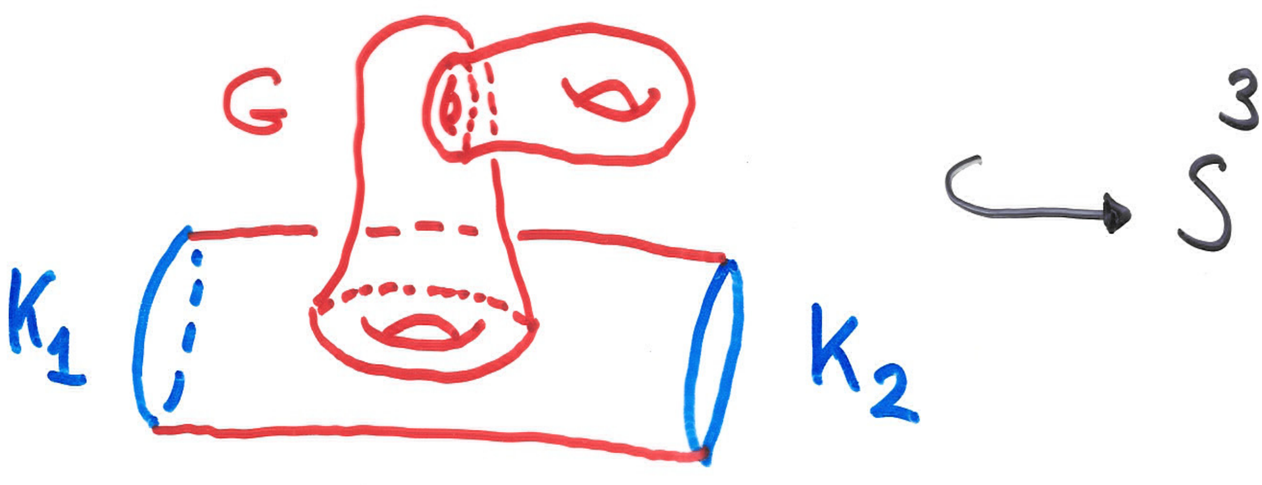
Gropes in 3-space

Lemma: $c: S^1 \to X$ represents an element in $\pi_1 X^{(k)}$ (resp. $\pi_1 X_k$)

$\Leftrightarrow$ c extends to $\bar{c}: G \to X$,

G a grope of height k (resp. class k).

Definition: Two knots $K_1, K_2 : S^1 \hookrightarrow S^3$ are grope cobordant of class k iff they cobound $G \hookrightarrow S^3$, G class k grope.



class 4

3D-Theorem

joint with
Jim Conant

(a) $\mathcal{K}_k := \dfrac{\{\text{knots}\}}{\text{class } k \text{ grope cobordism}}$

are finitely generated abelian groups

k	2	3	4	5	....
$\mathcal{K}_k$	0	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}/2$	....

(b) $\mathcal{K}_k \otimes \mathbb{Q} \cong \dfrac{\langle \text{Feynman Diagrams}_{\text{degree} < k} \rangle_{\mathbb{Q}}}{\text{AS, IHX - relations}}$

via Kontsevich integral. [Garoufalidis, Rozansky]

Feynman diagram: , , ...

graded by grope degree $:= v + b_1$.

(C) Two knots share the same Vassiliev invariants of degree $< k$

$\Longleftrightarrow$ they cobound a capped grope cobordism of class k.

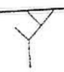
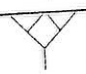
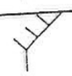
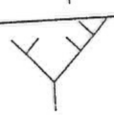
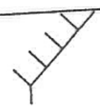
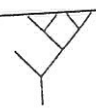
[Goussarov - Habiro theory of clasper surgery]

Remark: Finite type invariants like the Kontsevich integral were already known! In 4D, we won't be that fortunate and we'll have to invent new invariants .....

