm{TV}$ and $\mathrm{TV}(\mathbb{C},\mathbb{C}) = \mathbb{C}$.

For $d = 2$, the *extended partition function* of E is the function

$$Z_E\colon \mathbb{R}_+ \times \mathbb{R}^2_+ \longrightarrow \mathbb{C} \quad \text{defined by} \quad (\ell,\tau) \mapsto E^+(T^2_{\ell,\tau}),$$

where $T^2_{\ell,\tau} \in \mathrm{RB}^2(\emptyset,\emptyset)$ is the torus $\mathbb{R}^2/\ell(\mathbb{Z}\tau+\mathbb{Z}1)$ (see example 17). The *partition function* of E is the function $Z_E\colon \mathbb{R}^2_+ \to \mathbb{C}$ obtained by restricting to $\ell = 1$.

The upper half-plane $\mathbb{R}^2_+$ parametrizes all tori up to conformal equivalence (not uniquely); it does not parametrize all tori equipped with Riemannian metrics up to isometry. However, every flat torus is isometric to $T_{\ell,\tau}$ for some $(\ell,\tau) \in \mathbb{R}_+ \times \mathbb{R}^2_+$. Hence, while the upper half plane is the appropriate domain for partition functions for *conformal* field theories, for QFT's it is better to work with the larger domain $\mathbb{R}_+ \times \mathbb{R}^2_+$, since we obtain invariance for the extended partition function defined on $\mathbb{R}_+ \times \mathbb{R}^2_+$ (see Lemma 23).

It is well-known that for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ the tori $\mathbb{R}^2/(\mathbb{Z}\tau+\mathbb{Z}1)$ and $\mathbb{R}^2/(\mathbb{Z}g\tau+\mathbb{Z}1)$ are conformally equivalent, where $g\tau = \frac{a\tau+b}{c\tau+d}$. In particular, if E is a *conformal* field theory (i.e., the operators $E(\Sigma)$ depend only on the conformal structure on Σ), then its partition function $Z_E(\tau)$ is invariant under the $SL_2(\mathbb{Z})$-action on the upper half plane $\mathbb{R}^2_+$. However, these tori are not isometric, and hence the partition function of a QFT is usually not invariant.

What is still true is that for $g \in SL_2(\mathbb{Z})$ the torus $T_{g(\ell,\tau)}$ is isometric to $T_{\ell,\tau}$ (see part 4 of Lemma 18; the $SL_2(\mathbb{Z})$-action on $\mathbb{R}_+ \times \mathbb{R}^2_+$ is defined in equation (19)). This implies:

Lemma 23. *The extended partition function $Z_E\colon \mathbb{R}_+ \times \mathbb{R}^2_+ \to \mathbb{C}$ is invariant under the $SL_2(\mathbb{Z})$-action on $\mathbb{R}_+ \times \mathbb{R}^2_+$.*

In section 5 we will show that for a *super symmetric* QFT E, its extended partition function $Z_E(\ell,\tau)$ is in fact *independent* of ℓ. In particular, the above corollary implies that its partition function $Z_E(\tau) = Z_E(1,\tau)$ is invariant under the $SL_2(\mathbb{Z})$-action.

2.6 Consequences of the axioms

The goal of this subsection is to prove the following results which will be needed in the proof of our main theorem.

Proposition 24. *Let E be a d-dimensional QFT, and let $\Sigma \in \mathrm{RB}_d(Y_1,Y_2)$. Then the associated operator $E(\Sigma)^+\colon E(Y_1)^+ \to E(Y_2)^+$ is nuclear (see definition 26 below).*

Proposition 25. *Let E be a d-dimensional QFT, let $\Sigma \in \mathrm{RB}_d(Y,Y)$, and let $\widehat{\Sigma}$ be the closed Riemannian spin manifold obtained by gluing the two boundary copies of Σ. If we assume that the locally convex vector space $E(Y)^+$ has the approximation property (see Definition 27), then*

$$E(\widehat{\Sigma})^+ = \operatorname{str} E(\Sigma)^+,$$

where $\operatorname{str} E(\Sigma)^+$ *is the super trace of the operator* $E(\Sigma)^+$ *(see Definition 27).*

Definition 26. (Nuclear operators). Let V, W be locally convex vector spaces, and let V' be the dual of V. Then the *finite rank operators* are the continuous linear maps $T\colon V \to W$ which are in the image of the canonical map

$$\psi\colon W \otimes_{alg} V' \longrightarrow \mathcal{L}(V,W) \qquad w \otimes f \mapsto (v \mapsto wf(v)).$$

The map ψ is continuous (with the respect to the strong topology on V' and the projective topology on the algebraic tensor product $W \otimes_{alg} V'$). In particular, it extends uniquely to a continuous linear map

$$\bar{\psi}\colon W \otimes V' \longrightarrow \mathcal{L}(V,W)$$

of the projective tensor product (the completion of $W \otimes_{alg} V'$, see definition 4). The image of $\bar{\psi}$ consists of the *nuclear operators* $\mathcal{N}(V,W) \subset \mathcal{L}(V,W)$ (see [Koe, p. 214]). If W, V are Hilbert spaces, the nuclear operators $\mathcal{N}(V,W)$ are precisely the trace class operators (see [Koe, §42.6]) and the map $\bar{\psi}$ is injective.

Definition 27. (Approximation property and traces). In order to define the *trace* of a nuclear operator $T\colon V \to V$, the locally convex vector space V needs to satisfy a technical condition, namely that the finite rank operators (or equivalently the nuclear operators) are dense in $\mathcal{L}(V,V)$ (with respect to the topology of uniform convergence on all precompact subsets of V), see [Koe, §43.1]. This condition, the *approximation property*, guarantees that the map $\bar{\psi}$ is injective for any locally convex vector space W (see [Koe, §43]). Most locally convex spaces have the approximation property, for example Hilbert spaces and nuclear spaces; in fact, as Koethe mentions on p. 235 of his book, the construction of Banach spaces without the approximation property is quite involved. Grothendieck conjectured the existence of such spaces, but the first examples were constructed only in 1973.

If V is a locally convex vector space that has the approximation property, one can associate to any nuclear operator $T \in \mathcal{N}(V,V)$ a *trace* $\operatorname{tr} T \in \mathbb{C}$ (see [Koe, §42.7]); the map $\operatorname{tr}\colon \mathcal{N}(V,V) \to \mathbb{C}$ is simply the unique map making the following diagram commutative

$$\begin{tikzcd} V \otimes V' \arrow[r, hook, two heads, "\bar{\psi}"] \arrow[d, "\tau"'] & \mathcal{N}(V,V) \arrow[d, "\operatorname{tr}"] \\ V' \otimes V \arrow[r, "ev"] & \mathbb{C} \end{tikzcd}$$

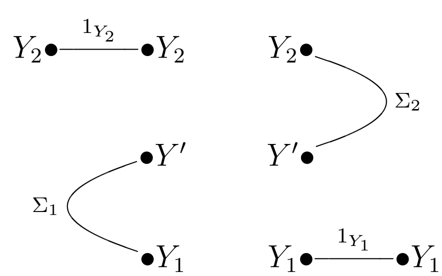
Here τ is the obvious symmetry isomorphism, and $ev\colon V' \otimes V \to \mathbb{C}$, $f \otimes v \mapsto f(v)$ is the evaluation map.

If V is equipped with a $\mathbb{Z}/2$-grading, one defines the *super trace*

$$\operatorname{str}\colon \mathcal{N}(V,V) \longrightarrow \mathbb{C}$$

as the trace above, but using the symmetry isomorphism $\tau\colon V \otimes V' \to V' \otimes V$ appropriate in the graded context, namely $v \otimes f \mapsto (-1)^{|v||f|} f \otimes v$ (where $|v|, |f| \in \{0,1\}$ is the degree of v resp. f). This agrees with the ususal definiton of super trace if V is finite dimensional.

Proof of Proposition 24. We factor the Riemannian spin bordism Σ from Y_1 to Y_2 in the following way



(28)

Applying the symmetric monoidal functor $E\colon \mathrm{RB}_d \to \mathrm{TV}^{\pm}$ we obtain an analogous factorization of $E(\Sigma)$ in the form

$$E(Y_1) = \mathbb{C} \otimes E(Y_1) \xrightarrow{T_2 \otimes 1} E(Y_2) \otimes E(Y') \otimes E(Y_1) \xrightarrow{1 \otimes T_1} E(Y_2) \otimes \mathbb{C} = E(Y_2).$$

Then the following algebraic lemma implies the proposition. $\square$

Lemma 29. *Let V, W be objects of $\mathrm{TV}^{\pm}$ and let $T = (T^+, T^-)$ be a morphism from V to W. Assume that T in the symmetric monoidal category $\mathrm{TV}^{\pm}$ can be factored in the form*

$$V = \mathbb{C} \otimes V \xrightarrow{T_2 \otimes 1_V} W \otimes U \otimes V \xrightarrow{1_W \otimes T_1} W \otimes \mathbb{C} = W.$$

Then $T^+ \colon V^+ \to W^+$ is a nuclear map.

For the proof of this, we will need the following direct consequence of the duality relation **??**.

Lemma 30. *If $T = (T^+, T^-)$ is a morphism in $\mathrm{TV}^{\pm}$ from $V = (V^+, V^-, \mu_V)$ to $\mathbb{C} = (\mathbb{C}, \mathbb{C}, \mu_{\mathbb{C}})$, then T^- determines T^+ via the formula*

$$T^+ v^+ = \mu_V(T^-(1) \otimes v^+).$$

Proof of Lemma 29. Let $\check{T} \in W^+ \otimes V^-$ be the image of $T_1^-(1) \otimes T_2^+(1)$ under the map

$$U^- \otimes V^- \otimes W^+ \otimes U^+ \xrightarrow{\tau} V^- \otimes U^- \otimes U^+ \otimes W^+ \xrightarrow{1 \otimes \mu_U \otimes 1} W^+ \otimes V^-,$$

where τ is the usual (graded) symmetry isomorphism. We claim that T^+ is image of $\check{T}$ under the map $\bar{\psi}$. To check this, we may assume that $T_1^-(1)$, $T_2^+(1)$ are of the form

$$T_1^-(1) = u^- \otimes v^- \in U^- \otimes V^- \qquad T_2^+(1) = w^+ \otimes u^+ \in W^+ \otimes U^+.$$

Then for $v^+ \in V^+$ we have

$$\begin{aligned} T^+(v^+) &= (1 \otimes T_1^+) \circ (T_2^+ \otimes 1)(v^+) = (1 \otimes T_1^+)(w^+ \otimes u^+ \otimes v^+) \\ &= (-1)^{|w^+|(|u^-|+|v^-|)} w^+ \otimes T_1^+(u^+ \otimes v^+) \\ &= (-1)^{|w^+|(|u^-|+|v^-|)} w^+ \mu_{U \otimes V}(u^- \otimes v^- \otimes u^+ \otimes v^+) \\ &= (-1)^{|w^+|(|u^-|+|v^-|)+|v^-||u^+|} w^+ \mu_U(u^- \otimes u^+) \mu_V(v^- \otimes v^+) \end{aligned}$$

Here we use Lemma 30 to express T_1^+ in terms of $T_1^-(1)$.

Now let us express $\check{T}$ explicitly:

$$\begin{aligned} \check{T} &= (1 \otimes \mu_U \otimes 1) \circ \tau(u^- \otimes v^- \otimes w^+ \otimes u^+) \\ &= (-1)^{|w^+|(|u^-|+|v^-|)+|v^-||u^+|}(1 \otimes \mu_U \otimes 1)(w^+ \otimes u^- \otimes u^+ \otimes v^-) \\ &= (-1)^{|w^+|(|u^-|+|v^-|)+|v^-||u^+|} w^+ \otimes \mu_U(u^- \otimes u^-) v^- \end{aligned}$$

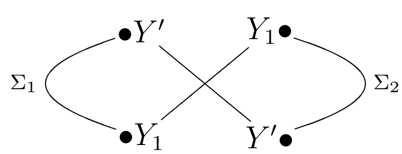
This implies that

$$\bar{\psi}(\check{T})(v^+) = (-1)^{|w^+|(|u^-|+|v^-|)+|v^-||u^+|} w^+ \mu_U(u^- \otimes u^-)\mu_V(v^- \otimes v^+),$$

which proves our claim $T^+ = \bar{\psi}(\check{T})$. In particular T^+ is in the image of $\bar{\psi}$ and hence nuclear. $\square$

Remark 31. In the above proof we are more careful with signs than we need to be at this point; e.g., the map $\mu_U \colon U^- \otimes U^+ \to \mathbb{C}$ is grading preserving, and hence $\mu_U(u^- \otimes u^+) = 0$ unless $|u^-| = |u^+|$, and similarly for $v^\pm$. So in the above proof (as well as in the proof of Lemma 32 below) we could make things a little easier by assuming $|u^-| = |u^+|$ and $|v^-| = |v^+|$ without loss of generality. However, we will need more general versions of these results, where the complex vector spaces are replaced by modules over a $\mathbb{Z}/2$-graded algebra Λ, and μ_U, μ_V take values in Λ. Again, μ_U, μ_V are *even*, but this no longer implies $\mu_U(u^- \otimes u^+) = 0$ unless $|u^-| = |u^+|$ (and the analogous statement for $v^\pm$). As long as we avoid the simplifying assumptions $|u^-| = |u^+|$ and $|v^-| = |v^+|$, the proof of these more general results is the same.

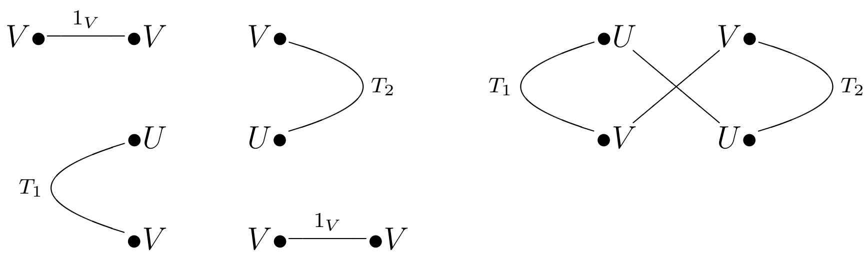
Proof of Proposition 25. Like in the proof of Proposition 24, we decompose the morphism $\Sigma \in \mathrm{RB}_d(Y_1, Y_2)$ as shown in the figure 28. Then the closed manifold $\widehat{\Sigma} \in \mathrm{RB}_d(\emptyset, \emptyset)$ can be written as the following composition in RB_d:



Applying the functor E results in corresponding decompositions of the morphisms $E(\Sigma)$ (resp. $E(\widehat{\Sigma})$) in the category $\mathrm{TV}^\pm$. The following algebraic result then implies the desired statement $E(\widehat{\Sigma})^+ = \operatorname{str} E(\Sigma)^+$, provided $E(Y)^+$ has the approximation property. $\square$

Lemma 32. *Let U, V be objects of $\mathrm{TV}^\pm$, let $\tau \colon U \otimes V \cong V \otimes U$ be the symmetry isomorphism, and let $T_1 \colon U \otimes V \to \mathbb{C}$, $T_2 \colon \mathbb{C} \to V \otimes U$ be morphisms in $\mathrm{TV}^\pm$. Let $T \colon V \to V$ (resp. $\widehat{T} \colon \mathbb{C} \to \mathbb{C}$) be the following compositions in*

$\mathrm{TV}^{\pm}$ *(these pictures are to be read right to left):*



If V^+ has the approximation property, then

$$\widehat{T}^{+} = \operatorname{str}\left(T^{+} \colon V^{+} \to V^{+}\right).$$

Proof. As in the proof of Lemma 29 we can assume

$$T_1^{-}(1) = u^{-} \otimes v^{-} \in U^{-} \otimes V^{-} \qquad T_2^{+}(1) = v^{+} \otimes u^{+} \in V^{+} \otimes U^{+};$$

here we write v^+ instead of w^+, since now we have $W = V$. We recall from that proof also that $T = \bar{\psi}(\check{T})$, where

$$\check{T} = (-1)^{|v^+|(|u^-|+|v^-|)+|v^-||u^+|} v^{+} \otimes \mu_U(u^{-} \otimes u^{-}) v^{-} \in V^{+} \otimes V^{-}$$

This implies that

$$\tau(\check{T}) = (-1)^{|v^+||u^+|+|v^-||u^+|} \mu_U(u^{+} \otimes u^{-}) v^{-} \otimes v^{+} \in V^{-} \otimes V^{+}$$

It follows that

$$\operatorname{str} T = \mu_V(\tau(\check{T})) = (-1)^{|v^+||u^+|+|v^-||u^+|} \mu_U(u^{+} \otimes u^{-}) \mu_V(v^{-} \otimes v^{+}).$$

On the other hand, we calculate the composition $\widehat{T}$ as follows:

$$\begin{aligned}
\widehat{T}(1) &= T_1^{+} \circ \tau \circ T_2^{+}(1) = T_1^{+} \circ \tau(v^{+} \otimes u^{+}) = (-1)^{|v^+||u^+|} T_1^{+}(u^{+} \otimes v^{+}) \\
&= (-1)^{|v^+||u^+|} \mu_{U \otimes V}(u^{-} \otimes v^{-} \otimes u^{+} \otimes v^{+}) \\
&= (-1)^{|v^+||u^+|+|v^-||u^+|} \mu_U(u^{-} \otimes u^{+}) \mu_V(v^{-} \otimes v^{+});
\end{aligned}$$

Here the second to last equality sign is a consequence of Lemma 30 applied to T_1; the last equality is the definition of $\mu_{U \otimes V}$. $\square$

3 QFT's as smooth functors

The goal of this section is to refine our preliminary definition of a quantum field theory (see definition 43) by requiring that the functor $E\colon \mathrm{RB}_d \longrightarrow \mathrm{TV}$ describing the QFT is *smooth* in the sense explained below. Strictly speaking, this is not a requirement on the functor E, but rather an extension of the functor E to a larger category $\mathrm{RB}_d^{\mathrm{fam}}$ whose objects are *smooth families* of objects of RB_d.