Refinements: For each tree type T, may study knots modulo

(a) capped grope cobordism
(b) grope cobordism
(c) grope concordance (in $S^3 \times I$)

} of type T

Tree Type T	(a) $\mathcal{K}/cT$	(b) $\mathcal{K}/T$	(c) $\mathcal{K}/T^4$
	$\{0\}$	$\{0\}$	$\{0\}$
	$\mathbb{Z}(c_2)$	$\mathbb{Z}_2(\mathrm{arf})$	$\mathbb{Z}_2(\mathrm{arf})$
	$\mathbb{Z}(c_3) \oplus \mathbb{Z}(c_2)$	$\mathbb{Z}(c_2)$	$\mathbb{Z}_2(\mathrm{arf})$
	$\mathbb{Z}(c_3) \oplus \mathbb{Z}(c_2)$	$\mathbb{Z}(c_2)$	$\mathbb{Z}_2(\mathrm{arf})$
	$\mathbb{Z}(c_4) \oplus \mathbb{Z}(c_4') \oplus \mathbb{Z}(c_3) \oplus \mathbb{Z}(c_2)$	$\mathbb{Z}_2(c_3) \oplus \mathbb{Z}(c_2)$	$\mathbb{Z}_2(\mathrm{arf})$
	$\mathbb{Z}(c_4) \oplus \mathbb{Z}(c_4') \oplus \mathbb{Z}(c_3) \oplus \mathbb{Z}(c_2)$	$\mathbb{Z}_2(c_3) \oplus \mathbb{Z}(c_2)$	$\mathbb{Z}_2(\mathrm{arf})$
	?	S-equivalence or Bl-forms	cobordism of Bl-forms