3.1 Smooth categories and functors

To motivate our formal definition of *smooth functor* and *smooth category* below (see Definition 36), we start with an informal discussion of smoothness of a functor $E\colon \mathrm{RB}_d \longrightarrow \mathrm{TV}$. Heuristically, smoothness means that the vector space $E(Y)$ associated to a $(d-1)$-manifold should depend 'smoothly' on Y (and a similar statement for morphisms); in particular, given a smooth family Y of closed $(d-1)$-manifolds parametrized by some manifold S, we should obtain a smooth family of vector spaces parametrized by S (by applying E to each member of the family), and similarly for smooth families of bordisms. More formally, we expect that E can be extended to a functor E^{fam} fitting into a commutative diagram

$$\begin{tikzcd} \mathrm{RB}_d^{\mathrm{fam}} \arrow[rr, "E^{\mathrm{fam}}"] \arrow[dr] & & \mathrm{TV}^{\mathrm{fam}} \arrow[dl] \\ & \mathrm{MAN} & \end{tikzcd} .$$

The categories $\mathrm{TV}^{\mathrm{fam}}$ and $\mathrm{RB}_d^{\mathrm{fam}}$ will be described in detail in the next subsection; they are 'family versions' of the categories TV resp. RB_d considered in the last section. E.g., the objects of $\mathrm{TV}^{\mathrm{fam}}$ are smooth families of objects of TV, i.e., pairs (S,V), where S is a smooth manifold, and $V \to S$ is a smooth vector bundle over S whose fibers are complete locally convex topological vector spaces. The slanted arrows are the natural forgetful functors to the category MAN of smooth manifolds, which map e.g. an object (S,V) of $\mathrm{TV}^{\mathrm{fam}}$ to the object S of MAN. We expect that the functor E^{fam} is 'compatible with pull-backs' of families via smooth maps $f\colon S \to T$ of manifolds (in a sense to be made precise below).

What is the correct categorical context to talk about *smooth* functors? The above heuristic discussion of 'smooth' suggests that if $\mathcal{C}$, $\mathcal{D}$ are categories over MAN (i.e., equipped with a functors $p_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{MAN}$ resp. $p_{\mathcal{D}}\colon \mathcal{D} \to \mathrm{MAN}$), a functor $F\colon \mathcal{C} \to \mathcal{D}$ should be called *smooth* if it makes the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "p_{\mathcal{C}}"'] & & \mathcal{D} \arrow[dl, "p_{\mathcal{D}}"] \\ & \mathrm{MAN} & \end{tikzcd}$$

commutative and is 'compatible with pull-backs'. To make precise what this means, we recall the universal property of the pull-back and the notion of Grothendieck fibrations. An excellent reference is [**?**] but we briefly recall the definition for the convenience of the reader who is not familiar with this language.

For a category $\mathcal{S}$ one can study *categories over* $\mathcal{S}$, i.e. categories $\mathcal{C}$ equipped with a functor $p_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{S}$. In the examples above we have $\mathcal{S} = \mathrm{MAN}$ and $\mathcal{C}$ is either the category $\mathrm{TV}^{\mathrm{fam}}$ of smooth vector bundles or the category $\mathrm{RB}_d^{\mathrm{fam}}$ (whose objects are smooth bundles of Riemannian spin manifolds), and $p_{\mathcal{C}}$ is the obvious forgetful functor.

In the following diagrams, an arrow going from an object ξ of $\mathcal{C}$ to an object S of $\mathcal{S}$, written as $\xi \mapsto S$, will mean that $p_{\mathcal{C}}\xi = S$. Furthermore, the commutativity of the diagram

$$\begin{tikzcd} \xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ S \arrow[r, "f"] & T \end{tikzcd}$$

will mean that $p_{\mathcal{C}}\phi = f$.

Definition 33. Let $\mathcal{C}$ be a category over $\mathcal{S}$. An arrow $\phi\colon \xi \to \eta$ of $\mathcal{C}$ is *cartesian* if for any arrow $\psi\colon \zeta \to \eta$ in $\mathcal{C}$ and any arrow $g\colon p_{\mathcal{C}}\zeta \to p_{\mathcal{C}}\xi$ in $\mathcal{S}$ with $p_{\mathcal{C}}\phi \circ g = p_{\mathcal{C}}\psi$, there exists a unique arrow $\theta\colon \zeta \to \xi$ with $p_{\mathcal{C}}\theta = g$ and $\phi \circ \theta = \psi$, as in the commutative diagram

$$\begin{tikzcd} \zeta \arrow[dr, dashed, "\theta"'] \arrow[drr, bend left, "\psi"] \arrow[dd, mapsto] & & \\ & \xi \arrow[r, "\phi"'] \arrow[dd, mapsto] & \eta \arrow[dd, mapsto] \\ R \arrow[dr, "g"'] \arrow[drr, "h"] & & \\ & S \arrow[r, "f"'] & T \end{tikzcd}$$

If $\xi \to \eta$ is a cartesian arrow of $\mathcal{C}$ mapping to an arrow $S \to T$ of $\mathcal{S}$, we also say that ξ is *a pullback of η to S*. From the definition, a pullback is unique, up to a unique isomorphism.

Definition 34. A *fibered category over $\mathcal{S}$* is a category $\mathcal{C}$ over $\mathcal{S}$, such that given an arrow $f\colon S \to T$ in $\mathcal{S}$ and an object η of $\mathcal{C}$ mapping to T, there is a cartesian arrow $\phi\colon \xi \to \eta$ with $p_{\mathcal{C}}\phi = f$. In other words, the requirement is that there is a pull-back for every object $\eta \in \mathcal{C}$ and every arrow $S \to T = p_{\mathcal{C}}\eta$. A fibered category $\mathcal{C} \to \mathcal{S}$ is sometimes also referred to as a *Grothendieck fibration*.

If $\mathcal{C}$ and $\mathcal{D}$ are fibered categories over $\mathcal{S}$, then a *morphism of fibered categories* $F\colon \mathcal{C} \to \mathcal{D}$ is a functor such that:

1. F is *base-preserving*, i.e. $p_{\mathcal{D}} \circ F = p_{\mathcal{C}}$;

2. F sends cartesian arrows to cartesian arrows.

There is also a definition of base-preserving natural transformations between two morphisms $\mathcal{C} \to \mathcal{D}$ but it won't concern us in this paper.

Example 35. Let Bun be the category of smooth fiber bundles; i.e., the objects are smooth fiber bundles $\xi = (p\colon E \to S)$, and a morphisms ϕ from $\xi = (p\colon E \to S)$ to $\eta = (q\colon F \to T)$ is a pair of smooth maps $(f\colon S \to T, \widehat{f}\colon E \to F)$ which makes the diagram

$$\begin{tikzcd} E \arrow[r, "\widehat{f}"] \arrow[d, "p"'] & F \arrow[d, "q"] \\ S \arrow[r, "f"] & T \end{tikzcd}$$

commutative. Then the forgetful functor Bun $\to$ MAN given by $(p\colon E \to S) \mapsto S$ and $(f, \widehat{f}) \mapsto f$ makes Bun a *category over* MAN.

Given a smooth map $f\colon S \to T$ and a bundle η over T, let $f^*\eta$ be the usual pull-back bundle over S. Then it is straightforward to see that the tautological bundle map $\phi\colon \xi \overset{\text{def}}{=} f^*\eta \to \eta$ (covering $f\colon S \to T$ on the base) is cartesian. In other words, the usual pull-back of bundles satisfies the universal property that characterizes pull-backs in the categorical setting of definition 33. In particular, Bun $\to$ MAN is a Grothendieck fibration.

We are now ready to make the central definition of this section.

Definition 36. A *smooth category* is a category $\mathcal{C}$ fibered over the category MAN of smooth manifolds, and a *smooth functor* between two smooth categories $\mathcal{C}$ and $\mathcal{D}$ is a morphism of fibered categories $F : \mathcal{C} \to \mathcal{D}$ as in Definition 34.

The main examples we have in mind are the smooth categories $\mathrm{RB}_d^{\mathrm{fam}} \to \mathrm{MAN}$ and $\mathrm{TV}^{\mathrm{fam}} \to \mathrm{MAN}$ to be introduced in subsection **??**; a QFT is then simply a smooth functor $\mathrm{RB}_d^{\mathrm{fam}} \to \mathrm{TV}^{\mathrm{fam}}$ compatible with the symmetric monoidal structure (see definition 43. When we introduce our notion of super symmetric QFT's in section 4.5, we will replace the base category MAN by the category SMAN of super manifolds, or more precisely, cs-manifolds.

Example 37. Here is a simple minded example of smooth functors that are related to smooth maps. More precisely, we will show that there is a natural bijection between the set of smooth maps from a manifold X to a manifold Y and the set of smooth functors

$$\mathrm{MAN}_X \longrightarrow \mathrm{MAN}_Y$$

between associated smooth categories. Here the category MAN_X is the *over-category* of $X \in \mathrm{MAN}$, i.e., objects are morphisms $g\colon S \to X$ in MAN, and morphisms from $g\colon S \to X$ to $h\colon T \to X$ are morphims $f \in \mathrm{MAN}(S,T)$ making the diagram

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "g"'] & & T \arrow[dl, "h"] \\ & X & \end{tikzcd}$$

commutative. There is an obvious forgetful functor $\mathrm{MAN}_X \to \mathrm{MAN}$ which makes MAN_X a category over MAN; in fact this is a Grothendieck fibration over MAN (pull-back's are given by precomposition by $f\colon S \to T$), and hence MAN_X is a smooth category. If $f\colon X \to Y$ is a smooth map, it induces a functor

$$\widehat{f}\colon \mathrm{MAN}_X \longrightarrow \mathrm{MAN}_Y \qquad \text{given by} \qquad (g\colon S \to X) \mapsto (S \xrightarrow{g} X \xrightarrow{f} Y).$$

This is a *smooth* functor (every arrow in MAN_Y is cartesian). Conversely, given a smooth functor $F\colon \mathrm{MAN}_X \to \mathrm{MAN}_Y$, we can apply F to the object $\mathrm{id}\colon X \to X$ of MAN_X and obtain a smooth map $f\colon X \to Y$. It is easy to check that these two constructions are inverse to each other and hence we obtain the claimed bijection.

3.2 Stacks

All of these fibered categories (of bundles) have the property that one can get objects on a union of open sets $S = \cup_i U_i$ by specifying objects on U_i and isomorphisms (or clutching maps) on $U_i \cap U_j$ that satisfy the cocycle conditions on $U_i \cap U_j \cap U_k$. Moreover, these *descent data* can be reconstructed from a given bundle on all of S.

This important property of bundles can be formalized in the following notion of a *stack*. We will not use this language much in this paper since we are only interested in very particular examples of stacks, however, we feel that it will be important for future use to have the right notion of a *smooth category* respectively a *super category*. A fibered category over $\mathcal{S}$ can be thought of as a weak version of a functor from $\mathcal{S}$ to CAT, i.e. as a presheaf of categories over $\mathcal{S}$. A stack is, morally, a sheaf of categories over $\mathcal{S}$.

Let $\mathcal{S}$ be a *Grothendieck site*, which means that every object S of $\mathcal{S}$ comes with a notion of a *covering* that is a collection of morphisms $\{U_i \to S\}_{i \in I}$ in $\mathcal{S}$ for some index set I. In our case of $\mathcal{S} = \mathrm{MAN}$ these are just the usual coverings of a manifold S by (inclusions of) open sets U_i.

Let $\mathcal{C}$ be a category fibered over $\mathcal{S}$. By the axiom of choice, we may fix a *cleavage* which consists of a class K of cartesian arrows in $\mathcal{C}$ such that for each arrow $f\colon S \to T$ in $\mathcal{S}$ and each object η in $\mathcal{C}(T)$ there exists a unique arrow in K with target η mapping to f in $\mathcal{S}$. This choice is just a matter of convenience (and for keeping the language slightly under control) since all the definitions can be given without resorting to the choice of a cleavage.

Given a covering $\{\sigma\colon U_i \to S\}$, set $U_{ij} = U_i \times_S U_j$ and $U_{ijk} = U_i \times_S U_j \times_S U_k$ for each triple of indices i, j and k.

Definition 38. Let $\mathcal{U} = \{\sigma_i\colon U_i \to S\}$ be a covering in $\mathcal{S}$. An *object with descent data* $(\{\xi_i\}, \{\phi_{ij}\})$ on $\mathcal{U}$, is a collection of objects $\xi_i \in \mathcal{C}(U_i)$, together with isomorphisms $\phi_{ij}\colon \mathrm{pr}_2^*\xi_j \simeq \mathrm{pr}_1^*\xi_i$ in $\mathcal{C}(U_i \times_S U_j)$, such that the following cocycle condition is satisfied.

For any triple of indices i, j and k, we have the equality

$$\mathrm{pr}_{13}^*\phi_{ik} = \mathrm{pr}_{12}^*\phi_{ij} \circ \mathrm{pr}_{23}^*\phi_{jk}\colon \mathrm{pr}_3^*\xi_k \to \mathrm{pr}_1^*\xi_i$$

where the pr_{ab} and pr_a are projections on the a-th and b-th factor, or the a-th factor respectively.

The isomorphisms ϕ_{ij} are called *transition isomorphisms* of the object with descent data.

An arrow between objects with descent data

$$\{\alpha_i\}\colon (\{\xi_i\}, \{\phi_{ij}\}) \to (\{\eta_i\}, \{\psi_{ij}\})$$

is a collection of arrows $\alpha_i \colon \xi_i \to \eta_i$ in $\mathcal{C}(U_i)$, with the property that for each pair of indices i, j, the diagram

$$\begin{tikzcd} \mathrm{pr}_2^*\xi_j \arrow[r, "\mathrm{pr}_2^*\alpha_j"] \arrow[d, "\phi_{ij}"] & \mathrm{pr}_2^*\eta_j \arrow[d, "\psi_{ij}"] \\ \mathrm{pr}_1^*\xi_i \arrow[r, "\mathrm{pr}_1^*\alpha_i"] & \mathrm{pr}_1^*\eta_i \end{tikzcd}$$

commutes. There is an obvious way of composing morphisms, which makes objects with descent data the objects of a category, denoted by $\mathcal{C}(\mathcal{U}) = \mathcal{C}(\{U_i \to S\})$.

For each object ξ of $\mathcal{C}_S$ we can construct an object with descent data on a covering $\{\sigma_i \colon U_i \to S\}$ as follows. The objects are the pullbacks $\sigma_i^*\xi$; the isomorphisms $\phi_{ij} \colon \mathrm{pr}_2^*\sigma_j^*\xi \simeq \mathrm{pr}_1^*\sigma_i^*\xi$ are the isomorphisms that come from the fact that both $\mathrm{pr}_2^*\sigma_j^*\xi$ and $\mathrm{pr}_1^*\sigma_i^*\xi$ are pullbacks of ξ to U_{ij}. If we identify $\mathrm{pr}_2^*\sigma_j^*\xi$ with $\mathrm{pr}_1^*\sigma_i^*\xi$, as is commonly done, then the ϕ_{ij} are identities.

Given an arrow $\alpha \colon \xi \to \eta$ in $\mathcal{C}_S$, we get arrows $\sigma_i^* \colon \sigma_i^*\xi \to \sigma_i^*\eta$, yielding an arrow from the object with descent associated with ξ to the one associated with η. This defines a functor $\mathcal{C}_S \to \mathcal{C}(\{U_i \to S\})$.

It is important to notice that these construction do not depend on the choice of a cleavage, in the following sense. Given a different cleavage, for each covering $\{U_i \to S\}$ there is a canonical isomorphism of the resulting categories $\mathcal{C}(\{U_i \to S\})$; and the functors $\mathcal{C}_S \to \mathcal{C}(\{U_i \to S\})$ commute with these equivalences.

Definition 39. A fibered category $\mathcal{C} \to \mathcal{S}$ is a *stack* over the Grothendieck site $\mathcal{S}$ if for each covering $\{U_i \to S\}$ in $\mathcal{S}$, the functor $\mathcal{C}_S \to \mathcal{C}(\{U_i \to S\})$ is an equivalence of categories.

3.3 QFT's of dimension d

Definition 40. (The smooth category $\mathrm{TV}^{\mathrm{fam}}$) This is the smooth family version the category TV of complete locally convex topological vector spaces. More precisely, objects are locally trivial smooth vector bundles $V \to S$ over smooth manifolds S whose fibers are complete locally convex topological vector spaces. Morphisms from $V \to S$ to $W \to T$ are pairs $(f, \widehat{f})$ where $f\colon S \to T$ is a smooth map, and $\widehat{f}$ is a smooth fiberwise linear map from V to the pull-back vector bundle f^*W. In terms of smooth sections, $\widehat{f}$ can be described as a continuous $C^\infty(S)$-linear map $C^\infty(S,V) \to C^\infty(S, f^*W)$ between the spaces of smooth sections of V and f^*W equipped with the Frechet topology. It is this point of view that will generalize to the case where S is super manifold (see Definition 73).

Mapping a vector bundle $V \to S$ to its base space S gives a forgetful functor $p\colon \mathrm{TV}^{\mathrm{fam}} \to \mathrm{MAN}$. The ordinary pull-back of vector bundles shows that $\mathrm{TV}^{\mathrm{fam}} \to \mathrm{MAN}$ is a Grothendieck fibration; in other words, $\mathrm{TV}^{\mathrm{fam}}$ is a smooth category.

We recall that the Riemannian bordism category RB_d was defined in terms of the category Riem_d (see definitions 11 and 10). Similarly, the family version $\mathrm{RB}_d^{\mathrm{fam}}$ will be defined in terms of a family version $\mathrm{Riem}_d^{\mathrm{fam}}$ of the category of Riemannian spin manifolds and isometric embeddings.

Definition 41. We define $\mathrm{Riem}_d^{\mathrm{fam}}$ to be the category of smooth families of Riemannian spin manifolds. More precisely,

objects of $\mathrm{Riem}_d^{\mathrm{fam}}$ are pairs

Definition 42. (The smooth category $\mathrm{RB}_d^{\mathrm{fam}}$**)** Let S be a smooth manifold. We want to define the category RB_d of S-families of Riemannian bordisms of dimension d in such a way that RB_d agrees with the category RB_d of definition 11. We observe that objects and morphisms of the category RB_d were defined in terms of the category Riem_d whose objects are Riemannian spin manifolds of dimension d and morphisms are isometric spin embeddings. Here we need only replace Riem_d be the appropriate category $\mathrm{Riem}_d^{\mathrm{fam}}$ whose objects (resp. morphisms) are families of objects (resp. morphisms) of Riem_d parametrized by smooth manifolds. More precisely,

- an object of Riem_d is a smooth quasi bundle $U \to S$ with d-dimensional fibers, equipped with a fiberwise Riemannian metric and spin structure (i.e., a spin structure on the vertical tangent bundle).
- A morphism from U to U' is a smooth quasi bundle map $f\colon U \to U'$ preserving the fiberwise Riemannian metric and spin structure.

Definition 43. A *quantum field theory of dimension d* is a smooth symmetric monoidal functor

$$E\colon \mathrm{RB}_d \longrightarrow \mathrm{TV}^{\pm},$$

which is compatible with the involution $^-$ and the anti-involution $^\vee$. Here RB_d and $\mathrm{TV}^{\pm}$ are smooth categories (which we don't indicate in our notation).

Remark 44. If E is a QFT of dimension 1 we can apply E_S to the universal family of circles $S^1_\ell \in \mathrm{RB}^1_S(\emptyset,\emptyset)$ parametrized by $S = \mathbb{R}_+$, where $\ell \in \mathbb{R}_+(\mathbb{R}_+)$ is the identity function. The result is a smooth function $E(S^1_\ell)\colon \mathbb{R}_+ \to \mathbb{C}$; the compatibility condition (**??**) (for $S' = \mathrm{pt}$) implies that this is the partition function Z_E of definition 22; in particular Z_E is *smooth.*

Similarly, applying a 2-dimensional QFT E to the universal family of tori $T^2_{\ell,\tau} \in \mathrm{RB}^2_S(\emptyset,\emptyset)$, where $S = \mathbb{R}_+ \times \mathbb{R}^2_+$ ($\mathbb{R}^2_+ \subset \mathbb{R}^2$ is the upper half-plane) and $\ell\colon S \to R_+$, $\tau\colon S \to \mathbb{R}^2_+$ are the projection maps, leads to a smooth function $\mathbb{R}_+ \times \mathbb{R}^2_+ \to \mathbb{C}$ which agrees with the extended partition function Z_E of definition 22. Again, we conclude that the smoothness of the functor E implies the smoothness of Z_E.

3.4 Examples of objects and morphisms of $\mathrm{RB}^{\mathrm{fam}}_d$

Definition 45. The following examples of objects and morphisms in RB_d will be parametrized by smooth maps from S to suitable smooth manifolds M (e.g., $M = \mathbb{R}_+$ or $M = \mathbb{R}^2_+$). It will be convenient to use the notation $M(S)$ for the set of smooth maps from S to M; elements of $M(S)$ are referred to as *S-points of M*. We will make use of this notation in particular later when we use the *functor of points* formalism to describe maps between super manifolds.

Example 46. (Examples of objects and morphisms of RB_d.) The following examples of objects and morphisms in the family bordism category RB_d will be important to us. They are *family versions* of the examples 13 and 17, parametrized by a smooth manifold S. We shall write these examples in a fairly formal way that will extend to super manifolds without additional work (see example 81). All fiber bundles over S involved here will be topologically trivially; they are either of the form $S \times \mathbb{R}^d \to S$ (with the obvious fiberwise metric and spin structure), or a quotient of this bundle by a discrete subgroup of the group $\mathbb{R}^d(S)$ (of smooth maps from S to $\mathbb{R}^d$, see definition 45). The group $\mathbb{R}^d(S)$ acts by structure preserving bundle automorphisms by associating to $f \in \mathbb{R}^d(S)$ the bundle automorphism

$$S \times \mathbb{R}^d \longrightarrow S \times \mathbb{R}^d \qquad \text{given by} \qquad (s,x) \mapsto (s, f(s)+x).$$

Abusing language, we will write again f for this bundle automorphism.

the point $\mathrm{pt}_S \in \mathrm{RB}_1$**.** The quadruple

$$\mathrm{pt}_S \stackrel{\text{def}}{=} (U, Y, U^+, U_-) = (S \times \mathbb{R}, S \times \{0\}, S \times (-\infty, 0), S \times (0, \infty))$$

is an object of RB_1.

the interval $I^1_\ell \in \mathrm{RB}_1(\mathrm{pt}_S, \mathrm{pt}_S)$**.** For $\ell \in \mathbb{R}_+(S)$ the pair of bundle maps

$$U = S \times \mathbb{R} \xhookrightarrow{\text{id}} \Sigma = S \times \mathbb{R} \xhookleftarrow{\ell} U = S \times \mathbb{R}\ ,$$

is a Riemannian spin bordism from pt_S to pt_S. We will use the notation I^1_ℓ for this S-family of intervals whose length is given by the function $\ell\colon S \to \mathbb{R}_+$.

the circle $S^1_\ell \in \mathrm{RB}_1(\emptyset, \emptyset)$**.** For $\ell \in \mathbb{R}_+(S)$ the circle bundle

$$S^1_\ell \stackrel{\text{def}}{=} (S \times \mathbb{R})/\mathbb{Z}\ell$$

is a Riemannian spin bordism from $\emptyset$ to $\emptyset$.

the circle $S^1_\ell \in \mathrm{RB}_2$**.** For $\ell \in \mathbb{R}_+(S)$ the quadruple

$$S^1_\ell \stackrel{\text{def}}{=} ((S \times \mathbb{R}^2)/\mathbb{Z}\ell, (S \times \mathbb{R})/\mathbb{Z}\ell, (S \times \mathbb{R}^2_+)/\mathbb{Z}\ell, (S \times \mathbb{R}^2_-)/\mathbb{Z}\ell)$$

is an object of RB_2.

the cylinder $C^2_{\ell,\tau} \in \mathrm{RB}^2(S^1_\ell, S^1_\ell)$**.** For $\ell \in \mathbb{R}_+(S)$ and $\tau \in \mathbb{R}^2(S)$, consider the following pair of bundle maps preserving the fiberwise metrics and spin structures

$$U_2 = (S \times \mathbb{R}^2)/\mathbb{Z}\ell \xhookrightarrow{\text{id}} \Sigma = (S \times \mathbb{R}^2)/\mathbb{Z}\ell \xhookleftarrow{\ell\cdot\tau} U_1 = (S \times \mathbb{R}^2)/\mathbb{Z}\ell\ .$$

For $\tau \in \mathbb{R}^2_+(S)$ conditions **??** are satisfied and this is a Riemannian spin bordism from S^1_ℓ to itself.

The torus $T^2_{\ell,\tau} \in \mathrm{RB}^2(\emptyset,\emptyset)$. For $t \in \mathbb{R}_+(S)$ and $\tau \in \mathbb{R}^2_+(S)$ the torus bundle

$$T^2_{\ell,\tau} \stackrel{\text{def}}{=} (S \times \mathbb{R}^2)/\ell(\mathbb{Z}\tau + \mathbb{Z}1).$$

is a Riemannian bordism from $\emptyset$ to $\emptyset$.

The following two lemmas are the family versions of the relations among morphisms in RB_d formulated in Lemmas 16 and 18.

Lemma 47. *Let S be a smooth manifold and $\ell, \ell' \in \mathbb{R}_+(S)$. Then the following relations hold in the category RB_1:*

1. $I^1_\ell \circ I^1_{\ell'} = I^1_{\ell+\ell'} \in \mathrm{RB}_1(\mathrm{pt},\mathrm{pt})$;
2. $\widehat{I^1_\ell} = S^1_\ell \in \mathrm{RB}_1(\emptyset,\emptyset)$;

Lemma 48. *Let S be a smooth manifold and $\ell \in \mathbb{R}_+(S)$, $\tau,\tau' \in \mathbb{R}^2_+(S)$. Then the following relations hold in the category RB_2:*

1. $C^2_{\ell,\tau} \circ C^2_{\ell,\tau'} = C^2_{\ell,\tau+\tau'} \in \mathrm{RB}_2(S^1_\ell, S^1_\ell)$;
2. $\widehat{C^2_{\ell,\tau}} = T^2_{\ell,\tau} \in \mathrm{RB}_2(\emptyset,\emptyset)$;
3. $C^2_{\ell,\tau+1} = C^2_{\ell,\tau} \in \mathrm{RB}_2(S^1_\ell, S^1_\ell)$;
4. $T^2_{g(\ell,\tau)} = T^2_{\ell,\tau} \in \mathrm{RB}_2(\emptyset,\emptyset)$ *for every* $g \in SL_2(\mathbb{Z})$;

We will only prove Lemma 48, since the proof two relations of Lemma 47 is completely analogous, but simpler than the proof of the first two relations of Lemma 48.

Proof. To prove the first relation, we use the notation of Definition 11 (where the composition was described) and we write $C^2_{\ell,\tau} = (\Sigma, \iota_2, \iota_1)$ and $C^2_{\ell,\tau'} = (\Sigma', \iota'_2, \iota'_1)$ and arrange this in the diagram

$$\begin{tikzcd}
V_3 = (S \times \mathbb{R}^2)/\mathbb{Z}\ell \arrow[r, hook, "\iota_2 = id"] & \Sigma = (S \times \mathbb{R}^2)/\mathbb{Z}\ell & (S \times \mathbb{R}^2)/\mathbb{Z}\ell = V_2 \arrow[l, hook', "\iota_1 = \ell\tau"'] \\
V_2 = (S \times \mathbb{R}^2)/\mathbb{Z}\ell \arrow[r, hook, "\iota'_2 = id"] & \Sigma' = (S \times \mathbb{R}^2)/\mathbb{Z}\ell \arrow[u, "\iota_1\circ(\iota_2)^{-1} = \ell\tau"'] & (S \times \mathbb{R}^2)/\mathbb{Z}\ell = V_1 \arrow[l, hook', "\iota'_1 = \ell\tau'"']
\end{tikzcd}$$

According to the construction, the composition is given by the bordism $\Sigma'' \stackrel{\text{def}}{=} \Sigma \cup \Sigma'$ (where a point $x' \in \Sigma'$ is identified with its image under $\iota_1 \circ (\iota_2)^{-1}$), together with the embeddings

$$\iota_3'' \colon V_3 \stackrel{\iota_3}{\hookrightarrow} \Sigma \subset \Sigma'' \qquad \text{and} \qquad \iota_1'' \colon V_1'^{\iota_1'} into \Sigma' \subset \Sigma''$$

The diagram above shows that we can identify $\Sigma'' \stackrel{\text{def}}{=} \Sigma \cup \Sigma'$ with Σ (by sending $x' \in \Sigma'$ to $\tau(x)$); with this identification, we have $\iota_3'' = id$ and $\iota_1'' = \ell\tau \circ \ell\tau' = \ell\tau + \ell\tau' = \ell(\tau + \tau')$. In other words, the composition $(\Sigma'', \iota_3'', \iota_1'')$ is equal to $C^2_{\ell,\tau+\tau'}$.

The second relation follows from the fact that the projection map

$$(S \times \mathbb{R}^2)/\mathbb{Z}\ell \longrightarrow (S \times \mathbb{R}^2)/\ell(\mathbb{Z}\tau + \mathbb{Z}1) = T^2_{\ell,\tau}$$

induces the desired bundle isomorphism $\widehat{C^2_{\ell,\tau}} \cong T^2_{\ell,\tau}$.

The third relation follows from the fact that the bundle automorphisms

$$\ell\tau, \ell(\tau+1) \colon S \times \mathbb{R}^2 \longrightarrow S \times \mathbb{R}^2$$

induce the *same* bundle automorphism on $(S \times \mathbb{R}^2)/\mathbb{Z}\ell$.

To prove the last relation, let us use the notation

$$\Lambda_{\ell,\tau} \stackrel{\text{def}}{=} \ell(\mathbb{Z}\tau + \mathbb{Z}1) \subset \mathbb{R}^2_+(S).$$

For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ let $R_a \colon \mathbb{R}^2(S) \to \mathbb{R}^2(S) = \mathbb{C}(S)$ be the rotation given by multiplication by $a = \frac{c\tau+d}{|c\tau+d|} \in S^1(S) \subset \mathbb{C}(S)$. We note that

$$\begin{aligned} R_a(\Lambda_{g(\ell,\tau)}) = \frac{c\tau+d}{|c\tau+d|}\Lambda_{g(\ell,\tau)} &= \frac{c\tau+d}{|c\tau+d|}\ell|c\tau+d|\left(\mathbb{Z}\frac{a\tau+b}{c\tau+d} + \mathbb{Z}1\right) \\ &= \ell\left(\mathbb{Z}(a\tau+b) + \mathbb{Z}(c\tau+d)\right) = \Lambda_{\ell,\tau}. \end{aligned}$$

Abusing notation, let us also write R_a for the bundle automorphism

$$R_a \colon S \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad (s,v) \mapsto (s, a(s)v).$$

The calculation above shows that R_a induces a bundle isomorphism

$$T^2_{g(\ell,\tau)} = (S \times \mathbb{R}^2)/\Lambda_{g(\ell,\tau)} \longrightarrow (S \times \mathbb{R}^2)/\Lambda_{g(\ell,\tau)} = T^2_{\ell,\tau}.$$

$\square$

A smooth map $f\colon S' \to S$ induces a pull-back functor $f^*\colon \mathrm{RB}_d \to \mathrm{RB}_d$ such that $(fg)^* = g^* f^*$. The fiberwise disjoint union gives RB_d the structure of a symmetric monoidal category; the involution $^-$, the anti-involution $^\vee$, and the adjunction transformation generalize from RB_d to $\mathrm{RB}_d^{\mathrm{fam}}$.