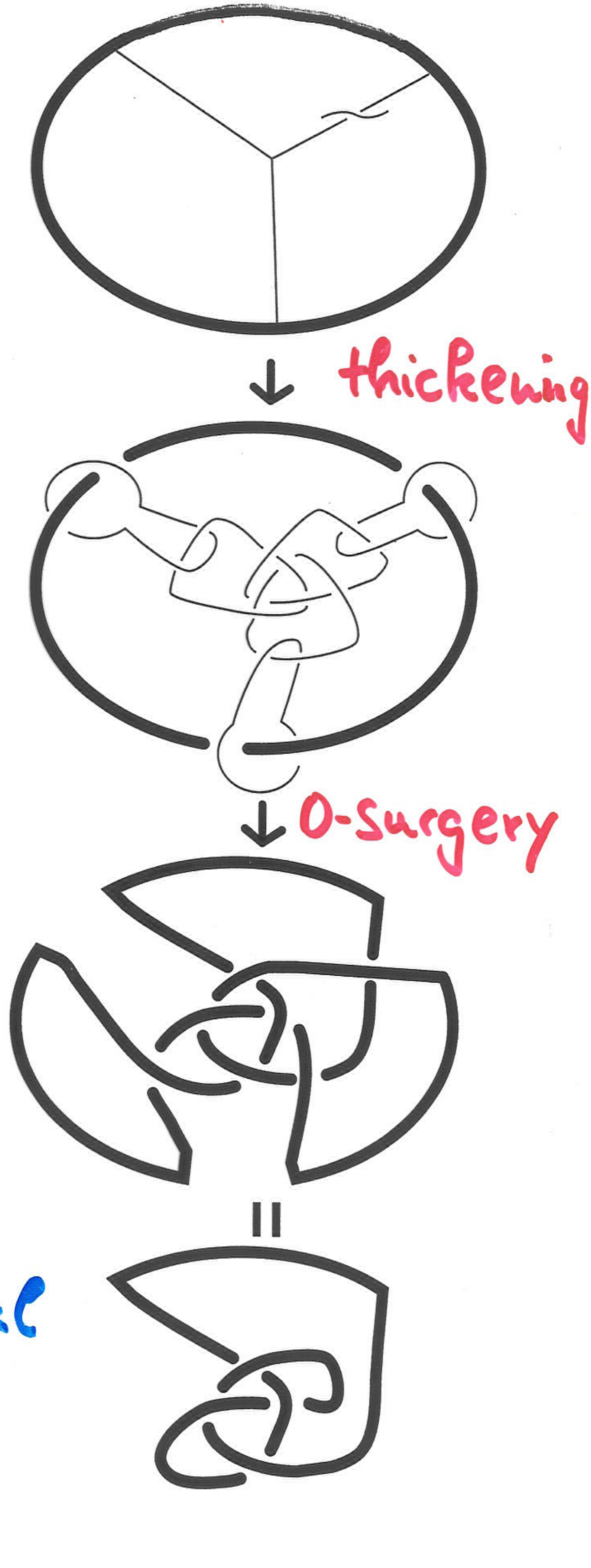
mapping diagrams to knots

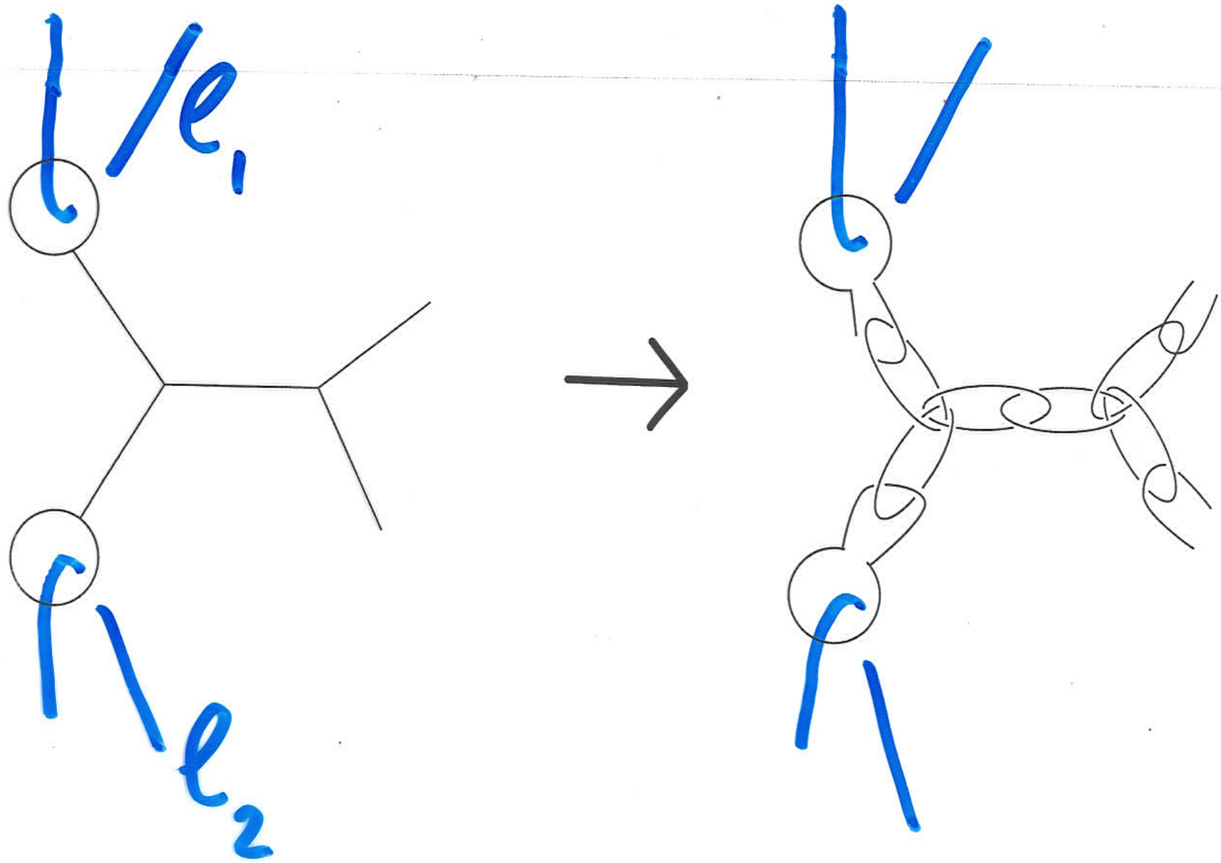
$$\mathcal{K}_k = \frac{\text{knots}}{G_k}$$

$$B_k \xrightarrow{\phi} \frac{G_k}{G_{k+1}}$$

is a geometric inverse to the Kontsevich integral

Habiro's clasper surgery.





Claspers are trivalent graphs with leaves at their tips.