Definition 49. The smooth category RB_d is the functor

$$\mathrm{MAN}^{op} \longrightarrow \mathrm{CAT}$$

which sends a smooth manifold S to the category RB and a smooth map $f\colon S' \to S$ to the pull-back functor $f^*\colon \mathrm{RB}_d \to \mathrm{RB}_d$.

4 Super symmetric quantum field theories

As mentioned in the introduction, super symmetric field theories are a variant of the field theories described in the previous section obtained by replacing the Riemannian bordism category RB_d by its super version SRB_d, whose objects are closed super manifolds of dimension $d-1|1$, and whose morphisms are super bordisms of dimension $d|1$ equipped with a super Riemannian metric. The goal of this section is to give a rapid introduction to super manifolds and to define super Riemannian metrics on super manifolds of dimension $d|1$ for $d = 1, 2$. For more details on super manifolds we refer the reader to [Va], [Fr] and [DM].

4.1 Super manifolds

Before giving the sheaf-theoretic definition of super manifolds, we make some motivational remarks. Like schemes, super manifolds are described in terms of their *functions*. In particular, associated to any super manifold M of dimension $n|m$ is a $\mathbb{Z}/2$-graded, graded commutative algebra $C^\infty(M)$, the elements of which we think of as *functions on M*. For example, there is a super manifold denoted $\mathbb{R}^{n|m}$ of dimension $n|m$ with

$$C^\infty(\mathbb{R}^{n|m}) = C^\infty(\mathbb{R}^n) \otimes \Lambda[\theta_1, \dots, \theta_m],$$

where $\Lambda[\theta_1, \dots, \theta_m]$ is the exterior algebra generated by m elements $\theta_1, \dots, \theta_m$ of odd degree. The super manifold $\mathbb{R}^{n|m}$ is the local model for a super manifold of dimension $n|m$ in the same way that $\mathbb{R}^n$ is the local model for a manifold of dimension n or that $\mathrm{spec}(R)$ for a commutative ring R is the local model for a scheme. We note that the algebra $C^\infty(\mathbb{R}^{n|m})$ can be interpreted as the global sections of a sheaf $\mathcal{O}^{n|m}$ of graded commutative algebras over $\mathbb{R}^n$, which on open subsets $U \subset \mathbb{R}^n$ is given by

$$\mathcal{O}^{n|m}(U) \stackrel{\text{def}}{=} C^\infty(U) \otimes \Lambda[\theta_1, \dots, \theta_m]$$

Definition 50. (Super manifolds.) A super manifold M of dimension $n|m$ is a Hausdorff space M_{red} with countable basis together with a sheaf $\mathcal{O}$ of graded commutative algebras which is locally isomorphic to the sheaf $\mathcal{O}^{n|m}$ over $\mathbb{R}^n$ described above. The sheaf $\mathcal{O}$ is called the *structure sheaf*, its global sections $\mathcal{O}(M_{red})$ is a $\mathbb{Z}/2$-graded algebra denoted $C^\infty(M)$, and the topological space M_{red} is called the *reduced manifold*. To see that M_{red} is in fact a smooth manifold, let $J \subset \mathcal{O}$ be the nilpotent ideal generated by the odd functions in $\mathcal{O}$, and let $\mathcal{O}_{red}$ be the quotient sheaf, which is a sheaf of commutative $\mathbb{R}$-algebras over the topological space M_{red}. We note that any section f of $\mathcal{O}_{red}$ can in fact be interpreted as a *continuous function* on M_{red}, whose value at $x \in M_{red}$ is the unique real number λ such that $f - \lambda$ is not invertible in any neighborhood of x. Since $(M_{red}, \mathcal{O}_{red})$ is locally isomorphic to $(\mathbb{R}^n, \mathcal{O}^{n|m}_{red})$, and $\mathcal{O}^{n|m}_{red}$ is the sheaf of smooth functions on $\mathbb{R}^n$, we see that $\mathcal{O}_{red}$ gives a smooth structure on M_{red}. In particular, if M is a super manifold of dimension $n|0$, then the sheaf $\mathcal{O}$ is equal to $\mathcal{O}_{red}$ and hence a super manifold of dimension $n|0$ can be identified with an ordinary smooth manifold of dimension n.

Example 51. (Main example of super manifolds.) Let $E \to M$ be a q-dimensional smooth vector bundle over a smooth n-manifold N. Then

$$\Pi E \stackrel{\text{def}}{=} (N, \Lambda E^*)$$

is a super manifold of dimension $n|q$ with $M_{red} = N$, where abusing language we write $\Lambda E^* = \bigoplus_k \Lambda^k E^*$ for the sheaf of smooth sections of the exterior algebra bundle generated by the vector bundle E^* dual to E. In particular, if E is the tangent bundle of N, then E^* is the cotangent bundle of N and $C^\infty(\Pi TN)$ can be identified with the differential forms on N. The symbol Π stands for 'parity reversal'; it means that the fibers of E should be interpreted as 'odd' which makes ΠE a super manifold of dimension $n|q$ while the total space of E is an ordinary manifold of dimension $n + q$.

It can be shown that *every* super manifold is of the above form. In other words, the functor $E \mapsto \Pi E$ from the category of vector bundles over manifolds to the category of super manifolds is surjective on isomorphism classes of objects. It is however *not* an equivalence of categories, since in general there are many morphisms $\Pi E \to \Pi E'$ that are not induced by vector bundle maps $E \to E'$.

Many constructions and definitions for manifolds have analogs for super manifolds. To formulate these analogs, we typically try to express the original notion for ordinary manifolds in terms of their smooth functions, and then generalize to super manifolds. For example:

vector field A vector field X on an ordinary manifold N can be viewed as a derivation of the algebra $C^\infty(N)$ of smooth functions on N. A vector field X on a super manifold M is defined as a graded derivation of $C^\infty(M)$, i.e., $X\colon C^\infty(M) \to C^\infty(M)$ is a linear map with the derivation property

$$X(fg) = X(f)g + (-1)^{|X||f|} fX(g) \qquad \text{for} \quad f, g \in C^\infty(M)^\pm.$$

Here $|X|, |f| \in \{0, 1\}$ is the degree of X and f, respectively. More precisely, this defines even vector fields ($|X| = 0$) and odd vector fields ($|X| = 1$); a general vector field is the sum of an even and an odd vector field.

vector bundle We recall that the category of smooth vector bundles over an ordinary manifold N is equivalent to the category of sheaves of locally free modules over the sheaf of smooth functions on N (by associating to a vector bundle $E \to N$ its sheaf of smooth sections). We will use this equivalence to identify smooth vector bundles with sheaves of locally free modules. Moreover, if M is a super manifold, we *define* a vector bundle of dimension $p|q$ to be a sheaf E of graded modules over the structure sheaf $\mathcal{O}_M$, which is locally free of rank $p|q$ (i.e., for a sufficiently small open subset U the graded $\mathcal{O}_M(U)$-module $E(U)$ has a basis consisting of p even and q odd elements). In that situation, the quotient $E_{red} \stackrel{\text{def}}{=} E/JE$ is a sheaf of locally free $\mathbb{Z}/2$-graded modules over $\mathcal{O}_{red}$, i.e., $E_{red} = E_{red}^{ev} \oplus E_{red}^{odd}$ is a $\mathbb{Z}/2$-graded vector bundle over the reduced manifold M_{red}.

tangent bundle For an ordinary n-manifold N, the tangent bundle TN interpreted as a sheaf is equal to $\mathrm{Der}(\mathcal{O}_N)$, the sheaf of derivations of the structure sheaf $\mathcal{O}_N$. This is a sheaf of locally free modules of rank n. If M is a super manifold of dimension $n|m$, let $TM \stackrel{\text{def}}{=} \mathrm{Der}(\mathcal{O}_M)$ be the sheaf of vector fields (aka graded derivations) on M; this a vector bundle (aka sheaf of locally free modules over $\mathcal{O}_M$) of rank $p|q$. The associated reduced bundle $TM_{red} = TM_{red}^{ev} \oplus TM_{red}^{odd}$ over M_{red} is a vector bundle of dimension $p|q$; moreover, its even part TM_{red}^{ev} can be identified with the tangent bundle of M_{red}.

Example 52. (Vector fields on $\mathbb{R}^{n|m}$). Now let us consider vector fields on the super manifold $\mathbb{R}^{n|m}$. We recall that $C^\infty(\mathbb{R}^{n|m}) = C^\infty(\mathbb{R}^n) \otimes \Lambda[\theta_1, \dots, \theta_m]$. Let $x_1, \dots, x_n \in C^\infty(\mathbb{R}^n)$ be the coordinate functions, and let $\partial_{x_1}, \dots, \partial_{x_n} \in \mathrm{Der}(C^\infty(\mathbb{R}^n))$ be the corresponding partial derivatives. These can be extended to even derivations of $C^\infty(\mathbb{R}^{n|m})$ which commute with the action of $\Lambda[\theta_1, \dots, \theta_m]$. Similarly, there are odd derivations

$$\partial_{\theta_i} \colon C^\infty(\mathbb{R}^{n|m}) \longrightarrow C^\infty(\mathbb{R}^{n|m});$$

these commute with the action of $C^\infty(\mathbb{R}^n)$ and $\partial_{\theta_i}(\theta_i) = 1$, $\partial_{\theta_i}(\theta_j) = 0$ for $i \neq j$; in fact, these properties characterize ∂_{θ_i}. We note that ∂_{x_i} preserves the $\mathbb{Z}/2$-grading of $C^\infty(\mathbb{R}^{n|m})$, while ∂_{θ_i} reverses it. In other words, ∂_{x_i} (resp. ∂_{θ_i}) is an even (resp. odd) element of $\mathrm{Der}(C^\infty(\mathbb{R}^{n|m})$; or, geometrically speaking, ∂_{x_i} is an even vector field on $\mathbb{R}^{n|m}$, while ∂_{θ_i} is an odd vector field.

Now consider $\mathbb{R}^{1|1}$ and let us write $t, \theta \in C^\infty(\mathbb{R}^{1|1})$ for the even (resp. odd) coordinate function. Any function $f \in C^\infty(\mathbb{R}^{1|1}) = C^\infty(\mathbb{R}) \otimes \Lambda[\theta]$ is then of the form

$$f = f_0 + \theta f_1 \qquad \text{with } f_0, f_1 \in C^\infty(\mathbb{R}).$$

Let D be the odd vector field $D = \partial_\theta - \theta\partial_t$. Then

$$Df = (\partial_\theta - \theta\partial_t)(f_0 + \theta f_1) = \partial_\theta \theta f_1 - \theta\partial_t f_0 = f_1 - \theta\partial_t f_0,$$

since the terms $\partial_\theta f_0$ and $\theta\partial_t(\theta f_1) = \theta^2(\partial_t f_1)$ both vanish. In particular,

$$D^2(f_0 + f_1\theta) = -\partial_t f_0 - \partial_t f_1 \theta = -\partial_t(f_0 + f_1\theta).$$

In other words, $D^2 = -\partial_t$.

Definition 53. (cs-manifolds.) In the next subsection we will define *super Riemannian structures* on super manifolds of dimension $d|1$ for $d = 1, 2$. More precisely, these structures will be defined for a variant of super manifolds that Deligne and Morgan [DM, §4.8] refer to as cs-manifolds which stands for *complex super*. This terminology might be somewhat misleading, since it suggests that the associated reduced manifold is a complex manifold. This is not the case; rather, the adjective *complex* means that we describe the super manifold in terms of its complex valued functions instead of its real valued functions. The precise definition is this: a *cs-manifold* of dimension $n|m$ is a topological space M_{red} together with a sheaf $\mathcal{O}_M$ of graded commutative algebras over the *complex numbers*, which is locally isomorphic to $\mathbb{R}^{n|m}_{cs} \stackrel{\text{def}}{=} (\mathbb{R}^n, \mathcal{O}^{n|m} \otimes \mathbb{C})$. If $M = (M_{red}, \mathcal{O})$ is a cs-manifold, we denote by $\bar{M} \stackrel{\text{def}}{=} (M_{red}, \bar{\mathcal{O}}_{red})$ the *complex conjugate* cs-manifold (where $\bar{\mathcal{O}}_{red}$ is the complex conjugate structure sheaf obtained by replacing all the complex vector spaces $\mathcal{O}(U)$ for $U \subset M_{red}$ by the complex conjugate vector spaces; this is compatible with the algebra structure).

We note that a super manifold of dimension $n|m$ leads to a cs-manifold by complexifying its structure sheaf. In fact, we can interpret a super manifold as a cs-manifold M equipped with a *real structure*, i.e., a complex anti-linear involution $\bar{\ }$ on its (complex) structure sheaf $\mathcal{O}_M$. We note that the reduced structure sheaf $\mathcal{O}_{red} = \mathcal{O}/J$ has a *canonical* real structure, since it can be identified with the sheaf of $\mathbb{C}$-valued smooth functions on M_{red} (with respect to the smooth structure on M_{red} determined by this sheaf), which has the complex conjugation involution. In particular, if M is a cs-manifold of dimension $n|0$, then there are no odd elements in the sheaf $\mathcal{O}$ and hence $\mathcal{O} = \mathcal{O}_{red}$ has a canonical real structure. In other words, cs-manifolds of dimension $n|0$ are just ordinary smooth manifolds of dimension n.

Definition 54. (Maps between super manifolds) If M, N are super manifolds, the morphisms from M to N are simply grading preserving algebra homomophisms

$$C^\infty(N) \longrightarrow C^\infty(M).$$

If M, N are cs-manifolds, we require in addition that the reduced map $C^\infty(N_{red}; \mathbb{C}) \longrightarrow C^\infty(M_{red}; \mathbb{C})$ (obtained by modding out the ideal generated by odd functions) is *real*, i.e., compatible with complex conjugation (this condition guarantees that a map $f\colon M \to N$ between cs-manifolds induces a smooth map $M_{red} \to N_{red}$ between the reduced manifolds).

A convenient way to describe describe maps between cs-manifolds is the *functor of points approach* (see [DM, §2.8, 2.9]). If M, S are cs-manifolds, the S-points of M are the set of morphisms $S \to M$ of cs-manifolds; we will use the notation $M_S \stackrel{\text{def}}{=} \text{SMAN}(S, M)$ for the S-points of M. For example, an S-point of the cs-manifold $\mathbb{R}^{p|q}$ can be identified with a collection $(x_1, \dots, x_p, \theta_1, \dots, \theta_q)$ of even (resp. odd) functions on S such that the reductions $(x_k)_{red} \in C^\infty(S_{red}; \mathbb{C})$ are *real valued functions.*

4.2 Super Riemannian structures on $1|1$-manifolds

Now we are ready to define super Riemannian structures on cs-manifolds of dimension $d|1$ for $d = 1, 2$. There should be a general notion of super Riemannian structures on general super manifolds (for example along the lines of the paper by John Lott [Lo]), but for the purposes of this paper, the authors prefer the pedestrian approach of first defining this notion for $d = 1$, and then for $d = 2$. We hope that the terminology *super Riemannian structure* won't tempt the reader into thinking that this is some kind of inner product on the tangent bundle of the super manifold at hand. Rather, we think of a Riemannian metric as the structure needed to define an action functional (the usual energy); similarly, the structure on super manifolds of dimension $1|1$ and $2|1$ described below is the structure needed to define analogous action functionals (see Remarks 57 and 65). This motivates us to call this structure a super Riemannian structure. An additional motivation is provided by the fact that a super Riemannian structure on a cs-manifold M of dimension $d|1$ induces a Riemannian metric on the reduced manifold M_{red} (an ordinary manifold of dimension d). In both cases ($d = 1, 2$), we will first give a preliminary definition, provide the standard example of this structure, and motivate this structure by considerations from physics. Then we give a more elaborate definition which allows for more flexibility (which will be needed in the proof of our main result).

Definition 55. (Preliminary definition). A *super Riemannian structure* on a cs-manifold M of dimension $1|1$ is an odd vector field D on M such that

(i) the reduction of the even vector field D^2 gives a nowhere vanishing (complex) vector field $(D^2)_{red}$ on M_{red}.

(ii) The complex conjugate of $(D^2)_{red}$ is $-(D^2)_{red}$.

Example 56. (The standard super Riemannian structure on $\mathbb{R}^{1|1}_{cs}$) Let us write y resp. θ for the even resp. odd coordinate function on $\mathbb{R}^{1|1}_{cs}$ and ∂_y, ∂_θ for the corresponding vector fields. Then the calculation of example 52 shows that the odd vector field

$$D \stackrel{\text{def}}{=} \partial_\theta - i\theta\partial_y$$

squares to $D^2 = -i\partial_y$. In particular, it satisfies condition (i) and (ii) above. We note that $i(D^2)_{red}$ is the vector field ∂_y on $(\mathbb{R}^{1|1}_{cs})_{red} = \mathbb{R}$; this shows that this super Riemannian structure on $\mathbb{R}^{1|1}_{cs}$ induces the standard Riemannian metric and the standard spin structure on $\mathbb{R}$.

Remark 57. (Physics motivation). From the Lagrangian point of view, a particle moving on geodesics in a Riemannian manifold X can be described by minimizing the energy functional $S(f)$, where $f\colon \mathbb{R} \to X$ describes the *world line* of the particle. From an abstract point of view, the world lines don't need to be parametrized by $\mathbb{R}$; any 1-manifold Σ equipped with a Riemannian metric is sufficient to define the energy of $f\colon \Sigma \to X$.

Similarly, the world line of a super particle moving in X is described by a map $f\colon \Sigma \to X$, where now Σ is a super manifold of dimension $1|1$. If D is an odd vector field such that $(D^2)_{red}$ is nowhere vanishing, there is a well-defined action

$$S(f) \stackrel{\text{def}}{=} \frac{1}{2}\int_\Sigma \langle D^2 f, Df\rangle \,\mathrm{vol}_D\,.$$

Here $D^2 f$ (resp. Df) is the derivative of f in the direction of the vector field D^2 (resp. D); these are sections of the pull-back bundle f^*TX which can paired using the Riemannian metric on X to obtain the function $\langle D^2 f, Df\rangle \in C^\infty(\Sigma)$. We recall that on the super manifold Σ, it is sections of the line bundle $\mathrm{Ber}(T\Sigma)^*$ (the dual of the *Berezinian line bundle* associated to the tangent bundle) which can be integrated over Σ (after an orientation on the reduced manifold Σ_{red} is fixed; see [DM, Prop. 3.10.5]). A short calculation similar to that on p. 663 of [Wi3] shows that there is a canonical isomorphism $\mathrm{Ber}(T\Sigma) \cong \mathcal{D}$, where $\mathcal{D} \stackrel{\text{def}}{=} \langle D\rangle \subset T\Sigma$ is the odd line bundle spanned by D. In particular, D determines a dual section $\mathrm{vol}_D \in \mathrm{Ber}(T\Sigma)^*$. We note that replacing D by $-D$ doesn't change the action.

If $\Sigma = \mathbb{R}^{1|1}$ and $D = \partial_\theta - \theta\partial_t$, then the action takes the usual form for the action of a super particle moving in a Riemannian manifold X (see [Wi3, Problem FP2] and [Fr, p. 41-43]):

$$S(f) = -\frac{1}{2}\int_{\mathbb{R}^{1|1}} dtd\theta \langle \dot{f}, Df \rangle.$$

The parameter t here should be thought of as *time*; we would like to apply *Wick rotation* and express all quantities involved in terms of $y = it$ (*imaginary time*). In particular, D becomes our standard vector field $\partial_\theta - i\theta\partial_y$ on the cs-manifold $\mathbb{R}^{1|1}_{cs}$ with coordinates (y, θ) (see Example 56). We note that the appearance of i in the formula for D explains our preference for cs-manifolds. Why do we Wick rotate? One reason is that in the quantum theory we want trace class operators e^{-yQ^2} rather than unitary operators e^{itQ^2}.

We note that the action functional above is invariant under any automorphisms of $\mathbb{R}^{1|1}$ that preserve D (up to sign). It is easy to check that the vector field $Q = \partial_\theta + \theta\partial_t$ commutes with D (in the graded sense); in other words, Q is the infinitesimal generator of a group of automorphisms of $\mathbb{R}^{1|1}$ which preserve D. This symmetry of the classical action is a *super* symmetry in the sense that Q is an *odd* vector field. In particular, upon quantization, Q leads to an odd operator acting on the $\mathbb{Z}/2$-graded Hilbert space of the theory.

Definition 58 needs to be modified in two ways:

(a) We want to consider the super Riemannian structure given by an odd vector field D the *same* as the structure given by $-D$.

(b) A super Riemannian structure is only locally given by an odd vector field.

This suggests to define a super Riemannian structure is an *equivalence class* of such D's where we identify D and $-D$. The defect of that definition would be that it is *not local* in the following sense: suppose D_1 and D_2 are sections of the tangent sheaf $TM \to M_{red}$ restricted to open subsets $U_1 \subset M_{red}$ (resp. $U_2 \subset M_{red}$); suppose that the super Riemannian structures determined by D_1 resp. D_2 agree on the intersection $U_1 \cap U_2$ (i.e., $D_1 = \pm D_2$ on $U_1 \cap U_2$). Then there might not be any D on $U_1 \cup U_2$ which agrees with $\pm D_i$ on U_i. In other words, these structure might not fit together to give a structure on $U_1 \cup U_2$ restricting to the given structures on U_1 and U_2. This defect can be fixed by the following more elaborate definition:

Definition 58. Let M be a cs-manifolds of dimension $1|1$. A *super Riemannian structure* on a M is given by a collection of pairs (U_i, D_i) indexed by some set I, where

- the U_i's are open subsets of M_{red} whose union is all of M_{red};
- the D_i's are sections of the tangent sheaf TM restricted to U_i satisfying the conditions (i) and (ii) of Definition 55;
- the restrictions of D_i and D_j to $U_i \cap U_j$ are equal up to a possible sign.

Two such collection define the *same* structure if their union is again such a structure (this is analogous to saying that two smooth atlases define the same smooth structure if the union of these atlases is again an atlas).

Proposition 59. *There is a functor* $\mathrm{Riem}_1 \to \mathrm{Riem}_{1|1}$.

We remark that the functor we construct below is in fact an equivalence of categories, but we don't need this statement in this paper. In any case, this result shows that for $d = 1$ super Riemannian structures on super manifolds of dimension $d|1$ are closely related to Riemannian structures plus spin structures on manifolds of dimension d. In the next subsection we will show that the same holds for $d = 2$. This is another motivation for the terminology 'super Riemannian structure'.

Proof. Let M be a Riemannian spin 1-manifold. The functor F sends M to the super manifold $\Pi L_{\mathbb{C}}$. $\square$

We note that Lemma **??** continues to hold with respect to this more sophisticated notion of *super Riemannian structure*, since while the vector fields D_i might not fit together to give a vector field D on M, the even vector fields D_i^2 do agree on the intersections and hence we have a globally defined even vector field D^2 on M.

We will need some information about the automorphism super group of $\mathbb{R}^{1|1}_{cs}$ equipped with its standard super Riemannian structure. The analogous non-super statement is that the isometry group of $\mathbb{R}$ (equipped with its standard metric) contains $\mathbb{R}$ (acting by translations on itself). To state the analog for $\mathbb{R}^{1|1}_{cs}$, we first define the multiplication

$$\mu\colon \mathbb{R}^{1|1}_{cs} \times \mathbb{R}^{1|1}_{cs} \longrightarrow \mathbb{R}^{1|1}_{cs} \qquad (y_1, \theta_1), (y_2, \theta_2) \mapsto (y_1 + y_2 + \theta_1\theta_2, \theta_1 + \theta_2). \quad (60)$$

This gives $\mathbb{R}^{1|1}_{cs}$ the structure of a *super Lie group*, i.e., a group object in the category of super manifolds.

Lemma 61. *The left translation action of $\mathbb{R}^{1|1}_{cs}$ on itself preserves the standard super Riemannian structure of Example 56.*

4.3 Super Riemannian structures on $2|1$-manifolds

Definition 62. Let M be a cs-manifold of dimension $2|1$. A super Riemannian structure on M is given locally by a pair (D, B), consisting of an odd vector field D and an even vector field B on M such that

(i) The vector fields $(D^2)_{red}$, B_{red} on the reduced manifold M_{red} are linearly independent at every point of M_{red}.

(ii) The vector field B_{red} is the complex conjugate of $(D^2)_{red}$.

It should be emphasized that $(D^2)_{red}$ and B_{red} are *complex* vector fields on the reduced manifold M_{red}; i.e., sections of the complexified tangent bundle of M_{red}. As in the case of dimension $1|1$, this needs to be modified by describing an appropriate equivalence relation on such pairs and by making the definition *local* in M_{red}. We call two such pairs (D, B), (D', B') *equivalent* if there are $f, g \in C^\infty(M)$ whose reduced functions $f_{red}, g_{red} \colon M \to \mathbb{C}$ have values in $S^1 \subset \mathbb{C}$ such that $D' = gD$ and $B' = fB$. We note that condition (ii) above for the pairs (D, B), (D', B') implies that $\bar{f}_{red} = g^2_{red}$.

A *super Riemannian structure* on a cs-manifold M of dimension $2|1$ is given by a collection of triples (U_i, D_i, B_i), $i \in I$, where

- the U_i's are open subsets of M_{red} whose union is all of M_{red};
- the D_i's (resp. B_i's) are sections of the tangent sheaf TM restricted to U_i satisfying conditions (i) and (ii) above;
- the restrictions of (D_i, B_i) and (D_j, B_j) to $U_i \cap U_j$ are equivalent in the sense above.

Two such collection define the *same* structure if their union is again such a structure.

Example 63. (The standard super Riemannian structure on $\mathbb{R}^{2|1}_{cs}$.) We define a super Riemannian structure on the cs-manifold $\mathbb{R}^{2|1}_{cs} = (\mathbb{R}^2, \mathcal{O}^{2|1} \otimes \mathbb{C})$ as follows. Let us write $x, y, \theta \in C^\infty(\mathbb{R}^{2|1}_{cs})$ for the coordinate functions, set $z = x + iy \in C^\infty(\mathbb{R}^{2|1}_{cs})$, and consider the following vector fields on $\mathbb{R}^{2|1}_{cs}$:

$$B \overset{\text{def}}{=} \partial_z \overset{\text{def}}{=} \frac{1}{2}(\partial_x - i\partial_y) \qquad \partial_{\bar{z}} \overset{\text{def}}{=} \frac{1}{2}(\partial_x + i\partial_y) \qquad D \overset{\text{def}}{=} \partial_\theta + \theta\partial_{\bar{z}}$$

A calculation as in Example 52 shows that $D^2 = \partial_{\bar{z}}$; in particular, $D^2_{red} = \partial_{\bar{z}}$ is nowhere vanishing on $\mathbb{R}^2$ and $\overline{D^2_{red}} = \overline{\partial_{\bar{z}}} = \partial_z = B_{red}$ (to keep notation at bay we write ∂_z and $\partial_{\bar{z}}$ for the vector fields on $\mathbb{R}^{2|1}_{cs}$ as well as for their reductions, which are complex vector fields on $\mathbb{R}^2 = (\mathbb{R}^{2|1}_{cs})_{red}$). This shows that (D, B) is in fact a super Riemannian structure on $\mathbb{R}^{2|1}_{cs}$.

The terminology *super Riemannian structure* is motivated by the following result.

Lemma 64. *A super Riemannian structure on a cs-manifold M of dimension $2|1$ induces a Riemannian metric and a spin structure on the reduced manifold M_{red} (an ordinary 2-manifold). Replacing M by the complex conjugate cs-manifold $\bar{M}$ results in the same metric and the opposite spin structure.*

Proof. We first work locally. So let (D, B) as above be a pair of sections of the tangent sheaf $TM \to M_{red}$ over some open subset $U \subset M_{red}$; in particular, the (local) sections B_{red} and $\bar{B}_{red} = D^2_{red}$ of the complexified tangent bundle $TM_{red} \otimes \mathbb{C}$ are everywhere linearly independent. Hence there there is a unique hermitian metric on the complexified tangent bundle $TM_{red} \otimes \mathbb{C}$ with respect to which these sections are perpendicular to each other and both have length $\sqrt{2}$ (a choice of normalization motivated by the standard super Riemannian structure on $\mathbb{R}^{2|1}_{cs}$; see Example 63). This restricts to a *real valued* inner product on $TM_{red} \subset TM_{red} \otimes \mathbb{C}$. To see this, let X be a (complex) vector field on M_{red}; it can be written in the form $X = fB_{red} + g\bar{B}_{red}$ where f, g are smooth complex valued functions on M_{red}. Then

$$\bar{X} = \bar{f}\bar{B}_{red} + \bar{g}B_{red},$$

which shows that X is a *real* vector field if and only if $g = \bar{f}$. If $X = fB_{red} + \bar{f}\bar{B}_{red}$ and $Y = hB_{red} + \bar{h}\bar{B}_{red}$ are real vector fields, then

$$\langle X, Y \rangle = \langle fB_{red} + \bar{f}\bar{B}_{red}, hB_{red} + \bar{h}\bar{B}_{red} \rangle = 2(\bar{f}h + f\bar{h})$$

is real-valued.

We note that if (D', B') is another such pair equivalent to (D, B) (i.e., $B' = fB$ and $D' = gD$ for $f, g \in C^\infty(M)$ such that $f_{red}, g_{red} \colon M_{red} \to \mathbb{C}$ take values in S^1), then $B'_{red} = f_{red} B_{red}$ and $\bar{B}'_{red} = \bar{f}_{red} \bar{B}_{red}$ are again perpendicular vector fields of length $\sqrt{2}$, and hence the hermitian metrics these pairs determine on $TM_{red} \otimes \mathbb{C}$ agree. Hence our local arguments above are sufficient. $\square$

We remark that the standard super Riemannian structure on $\mathbb{R}^{2|1}_{cs}$ of example 63 induces the usual Riemannian metric on $\mathbb{R}^2 = (\mathbb{R}^{2|1}_{cs})_{red}$, since the vector fields $B_{red} = \partial_z$ and $\bar{B}_{red} = \partial_{\bar{z}}$ are perpendicular and of constant length $\sqrt{2}$ w.r.t. the usual metric.

Remark 65. (Physics Motivation.) A string moving in a Riemannian manifold X is described by a map $f \colon \Sigma^2 \to X$ from a Lorentz surface Σ^2 to X (in the simplest case $\Sigma = S^1 \times \mathbb{R}$, where S^1 parametrizes the string and $\mathbb{R}$ parametrizes time); a *super string* moving in X amounts to a map $f \colon \Sigma^{2|1} \to X$ from a super manifold Σ of dimension $2|1$ to X. Problem FP6 (p. 613) of Witten's homework collection in the IAS proceedings [Wi3] (see also the solution on p. 663) explains that an *action functional* (Lagrangian) for such maps can be defined, provided Σ comes equipped with a pair $(\mathcal{D}, \mathcal{B})$ of subbundles (distributions) $\mathcal{D}, \mathcal{B} \subset T\Sigma$ of dimension $0|1$ (resp. $1|0$). In other words, $\mathcal{D}$ (resp. $\mathcal{B}$) is generated locally by an odd vector field D (resp. and even vector field B); it is required that the vector fields D, D^2 and B span TX. Then the action of a map $f \colon \Sigma \to X$ is defined by

$$S(f) \overset{\text{def}}{=} \int_\Sigma \langle df_{|\mathcal{D}}, df_{|\mathcal{B}} \rangle.$$

Here $df_{|\mathcal{D}}$ (resp. $df_{|\mathcal{B}}$) is the differential of f restricted to $\mathcal{D} \subset T\Sigma$ (resp. $\mathcal{B} \subset T\Sigma$); these are sections of $\mathcal{D}^* \otimes f^*TX$ (resp. $\mathcal{B}^* \otimes f^*TX$) that can be paired using the Riemannian metric on X to obtain $\langle df_{|\mathcal{D}}, df_{|\mathcal{B}} \rangle$, which is a section of $\mathcal{D}^* \otimes \mathcal{B}^*$. The calculation on p. 663 of [Wi3] shows that this line bundle is canonically isomorphic to the Berezinian $\mathrm{Ber}(T\Sigma)^*$, and so its section $\langle df_{|\mathcal{D}}, df_{|\mathcal{B}} \rangle$ can be integrated over Σ.

As in the $1|1$-dimensional case, the automorphisms of $(\Sigma, \mathcal{D}, \mathcal{B})$ give rise to (super) symmetries of the theory. As is well-known, the automorphism group of $(\Sigma, \mathcal{D}, \mathcal{B})$ (super conformal group) is an infinite dimensional super Lie group. We prefer to work with the super Riemannian structure (given by

D, B) rather than the *super conformal structure* given by the distributions $\mathcal{D}$, $\mathcal{B}$ they generate, since the conformal invariance of the classical theory usually does not survive to the quantum theory (conformal anomaly).

A flat model is $\mathbb{R}^{2|1}$ with even coordinates u, v and odd coordinate θ and

$$D = \partial_\theta - \theta\partial_u \qquad B = \partial_v,$$

where, as Witten explains in [Wi2, §2.2], the coordinates u, v are the right- resp. left-moving light cone coordinates on $\mathbb{R}^2$ equipped with the Minkowski metric $ds^2 = du\,dv$. In terms of the more usual coordinates t (time) and x (space), for which the Minkowski metric takes the form $ds^2 = dt^2 - dx^2$, the light cone coordinates are given by

$$u = x + t \qquad v = t - x.$$

For the same reasons as in the $1|1$-dimensional case, we want to do a *Wick rotation* and express all quantities in terms of the *imaginary time* $y = it$. In particular, $u = x + t = x - iy = \bar{z}$ and $v = t - x = -(x + iy) = -z$. In particular, expressed in terms of $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, and ∂_θ we have:

$$D = \partial_\theta - \theta\partial_{\bar{z}} \qquad B = -\partial_z \tag{66}$$

Again the calculation in Example 52 shows that $D^2 = -\partial_{\bar{z}}$; in particular, the vector fields D^2 and B reduce to the everywhere linearly independent (complex) vectors fields $-\partial_{\bar{z}}$ (resp. $-\partial_z$) on $(\mathbb{R}^{2|1}_{cs})_{red} = \mathbb{R}^2$ which are complex conjugates of each other (of course with respect to the *new* real structure where the coordinate functions x, y are regarded as real functions). In other words, after a Wick rotation the vectors fields D, B define a super Riemannian structure on $\mathbb{R}^{2|1}_{cs}$.

Definition 67. (Flat super Riemannian structures on $2|1$-manifolds). Let M be a cs-manifold of dimension $2|1$. A super Riemannian structure on M is called *flat* if the pairs of local vector fields (D, B) defining it satisfy the following additional property:

(iii) The graded commutator $[D, B]$ vanishes.

We note that the standard super Riemannian structure (63) is flat and we believe that vice versa, a flat cs-manifold is locally isomorphic to the standard structure.

Lemma 68. *Let M be a cs-manifold of dimension $2|1$ equipped with a super Riemannian structure which is flat. Then the induced Riemannian metric on M_{red} is flat.*

Proof. Let ∇ be the unique connection on $TM_{red} \otimes \mathbb{C}$ for which the sections D^2_{red} and B_{red} are parallel. We claim that ∇ is the Levi-Civita connection for the induced metric on B_{red} (or more precisely, the connection on the complexified tangent bundle induced by the Levi-Civita connection). It is clear that ∇ is a metric connection since D^2_{red} and B_{red} are both vector fields of constant length. It remains to show that the torsion tensor T vanishes, which is given by

$$T(X,Y) \stackrel{\text{def}}{=} \nabla_X Y - \nabla_Y X - [X,Y]$$

for complex vector fields X, Y (the expression $\nabla_X Y$ is originally defined for real vector fields X and complex vector fields Y, but we can extend it to complex vector fields X by requiring $\nabla_X Y$ to depend complex linearly on X). Since $T(X,Y)$ is skew-symmetric and complex linear in both variables, to prove $T \equiv 0$, it suffices to show $T(D^2_{red}, B_{red}) = 0$; this is the case since $\nabla_X D^2_{red}$ and $\nabla_X B_{red}$ vanish for every vector field X (this is the condition that D^2_{red} and B_{red} are parallel w.r.t. ∇) and since our flatness-condition $[D,B] = 0$ implies $[D^2, B] = 0$ and hence $[D^2_{red}, B_{red}] = 0$. This shows that ∇ is the (complexified) Levi-Civita connection; in particular, the Levi-Civita connection is flat due to the existence of the parallel sections D^2_{red} and B_{red}. $\square$

We can give $\mathbb{R}^{2|1}_{cs}$ the structure of a super Lie group by defining the multiplication

$$\begin{aligned} \mu\colon \mathbb{R}^{2|1}_{cs} \times \mathbb{R}^{2|1}_{cs} &\longrightarrow \mathbb{R}^{2|1}_{cs} \\ (z_1, \bar{z}_1, \theta_1), (z_2, \bar{z}_2, \theta_2) &\mapsto (z_1 + z_2, \bar{z}_1 + \bar{z}_2 + \theta_1\theta_2, \theta_1 + \theta_2) \end{aligned} \tag{69}$$

Here $z = x + iy \in C^\infty(\mathbb{R}^{2|1}_{cs})$ and $\bar{z} = x - iy$, where $x, y \in C^\infty(\mathbb{R}^{2|1}_{cs})$ are the coordinate functions of the cs-manifold $\mathbb{R}^{2|1}_{cs}$; the subscript 1 (resp. 2) indicate the first (resp. second) copy of $\mathbb{R}^{2|1}_{cs}$. We note that we need to work with $\mathbb{R}^{2|1}_{cs}$ in order to make sense of the function z (and hence of the map μ). Moreover, after reduction $\bar{z}$ is the complex conjugate of z, which implies that μ satisfies the reality condition of Definition 54.

Lemma 70. *The left translation action of $\mathbb{R}^{2|1}_{cs}$ on itself preserves the standard super Riemannian structure.*

We will need the following result in the proof of our main result.

Lemma 71. *Let $a \in \mathbb{C}$ be of unit length. Then the automorphism*

$$R_a \colon \mathbb{R}^{2|1}_{cs} \longrightarrow \mathbb{R}^{2|1}_{cs} \qquad (z, \bar{z}, \theta) \mapsto (a^2 z, \bar{a}^2 \bar{z}, \bar{a}\theta)$$

preserves the standard super Riemannian structure.

Proof. A calculation shows

$$F_* \partial_z = a^2 \partial_z \qquad F_* \partial_{\bar{z}} = \bar{a}^2 \partial_{\bar{z}} \qquad F_* \partial_\theta = a \partial_\theta \qquad F_*(\partial_\theta - \theta \partial_{\bar{z}}) = \bar{a}(\partial_\theta - \theta \partial_{\bar{z}})$$

This implies that the pairs $(D, B) = (\partial_\theta - \theta\partial_{\bar{z}}, -\partial_z)$ and $(F_* D, F_* B)$ are equivalent in the sense of Definition 62 and hence represent the *same* super Riemannian structure on $\mathbb{R}^{2|1}_{cs}$. □

4.4 Super categories and functors

We recall that a QFT (see definition 43) is a smooth functor $\mathrm{RB}^{\mathrm{fam}}_d \to \mathrm{TV}^{\mathrm{fam}}$; here the objects of the domain and range category are 'smooth families' of suitable objects parametrized by smooth manifolds. In categorical language, domain and range categories are *smooth categories* in that sense that they are fibered over the category MAN of smooth manifolds (see Definitions 34 and 36). Similarly, the definition of a super symmetric QFT in subsection 4.5 will involve working with categories whose objects are 'families paramatrized by super manifolds'. We will refer to such categories as 'super categories'; the precise definition is the following.

Definition 72. A *super category* is a category $\mathcal{C}$ fibered over the category SMAN of super manifolds, and a *super functor* between two super categories $\mathcal{C}$ and $\mathcal{D}$ is a morphism of fibered categories $F \colon \mathcal{C} \to \mathcal{D}$ (see Definition 34).

Definition 73. (The super category STV**)** The objects of STV are 'vector bundles over super manifolds whose fibers are locally convex topological vector spaces', but we need to define what we mean by this. We recall that if S is a manifold, then the (local) sections of an n-dimensional complex vector bundle over S form a sheaf of modules over the structure sheaf $\mathcal{O}_S$, which is locally of the form $C^\infty(U) \otimes \mathbb{C}^n$. This motivated the definition of finite dimensional vector bundles over a super manifold S as sheaves of locally free modules over $\mathcal{O}_S$ in section **??**. For infinite dimensional vector bundles we need to be more careful: if V is a locally convex topological vector space, and S is an ordinary manifold, the smooth functions on S with values in V can be identified with $C^\infty(S) \otimes V$, but now the tensor product is not the algebraic tensor product, but rather the *projective* tensor product (see Definition 4) of $C^\infty(S)$ (equipped with the Frechet topology) and V; we note that there is no difference between these tensor products if V is finite dimensional.

This suggests the following definition of (possibly infinite dimensional) vector bundles over super manifolds. Let S be a super manifold, and let V be a locally convex $\mathbb{Z}/2$-graded topological vector space. Then a *vector bundle over S with fiber V* is a sheaf $\mathcal{E}$ over S_{red} of locally convex $\mathcal{O}_S$-modules which is locally of the form $\mathcal{O}_S(U) \otimes V$ (projective tensor product).

If $f\colon S \to T$ is a morphism of super manifolds, and $\mathcal{F}$ is a vector bundle over T, we can form the *pull-back* vector bundle $f^*\mathcal{F}$ over S, whose global sections $C^\infty(S, f^*\mathcal{F})$ are given by

$$C^\infty(S, f^*\mathcal{F}) \stackrel{\text{def}}{=} C^\infty(S) \otimes_{C^\infty(T)} C^\infty(T, \mathcal{F}).$$

If $\mathcal{E}$ is a vector bundle over S, we define a *vector bundle morphisms* from $\mathcal{E}$ to $\mathcal{F}$ to be a pair $(f, \widehat{f})$, where $f\colon S \to T$ is a morphism of super manifolds, and $\widehat{f}\colon \mathcal{E} \to f^*\mathcal{F}$ is a map of sheaves which is $\mathcal{O}_S$-linear and continuous. These are the morphisms in the category STV.

There is an obvious forgetful functor STV $\to$ SMAN. It can be shown that the pull-back $f^*\mathcal{F}$ as defined above is a pull-back in the sense of Definition 33. In particular, STV is fibered over the category of super manifolds, and hence STV is a *super category* in the sense of Definition 72.

4.5 Super symmetric quantum field theories

The goal of this subsection is definition of super symmetric quantum field theories (Definition 74) and their partition function (Definition **??**).

Definition 74. A *super symmetric quantum field theory of dimension d* for $d = 1, 2$ is a super functor

$$E\colon \mathrm{SRB}_d \longrightarrow \mathrm{TV},$$

compatible with the symmetric monoidal structure. Here the SRB_d is the super category defined below, which is the 'super' version of the smooth bordism category $\mathrm{RB}_d^{\mathrm{fam}}$.

The definition of SRB_d is completely analogous to Definition 11 of RB_d (resp. Definition 42 of $\mathrm{RB}_d^{\mathrm{fam}}$); we just need to replace Riemannian manifolds of dimension d by cs-manifolds of dimension $d|1$ equipped with super Riemannian structures and allow cs-manifolds as parameter spaces. More precisely, we recall that the objects and morphisms of the bordism categories RB_d (resp. $\mathrm{RB}_d^{\mathrm{fam}}$) are defined in terms of the category Riem_d (resp. $\mathrm{Riem}_d^{\mathrm{fam}}$) of Riemannian spin d-manifolds (resp. families of such manifolds parametrized by S). We obtain SRB_d by replacing Riem_d by the category SRiem_d defined as follows.

Definition 75. The super category SRiem_d

- the objects of SRiem_d are smooth locally trivial fibers bundles of cs-manifolds $Y \to S$ with fibers of dimension $d|1$ which are equipped with a fiberwise super Riemannian structure (fiberwise here means that the local vector fields defining the super Riemannian structure are *vertical*).
- A morphism from U to U' is an embedding $U \hookrightarrow U'$ of cs-manifolds that is a bundle map (i.e., commutes with the projection to S), and preserves the fiberwise super Riemannian structure.

Remark 76. There is a natural *reduction functor*

$$\mathrm{SRiem}_d \xrightarrow{red} \mathrm{Riem}_d$$

that sends a (quasi) bundle $U \to S$ of super manifolds equipped with a fiberwise super Riemannian structure to the (quasi) bundle $U_{red} \to S_{red}$ equipped with the induced fiberwise Riemannian metric and spin structure (cf. Lemmas **??** and 64). This induces a reduction functor between the corresponding bordism categories

$$\mathrm{SRB}_d \xrightarrow{red} \mathrm{RB}_d^{\mathrm{fam}}. \tag{77}$$

Example 78. (Example of a QFT of dimension $1|1$.) Let M be a compact spin manifold, let V the $\mathbb{Z}/2$-graded Hilbert space of L^2-spinors on M, $\mu\colon V \otimes \bar{V} \to \mathbb{C}$ in inner product and $D\colon V \to V$ be the Dirac operator on M (an unbounded self-adjoint operator). Then M determines a QFT E_M of dimension $1|1$ with the following properties: for any cs-manifold S and $t \in C^\infty(S)^{ev}$, $\theta \in C^\infty(S)^{odd}$ the functor

$$(E_M)_S\colon \mathrm{SRB}_1 \longrightarrow \mathrm{STV}$$

maps

$$\mathrm{spt}_S \mapsto (S \times V, S \times \bar{V}, \mu) \qquad \text{and} \qquad I_{t,\theta} \mapsto e^{-tD^2+\theta D},$$

where the operator valued function $e^{-tD^2+\theta D}$ is defined using spectral calculus.

Definition 79. (Partition function of a QFT of dimension $d|1$.) Let E be a QFT of dimension $d|1$ and $E^+\colon \mathrm{RB}^{2|1} \to \mathrm{TV}$ the composition of E with the forgetful functor $\mathrm{TV}^{\pm} \to \mathrm{TV}$ (see Remark **??**). Then its partition function is obtained by applying E to a suitable S-family Σ of closed Riemannian super manifolds of dimension $d|1$; in other words, Σ is an endomorphism of the object $\emptyset \in \mathrm{SRB}_2$. Then $E_S(\Sigma)$ is an even endomorphism of the trivial bundle $S \times \mathbb{C}$, which may be regarded as an element $E_S(\Sigma) \in C^\infty(S)^{ev}$.

For $d = 1$ we define

$$Z_E \stackrel{\text{def}}{=} E_S(S^{1|1}_\ell) \in C^\infty(S),$$

where $S = \mathbb{R}_+$ and $\ell\colon S \to \mathbb{R}_+$ is the identity.

For $d = 2$ we define

$$Z_E \stackrel{\text{def}}{=} E^+_S(T^{2|1}_{\ell,\tau}) \in C^\infty(S),$$

where $S = \mathbb{R}_+ \times \mathbb{R}^2_+$ and $\ell\colon S \to \mathbb{R}_+$, $\tau\colon S \to \mathbb{R}^2_+ \hookrightarrow \mathbb{R}^{2|1}_{cs,+}$ are the obvious maps.

Remark 80. It might seem that there is a better way to define the partition function of a super symmetric QFT of dimension $2|1$ by including odd parameters by setting

$$Z_E \stackrel{\text{def}}{=} E^+_S(T^2_{\ell,\tau}) \in C^\infty(S)^{ev},$$

where $S = \mathbb{R}_+ \times \mathbb{R}_+^{2|1}$ and $\ell\colon S \to \mathbb{R}_+$, $\tau\colon S \to \mathbb{R}_{cs,+}^{2|1}$ are the projection maps. However, $C^\infty(\mathbb{R}_+ \times \mathbb{R}_{cs,+}^{2|1}) = C^\infty(\mathbb{R}_+ \times \mathbb{R}_+^2;\mathbb{C}) \otimes \Lambda[\theta]$, where θ is odd, and hence

$$C^\infty(\mathbb{R}_+ \times \mathbb{R}_{cs,+}^{2|1})^{ev} = C^\infty(\mathbb{R}_+ \times \mathbb{R}_+^2),$$

which shows that the function associated to this more general family of super tori contains the same information as the partition function described above. An analogous remark applies to QFT's of dimension $1|1$.

4.6 Examples of objects and morphisms of SRB_d

Example 81. (Examples of objects and morphisms in $\mathrm{RB}_S^{d|1}$). The following examples extend the examples discussed in (46) to the super setting in the sense that their images under the reduction functor (77) yields the examples discussed there. All fiber bundles over the cs-manifold S described below are either of the form $S \times \mathbb{R}_{cs}^{d|1} \to S$ (with the obvious fiberwise super Riemannian structure), or a quotient of this bundle by a discrete subgroup of $\mathbb{R}_{cs}^{d|1}(S) = \mathrm{SMAN}(S, \mathbb{R}_{cs}^{d|1})$ (the group of smooth maps from S to $\mathbb{R}_{cs}^{d|1}$ with group structure induced by the multiplication map μ of equation (60) resp. (69)); this group acts by structure preserving bundle automorphisms by associating to $f\colon S \to \mathbb{R}_{cs}^{d|1}$ the bundle automorphism given by the composition

$$S \times \mathbb{R}_{cs}^{d|1} \xrightarrow{\Delta\times 1} S \times S \times \mathbb{R}_{cs}^{d|1} \xrightarrow{1\times f\times 1} S \times S \times \mathbb{R}_{cs}^{d|1} \xrightarrow{1\times\mu} S \times \mathbb{R}_{cs}^{d|1}.$$

Abusing language we again write f for this bundle automorphism. Lemma 61 resp. 70 imply that f preserves the fiberwise super Riemannian structure.

the super point $\mathrm{spt}_S \in \mathrm{SRB}_1$**.** The quadruple

$$\mathrm{spt} \overset{\mathrm{def}}{=} (U, Y, U^+, U_-) = (S \times \mathbb{R}_{cs}^{1|1}, S \times \mathbb{R}_{cs}^{0|1}, S \times \mathbb{R}_{cs,-}^{1|1}, S \times \mathbb{R}_{cs,+}^{1|1})$$

is an object of SRB_1; here $\mathbb{R}_{cs,\pm}^{1|1} \subset \mathbb{R}_{cs}^{1|1}$ is the super submanifold whose reduced manifold is $\mathbb{R}_\pm^1 \subset \mathbb{R}^1$.

the super interval $I_\ell^{1|1} \in \mathrm{SRB}_1(\mathrm{spt}_S, \mathrm{spt}_S)$**.** For $\ell \in \mathbb{R}_{cs,+}^{1|1}(S)$ the pair of bundle maps

$$U = S \times \mathbb{R}_{cs}^{1|1} \xhookrightarrow{\mathrm{id}} \Sigma = S \times \mathbb{R}_{cs}^{1|1} \xhookleftarrow{\ell} U = S \times \mathbb{R}_{cs}^{1|1},$$

is a super Riemannian bordism from spt_S to spt_S. We will use the notation $I_\ell^{1|1}$ for this morphism.

the super circle $S_\ell^{1|1} \in \mathrm{SRB}_1(\emptyset,\emptyset)$. For $\ell \in \mathbb{R}^{1|1}_{cs,+}(S)$ the bundle

$$S_\ell^{1|1} \stackrel{\text{def}}{=} (S \times \mathbb{R}^{1|1}_{cs})/\mathbb{Z}\ell$$

is a Riemannian bordism from $\emptyset$ to $\emptyset$.

the super circle $S_\ell^{1|1} \in \mathrm{SRB}_2$. For $\ell \in \mathbb{R}^1_{cs,+}(S)$ the quadruple

$$S_\ell^{1|1} \stackrel{\text{def}}{=} ((S \times \mathbb{R}^{2|1}_{cs})/\mathbb{Z}\ell, (S \times \mathbb{R}^{1|1})/\mathbb{Z}\ell, (S \times \mathbb{R}^{2|1}_{cs,+})/\mathbb{Z}\ell, (S \times \mathbb{R}^{2|1}_{cs,-})/\mathbb{Z}\ell)$$

is an object of SRB_2. We remark that we need to restrict ℓ to be an element of $\mathbb{R}^1_{cs,+}(S) \subset \mathbb{R}^{2|1}_{cs,+}(S)$ (rather than $\mathbb{R}^{1|1}_{cs,+}(S)$), since otherwise the translation ℓ doesn't preserve the subspace $S \times \mathbb{R}^{1|1}_{cs} \subset S \times \mathbb{R}^{2|1}_{cs}$.

the super cylinder $C^{2|1}_{\ell,f} \in \mathrm{SRB}_2(S_\ell^{1|1}, S_\ell^{1|1})$. For $\ell \in \mathbb{R}^1_{cs,+}(S)$, $f \in \mathbb{R}^{2|1}_{cs,+}(S)$ the bundle automorphism f commutes with ℓ and hence we have the following pair of morphisms in $\mathrm{Riem}^{d|1}_S$

$$V_2 = (S \times \mathbb{R}^{2|1}_{cs})/\mathbb{Z}\ell \xhookrightarrow{\mathrm{id}} \Sigma = (S \times \mathbb{R}^{2|1}_{cs})/\mathbb{Z}\ell \xhookleftarrow{\ell f} V_1 = (S \times \mathbb{R}^{2|1}_{cs})/\mathbb{Z}\ell \cdot$$

This is a super Riemannian bordism from $S_\ell^{1|1}$ to itself; we use the notion $C^{2|1}_{\ell,f}$ for this endomorphism of $S_\ell^{1|1} \in \mathrm{RB}^{2|1}_S$.

The super torus $T^{2|1}_{\ell,f} \in \mathrm{SRB}_2(\emptyset,\emptyset)$. For $\ell \in \mathbb{R}^1_{cs,+}(S)$ and $f \in \mathbb{R}^{2|1}_{cs,+}(S)$ the bundle

$$T^{2|1}_{\ell,f} \stackrel{\text{def}}{=} (S \times \mathbb{R}^{2|1}_{cs})/\ell(\mathbb{Z}f + \mathbb{Z}1).$$

is a super Riemannian bordism from $\emptyset \in \mathrm{SRB}_2$ to itself.

All the relations of Lemmas 47 and 48 generalize as follows.

Lemma 82. *Let S be a cs-manifold and $\ell, \ell' \in \mathbb{R}^{1|1}_{cs,+}(S)$. Then the following relations hold in the category SRB_1:*

1. $I_\ell^{1|1} \circ I_{\ell'}^{1|1} = I_{\ell+\ell'}^{1|1} \in \mathrm{SRB}_1(\mathrm{spt},\mathrm{spt})$*;*

2. $\widehat{I_\ell^{1|1}} = S_\ell^{1|1} \in \mathrm{SRB}_1(\emptyset,\emptyset)$*;*

Lemma 83. *Let S be a cs-manifold and $\ell \in \mathbb{R}_+(S)$, $f, f' \in \mathbb{R}^{2|1}_{cs,+}(S)$. Then the following relations hold in the category* SRB_2*:*

1. $C^{2|1}_{\ell,f} \circ C^{2|1}_{\ell,f'} = C^{2|1}_{\ell,\mu(f,f')} \in \mathrm{SRB}_2(S^{1|1}_\ell, S^{1|1}_\ell)$;

2. $\widehat{C^{2|1}_{\ell,f}} = T^{2|1}_{\ell,f} \in \mathrm{SRB}_2(\emptyset,\emptyset)$;

3. $C^{2|1}_{\ell,f+1} = C^{2|1}_{\ell,f} \in \mathrm{SRB}_2(S^{1|1}_\ell, S^{1|1}_\ell)$;

4. $T^{2|1}_{g(\ell,f)} = T^{2|1}_{\ell,f} \in \mathrm{RB}_2(\emptyset,\emptyset)$ *for every* $g \in SL_2(\mathbb{Z})$*;*

The proof of these results is completely analogous to the proof of Lemma 48. We only want to mention that the rotation $R_a \colon \mathbb{R}^2 \to \mathbb{R}^2$ now needs to be replaced by the structure preserving automorphism $R_a \colon \mathbb{R}^{2|1}_{cs} \to \mathbb{R}^{2|1}_{cs}$ of Lemma 71 (more precisely, a has to be replaced by its square root).

5 Partition functions of susy QFT's

The goal of this section is the proof of our main theorem 1 concerning the partition functions of super symmetric QFT's of dimension $2|1$; this is done in subsection 5.2. As a warm-up we prove in the first subsection an analogous result for super symmetric QFT's of dimension $1|1$.

5.1 Partition functions of QFT's of dimension $1|1$

Theorem 84. *Let E be a super symmetric quantum field theory of dimension $1|1$. Then its partition function $Z_E \colon \mathbb{R}_+ \to \mathbb{C}$ is an integer valued constant function.*

The proof of this result will be based on using the following algebraic data obtained by applying E or rather the composite functor

$$E^+ \colon \mathrm{RB}^{1|1} \xrightarrow{E} \mathrm{TV}^{\pm} \longrightarrow \mathrm{TV}$$

(see Remark **??**) to certain objects resp. morphisms of the Riemannian bordism category $\mathrm{RB}^{1|1}$:

- the locally convex vector space $H \overset{\mathrm{def}}{=} E^+_{\mathrm{pt}}(\mathrm{spt})$ associated to the *super point* spt (an object of the bordism category $\mathrm{RB}^{1|1}_{\mathrm{pt}}$);

- The function $E_S^+(S_f^{1|1}) \in \mathbb{R}_+^{1|1}(S) = \mathrm{SMAN}(S, \mathbb{R}_+^{1|1})$ associated to the family of *super circles* $S_f^{1|1} \in \mathrm{SRB}_1(\emptyset, \emptyset)$ determined by $f \in \mathbb{R}_+^{1|1}(S)$ (see Example 81).

- The function $E_S^+(S_f^{1|1}) \in \mathcal{N}(H,H)(S) = \mathrm{SMAN}(S, \mathcal{N}(H,H))$ associated to the family of *super intervals* $S_f^{1|1} \in \mathrm{SRB}_1(\emptyset, \emptyset)$ determined by $f \in \mathbb{R}_+^{1|1}(S)$ (see Example 81). We recall that $S_f^{1|1}$ is an endomorphism of $p_S^* \mathrm{spt} \in \mathrm{SRB}_1$; hence $E_S(S_f^{1|1})$ is an endomorphism of the trivial bundle $E_S(p_S^* \mathrm{spt}) = p_S^*(H) = S \times H$. This in turn can be reinterpreted as a smooth map $S \to \mathcal{N}(H,H)$ to the nuclear endomorphisms of H, i.e., $E_S^+(S_f^{1|1}) \in \mathcal{N}(H,H)(S) = \mathrm{SMAN}(S, \mathcal{N}(H,H))$.

Geometric relations then imply algebraic relations for the associated algebraic data. In particular, gluing the incoming with the outgoing super point in the family of super intervals $I_f^{1|1}$ results in the family of super circles $S_f^{1|1}$, and hence by proposition 25 (or rather its generalization to SRB_d) we have

$$E_S^+(S_f^{1|1}) = \operatorname{str} E_S^+(I_f^{1|1}). \tag{85}$$

Similarly, the geometric relation $I_f^{1|1} \circ I_{f'}^{1|1} = I_{\mu(f,f')}^{1|1}$ (see part 1 of Lemma 82) implies the algebraic relation

$$E_S^+(I_f^{1|1}) \circ E_S^+(I_g^{1|1}) = E_S^+(I_{\mu(f,g)}^{1|1}). \tag{86}$$

We note that the map

$$\mathbb{R}_{cs,+}^{1|1}(S) \longrightarrow \mathcal{N}(H,H)(S) \qquad \text{given by} \qquad f \mapsto E_S^+(S_f^{1|1})$$

depends functorially on S by commutativity of Diagram **??**. In other words, the above describes a map of (generalized) super manifolds $\mathbb{R}_{cs,+}^{1|1} \to \mathcal{N}(H,H)$ in the S-point formalism. Identifying elements $f \in \mathbb{R}_{cs,+}^{1|1}$ with pairs (y, θ) of functions $\theta \in C^\infty(S)^{odd}$, $y \in C^\infty(S)^{ev}$ with $y_{red}(s) \in \mathbb{R}_+$ for all $s \in S_{red}$, we can write $E_S^+(S_{y,\theta}^{1|1})$ in the form

$$E_S^+(S_{y,\theta}^{1|1}) = A(y) + \theta B(y).$$

Here $A, B \colon \mathbb{R}_+ \to \mathcal{N}(H,H)$ are smooth maps, described via the S-point formalism by

$$\mathbb{R}_+(S) = C^\infty(S)^{ev} \longrightarrow \mathcal{N}(H,H)(S) \qquad y \mapsto A(y) \quad (\text{resp. } B(y)).$$

Lemma 87. *The relation* (86) *implies the following relations for the operators* $A(y)$, $B(y)$*:*

$$\begin{aligned} A(y_1)A(y_2) &= A(y_1+y_2) \\ A(y_1)B(y_2) &= B(y_1)A(y_2) = B(y_1+y_2) \\ B(y_1)B(y_2) &= -A'(y_1+y_2) \end{aligned} \tag{88}$$

Proof. Writing out the left hand side of equation 28 for $f = (y_1, \theta_1)$ and $g = (y_2, \theta_2)$, we obtain

$$\begin{aligned} &E_S^+(I^{1|1}_{y_1,\theta_1}) \circ E_S^+(I^{1|1}_{y_1,\theta_1}) \\ =&(A(y_1)+\theta_1 B(y_1))(A(y_2)+\theta_2 B(y_2)) \\ =&A(y_1)A(y_2)+\theta_1 B(y_1)A(y_2)+\theta_2 A(y_1)B(y_2)-\theta_1\theta_2 B(y_1)B(y_2). \end{aligned}$$

Here the minus sign is a consequence of permuting the odd elements θ_1 and $B(y_1)$. In order to expand the right hand side, we recall from equation (60) that

$$\mu((y_1,\theta_1),(y_2,\theta_2)) = (y_1+y_2+\theta_1\theta_2, \theta_1+\theta_2).$$

It follows that the right hand side of equation 86 is equal to

$$\begin{aligned} &E_S^+(I^{1|1}_{\mu((y_1,\theta_1),(y_2,\theta_2))}) \\ =&A(y_1+y_2+\theta_1\theta_2)+(\theta_1+\theta_2)B(y_1+y_2+\theta_1\theta_2) \\ =&A(y_1+y_2)+A'(y_1+y_2)\theta_1\theta_2+(\theta_1+\theta_2)(B(y_1+y_2)+B'(y_1+y_2)\theta_1\theta_2) \\ =&A(y_1+y_2)+\theta_1 B(y_1+y_2)+\theta_2 B(y_1+y_2)+\theta_1\theta_2 A'(y_1+y_2). \end{aligned}$$

Comparing coefficients then yields the desired relations. $\square$

We note that

$$Z_E(y) = E^+_{\mathbb{R}_+}(S^{1|1}_{y,0}) = \operatorname{str} E^+_{\mathbb{R}_+}(I^{1|1}_{y,0}) = \operatorname{str} A(y),$$

where the first equality is the definition of the partition function (see Definiton 79), and the second equality is equation 85. Hence Theorem 84 is a consequence of the following algebraic result.

Proposition 89. *Let $A(y)$, $B(y)$ be smooth families of nuclear operators parametrized by $y \in \mathbb{R}_+$ satisfying relations* (88). *Let H_1 be the eigenspace of $A(1)$ with eigenvalue $+1$, and let* $\operatorname{sdim} H_1$ *be its super dimension. Then*

$$\operatorname{str} A(y) = \operatorname{sdim} H_1.$$

We recall that a nuclear operator is compact; in particular, any (generalized) eigenspace of a nuclear operator corresponding to a non-zero eigenvalue is finite dimensional.

Proof. The third of the relations 88 implies that

$$A'(y) = -B(\frac{y}{2})B(\frac{y}{2}) = -\frac{1}{2}[B(\frac{y}{2}), B(\frac{y}{2})],$$

where $[B(\frac{y}{2}), B(\frac{y}{2})]$ is the graded commutator of the odd operator $B(\frac{y}{2})$ with itself. Taking the super trace, we obtain

$$\frac{d}{dy} \operatorname{str} A(y) = -\frac{1}{2} \operatorname{str}[B(\frac{y}{2}), B(\frac{y}{2})] = 0,$$

since the super trace vanishes on graded commutators. This shows that $\operatorname{str} A(y)$ is independent of $y \in \mathbb{R}_+$.

To relate $\operatorname{str} A(1)$ to the super dimension of the eigenspace H_1, we apply the spectral calculus developed by Wrobel [Wr, Thm. 2.3] to the compact operator $A = A(1)$. The spectrum of any compact operator is a countable bounded subset $\sigma \subset \mathbb{C}$ whose only possible accumulation point is $0 \in \mathbb{C}$ (cf. [Ed, Thm. 9.10.2]). For $\lambda \in \sigma \setminus \{0\}$ the corresponding (generalized) eigenspace H_λ is finite dimensional. Since the eigenspace with eigenvalue 0 doesn't contribute to the super trace of A we have

$$\operatorname{str} A = \sum_{\lambda \in \sigma \setminus \{0\}} \operatorname{str}(A_{|H_\lambda})$$

To calculate $\operatorname{str}(A_{|H_\lambda})$, we choose a basis of H_λ such that the matrix corresponding to A is an upper triangular matrix with diagonal entries λ. Then $\operatorname{str}(A_{|H_\lambda}) = \lambda \operatorname{sdim} H_\lambda$ and similarly $\operatorname{str}(A^2_{|H_\lambda}) = \lambda^2 \operatorname{sdim} H_\lambda$.

Using Wrobel's spectral calculus (cf. [Wr, Thm. 2.3]), projection operators onto the generalized eigenspaces H_λ for $\lambda \neq 0$ can be constructed by functional calculus out of the operator A. Since $A = A(1)$ commutes with the operators $A(y)$, $B(y)$ for all $y \in \mathbb{R}_+$ (by the relations 88), also the projection operator onto H_λ commutes with $A(y)$ and $B(y)$. In particular, the operators $A(y)$, $B(y)$ map the subspace H_λ to itself and we can apply the argument above to the subspace H_λ to conclude that the super trace of $A(y)$ restricted to H_λ is independent of y.

Now let us calculate $\operatorname{str} A^2_{|H_\lambda}$ in two different ways. On one hand, $A^2 = A(1)A(1) = A(2)$ (by the first of the relations 88, and hence

$$\operatorname{str} A^2_{|H_\lambda} = \operatorname{str} A(2)_{|H_\lambda} = \operatorname{str} A_{|H_\lambda} = \lambda \operatorname{sdim} H_\lambda.$$

On the hand, we calculated above $\operatorname{str}(A^2_{|H_\lambda}) = \lambda^2 \operatorname{sdim} H_\lambda$. This implies $\operatorname{sdim} H_\lambda = 0$ for $\lambda \neq 1$ and hence $\operatorname{str} A = \operatorname{str} A_{|H_1} = \operatorname{sdim} H_1$. □

5.2 Partition functions of QFT's of dimension $2|1$

In this section we will prove our main result theorem 1 concerning the partition function of a super symmetric quantum field theory of dimension $2|1$. The proof is entirely analogous to the proof of the corresponding result 84 for field theories of dimension $1|1$; it is based by analyzing the algebraic data obtained by applying the QFT to the certain objects and morphisms of the Riemannian bordism category $\mathrm{RB}^{2|1}$. More precisely, if $E\colon \mathrm{RB}^{2|1} \to \mathrm{TV}^{\pm}$ is the QFT at hand, we evaluate the functor

$$E_S^+ \colon \mathrm{RB}_S^{2|1} \xrightarrow{E_S} \mathrm{TV}^{\pm} \longrightarrow \mathrm{TV}$$

on certain objects and morphisms of $\mathrm{RB}_S^{2|1}$ described in example 81.

- The locally convex vector space $H \overset{\text{def}}{=} E^+_{\mathrm{pt}}(S_1^{1|1})$ associated to the *super circle* of length 1 and the vector bundle over $\mathbb{R}_+$ given by $E^+_{\mathrm{pt}}(S_\ell^{1|1})$, where $\ell\colon \mathbb{R}_+ \to \mathbb{R}_+$ is the identity. The fiber of this bundle at $1 \in \mathbb{R}_+$ is H, and we fix a trivialization $E^+_{\mathrm{pt}}(S_\ell^{1|1}) \cong S \times H$ which is the identity at $1 \in \mathbb{R}_+$.

- The function $E_S^+(T^{2|1}_{\ell,f}) \in C^\infty(S)^{ev}$ associated to the family of *super tori* $T^{2|1}_{\ell,f} \in \mathrm{RB}_S^{2|1}(\emptyset,\emptyset)$ determined by $f \in \mathbb{R}^{2|1}_+(S)$.

- The function $E_S^+(C^{2|1}_{\ell,f}) \in \mathrm{SMAN}(S, \mathcal{N}(H,H))$ associated to the family of *super cylinders* $C^{2|1}_{\ell,f} \in \mathrm{RB}_S^{2|1}(S_\ell^{1|1}, S_\ell^{1|1})$ determined by $\ell \in \mathbb{R}_+(S)$, $f \in \mathbb{R}^{2|1}_{cs,+}(S)$.

We recall from part 2 of Lemma 83 that $T^{2|1}_{\ell,f} = \widehat{C^{2|1}_{\ell,f}}$ (i.e., the family of super tori $T^{2|1}_{\ell,f}$ is obtained by gluing the domain and range of the family of super cylinders $C^{2|1}_{\ell,f}$). Hence Proposition 25 (or rather its generalization to $\mathrm{RB}^{2|1}_S$) implies

$$E^+_S(T^{2|1}_{\ell,f}) = \operatorname{str} E^+_S(C^{2|1}_{\ell,f}). \tag{90}$$

The relation $C^{2|1}_{\ell,f} \circ C^{2|1}_{\ell,f'} = C^{2|1}_{\ell,\mu(f,f')}$ (see Lemma 83 implies

$$E^+_S(C^{2|1}_{\ell,f}) \circ E^+_S(C^{2|1}_{\ell,g}) = E^+_S(C^{2|1}_{\ell,\mu(f,g)}) \tag{91}$$

Identifying $f \in \mathbb{R}^{2|1}_+(S)$ with triples (x, y, θ) of functions $x, y \in C^\infty(S)^{ev}$, $\theta \in C^\infty(S)^{odd}$ with $y_{red}(s) > 0$ for all $s \in S_{red}$, we write $E^+_S(C_{\ell,x,y,\theta})$ in the form

$$E^+_S(C_{\ell,x,y,\theta}) = A(\ell, x, y) + \theta B(\ell, x, y), \tag{92}$$

where $A, B\colon \mathbb{R}_+ \times \mathbb{R}^2_+ \to \mathcal{N}(H, H)$ are smooth maps. Fixing ℓ for now, we will write $A(x, y)$, $B(x, y)$ instead of $A(\ell, x, y)$, $B(\ell, x, y)$.

Lemma 93. *Relation 91 implies the following relations for the functions A, B:*

$$\begin{aligned}
A(x_1, y_1)A(x_2, y_2) &= A(x_1 + x_2, y_1 + y_2)\\
A(x_1, y_1)B(x_2, y_2) &= B(x_1, y_1)A(x_2, y_2) = B(x_1 + x_2, y_1 + y_2)\\
B(x_1, y_1)B(x_2, y_2) &= -\frac{\partial A}{\partial \bar z}(x_1 + x_2, y_1 + y_2)
\end{aligned} \tag{94}$$

Proof. Writing out the left hand sice of equation (91) for $f = (x_1, y_1, \theta_1)$ and $g = (x_2, y_2, \theta_2)$ we obtain

$$\begin{aligned}
&E^+_S(C^{1|1}_{\ell,f}) \circ E^+_S(C^{1|1}_{\ell,g})\\
=&(A(\tau_1) + \theta_1 B(\tau_1))(A(\tau_2) + \theta_2 B(\tau_2))\\
=&A(\tau_1)A(\tau_2) + \theta_1 B(\tau_1)A(\tau_2) + \theta_2 A(\tau_1)B(\tau_2) - \theta_1\theta_2 B(\tau_1)B(\tau_2),
\end{aligned} \tag{95}$$

where we write τ_i instead of $(x_i, y_i) \in \mathbb{R}^2_+$.

In order to expand the right hand side, we need to write the multiplication map $\mu\colon \mathbb{R}^{2|1}_{cs} \times \mathbb{R}^{2|1}_{cs} \to \mathbb{R}^{2|1}_{cs}$ explicitly in terms of the coordinate functions

x, y, θ; rewriting equation (69) (which is written in terms of $z = x + iy$, $\bar{z} = x - iy$ and θ) we obtain:

$$\mu((x_1, y_1, \theta_1), (x_2, y_2, \theta_2)) = (x_1 + x_2 + \frac{1}{2}\theta_1\theta_2, y_1 + y_2 + \frac{i}{2}\theta_1\theta_2, \theta_1 + \theta_2).$$

It follows that the right hand side of equation 91 is equal to

$$\begin{aligned} & E_S^+(C^{2|1}_{\ell,\mu(f,g)}) \\ =& A(x + \frac{1}{2}\theta_1\theta_2, y + \frac{i}{2}\theta_1\theta_2) + (\theta_1 + \theta_2)B(x + \frac{1}{2}\theta_1\theta_2, y + \frac{i}{2}\theta_1\theta_2), \end{aligned}$$

where we abbreviate $x_1 + x_2$ by x and $y_1 + y_2$ by y. Now we use Taylor expansion around the point (x, y) to rewrite the first term as follows:

$$\begin{aligned} & A(x + \frac{1}{2}\theta_1\theta_2, y + \frac{i}{2}\theta_1\theta_2) \\ =& A(x, y) + \frac{\partial A}{\partial x}(x, y)\frac{1}{2}\theta_1\theta_2 + \frac{\partial A}{\partial y}(x, y)\frac{i}{2}\theta_1\theta_2 \\ =& A(x, y) + \frac{\partial A}{\partial \bar{z}}(x, y)\theta_1\theta_2, \end{aligned}$$

where as usual $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. We note that all higher order terms in the Taylor expansion vanish since $\theta_i^2 = 0$. Similarly, we obtain a Taylor expansion for $B(x + \frac{1}{2}\theta_1\theta_2, y + \frac{i}{2}\theta_1\theta_2)$. Putting the terms together, we obtain:

$$E_S^+(C^{2|1}_{\ell,\mu(f,g)}) = A(x, y) + \frac{\partial A}{\partial \bar{z}}(x, y)\theta_1\theta_2 + B(x, y)\theta_1 + B(x, y)\theta_2 \tag{96}$$

Comparing coefficients in equations (95) and (96) then yields the desired relations. □

The following algebraic result is the key step in the proof of our main theorem 1.

Proposition 97. *Let $A(\tau)$, $B(\tau)$ be smooth families of nuclear operators parametrized by $\tau \in \mathbb{R}^2_+ \subset \mathbb{C}$ satisfying relations* (94) *and $A(\tau + 1) = A(\tau)$. Then the function* $\operatorname{str} A(\tau)$ *is a holomorphic function with an expansion of the form*

$$\operatorname{str} A(\tau) = \sum_{k \in \mathbb{Z}} a_k q^k,$$

where $q = e^{2\pi i\tau}$, the coefficients a_k are integers and $a_k = 0$ for sufficiently negative k. More generally, if $A(\ell,\tau)$, $B(\ell,\tau)$ is a family of such operators depending smoothly on some parameter $\ell \in \mathbb{R}_+$, then $\operatorname{str} A(\ell,\tau)$ *is in fact independent of ℓ.*

Assuming this statement for now, we next prove our main theorem.

Proof of Theorem 1. We note that

$$Z_E(\ell,\tau) = E_S^+(T^{2|1}_{\ell,\tau,0}) = \operatorname{str} E_S^+(C^{2|1}_{\ell,\tau,0}) = \operatorname{str} A(\ell,\tau),$$

where the first equality is the definition of the partition function (see Definiton 79), and the second equality is equation 90.

If $A, B\colon \mathbb{R}_+ \times \mathbb{R}^2_+ \to \mathcal{N}(H,H)$ are the smooth families of nuclear operators defined by equation (92), these satisfy the relations (94) by Lemma 83. Moreover, the equality $C^{2|1}_{\ell,x+1,y,\theta} = C^{2|1}_{\ell,x,y,\theta} \in \mathrm{RB}^{2|1}_S(S^{1|1}_\ell, S^{1|1}_\ell)$ (see part 3 of Lemma 83) implies in particular $A(\ell, x+1, y, \theta) = A(\ell, x, y, \theta)$. Then the above proposition implies the holomorphicity of its partition function, the integrality of its q-expansion, and the fact that the coefficients a_k vanish for $k << 0$. To see that $Z_E(1,\tau)$ is invariant under the $SL_2(\mathbb{Z})$-action, we note that for $g \in SL_2(\mathbb{Z})$ we have

$$Z_E(\ell, g\tau) = Z_E(\ell|c\tau + d|, g\tau) = Z_E(g(\ell,\tau)) = Z_E(\ell,\tau).$$

Here the first equality comes from independence of $\operatorname{str} A(\ell,\tau)$ of ℓ, the second is by definition of the $SL_2(\mathbb{Z})$-action on $\mathbb{R}_+ \times \mathbb{R}^2_+$, and the third is Lemma 23 (or rather its generalization to QFT's of dimension $2|1$ which is straightforward). $\square$

Proof of Proposition 97. We first prove holomorphicity of $\operatorname{str} A(\tau)$. The third of the relations (94) implies that

$$\frac{\partial A}{\partial \bar{z}}(\tau) = -B^2(\frac{\tau}{2}) = -\frac{1}{2}[B(\frac{\tau}{2}), B(\frac{\tau}{2})]$$

(where $[\quad,\quad]$ is the *graded* commutator) is again a trace class operator and hence we can calculate:

$$\frac{\partial}{\partial \bar{z}} \operatorname{str} A(\tau) = -\operatorname{str}(B^2(\frac{\tau}{2})) = -\frac{1}{2}\operatorname{str}[B(\frac{\tau}{2}), B(\frac{\tau}{2})] = 0$$

This shows that $\operatorname{str} A(\tau)$ is a holomorphic function on the upper half plane $\mathbb{R}^2_+$.

We observe that the relations (94) imply that the compact operators $A(\tau)$ for various $\tau \in \mathbb{R}^2_+$ all commute with each other. In particular, we can consider simultaneous generalized eigenspaces H_λ for the family of operators $A(\tau)$, $\tau \in \mathbb{R}^2_+$, where $\lambda \colon \mathbb{R}^2_+ \to \mathbb{C}$ is the corresponding eigenvalue. We note that the first of the relations (94) imply that λ is an exponential map, i.e.,

$$\lambda(\tau_1 + \tau_2) = \lambda(\tau_1) \cdot \lambda(\tau_2).$$

It follows that λ is either *identically* equal to zero, or its image is contained in $\mathbb{C}^\times$ (the non-zero complex numbers); in the latter case, λ can be written in the form $\lambda(\tau) = e^{\widetilde{\lambda}(\tau)}$, where $\widetilde{\lambda} \colon \mathbb{R}^2_+ \to \mathbb{C}$ is a homomorphism (of additive semigroups). The continuity of $\widetilde{\lambda}$ (which follows from the fact that $A \colon \mathbb{R}^2_+ \to \mathcal{N}(H, H)$ is smooth) implies that $\widetilde{\lambda}$ is the restriction of an $\mathbb{R}$-linear map $\mathbb{C} \to \mathbb{C}$. It will be convenient to write $\widetilde{\lambda}$ in the form $\widetilde{\lambda}(\tau) = 2\pi i(a\tau - b\bar{\tau})$ for $a, b \in \mathbb{C}$; in other words,

$$\lambda(\tau) = e^{2\pi i(a\tau - b\bar{\tau})} = q^a \bar{q}^b \tag{98}$$

We note that the condition $A(\tau + 1) = A(\tau)$ implies $a - b \in \mathbb{Z}$. Let us denote by $H_{a,b} \subset H$ the generalized eigenspace corresponding to the eigenvalue function $\lambda(\tau)$ given by equation (98). We note that the spaces $H_{a,b}$ are *finite dimensional*, since the operators $A(\tau)$ are trace class and hence compact; in particular, any generalized eigenspace with non-zero eigenvalue is finite dimensional.

Since only the non-zero eigenspaces contribute to the super trace of $A(\tau)$, we have

$$\operatorname{str} A(\tau) = \sum_{a,b} \operatorname{str}(A(\tau)_{|H_{a,b}}).$$

It is straightforward to calculate the super trace of $A(\tau)$ restricted to $H_{a,b}$; $A(\tau)$ is an even operator and hence it maps the even (resp. odd) part of H to itself, and we can calculate the trace of $A(\tau)$ acting $H^\pm_{a,b}$ separately. There is a basis of $H^\pm_{a,b}$ such that the matrix corresponding to $A(\tau)$ is upper triangular with diagonal entries $\lambda_{a,b}(\tau)$. It follows that

$$\operatorname{str}\left(A(\tau)_{|H_{a,b}}\right) = \lambda_{a,b}(\tau) \operatorname{sdim} H_{a,b}.$$

We note that the argument proving the holomorphicity of $\operatorname{str} A(\tau)$ continues to hold if we restrict $A(\tau)$ to the subspace $H_{a,b}$ (the projection map onto $H_{a,b}$

is built by *functional calculus* from the operators $A(\tau)$; hence any operator that commutes with all $A(\tau)$'s – like $B(\tau/2)$ – will also commute with the projection operator and hence preserve the subspace $H_{a,b}$). We note that the function $\lambda_{a,b}(\tau)$ is holomorphic if and only if $b = 0$. It follows:

$$\operatorname{sdim} H_{a,b} = 0 \qquad \text{for} \quad b \neq 0.$$

In particular, the only contribution to the super trace of $A(\tau)$ comes from the space $H_{a,0}$, which forces a to be an integer. We conclude

$$\operatorname{str} A(\tau) = \sum_{k \in \mathbb{Z}} \operatorname{str}\left(A(\tau)_{|H_{k,0}}\right) = \sum_{k \in \mathbb{Z}} \lambda_{k,0}(\tau) \operatorname{sdim} H_{k,0} = \sum_{k \in \mathbb{Z}} q^k \operatorname{sdim} H_{k,0}.$$

We note that the eigenspaces $H_{k,0}$ must be trivial for sufficiently negative k (otherwise the corresponding eigenvalues q^k are arbitrarily large), and hence $a_k = \operatorname{sdim} H_{k,0}$ is zero. □

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