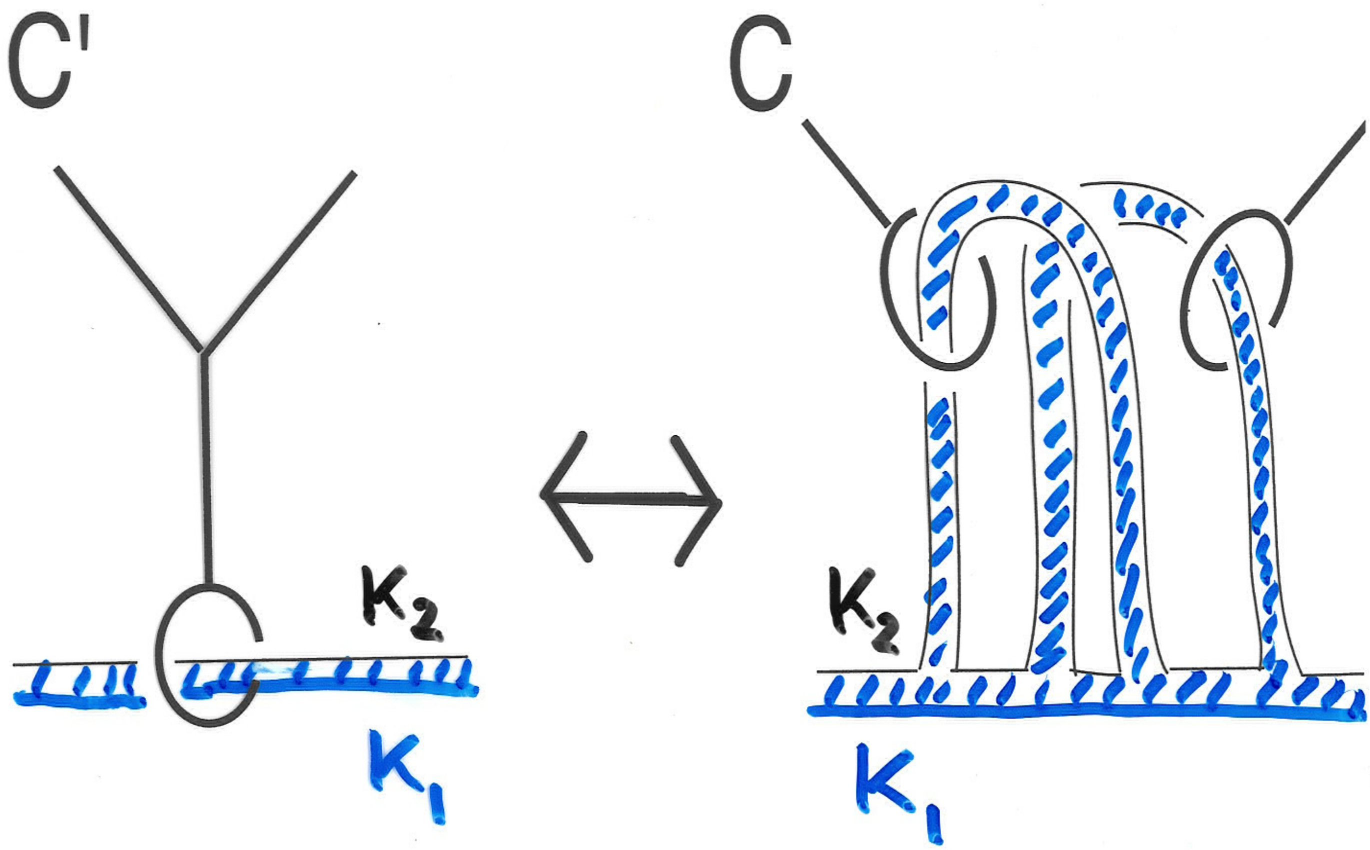
They act on links by the rules

edge $\rightsquigarrow$ Hopf link

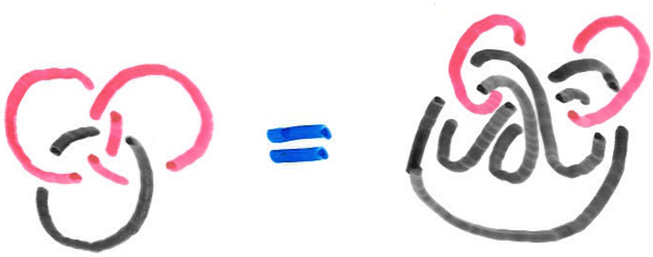
trivalent vertex $\rightsquigarrow$ Bor.

plus 0-surgery.

Relation to gropes



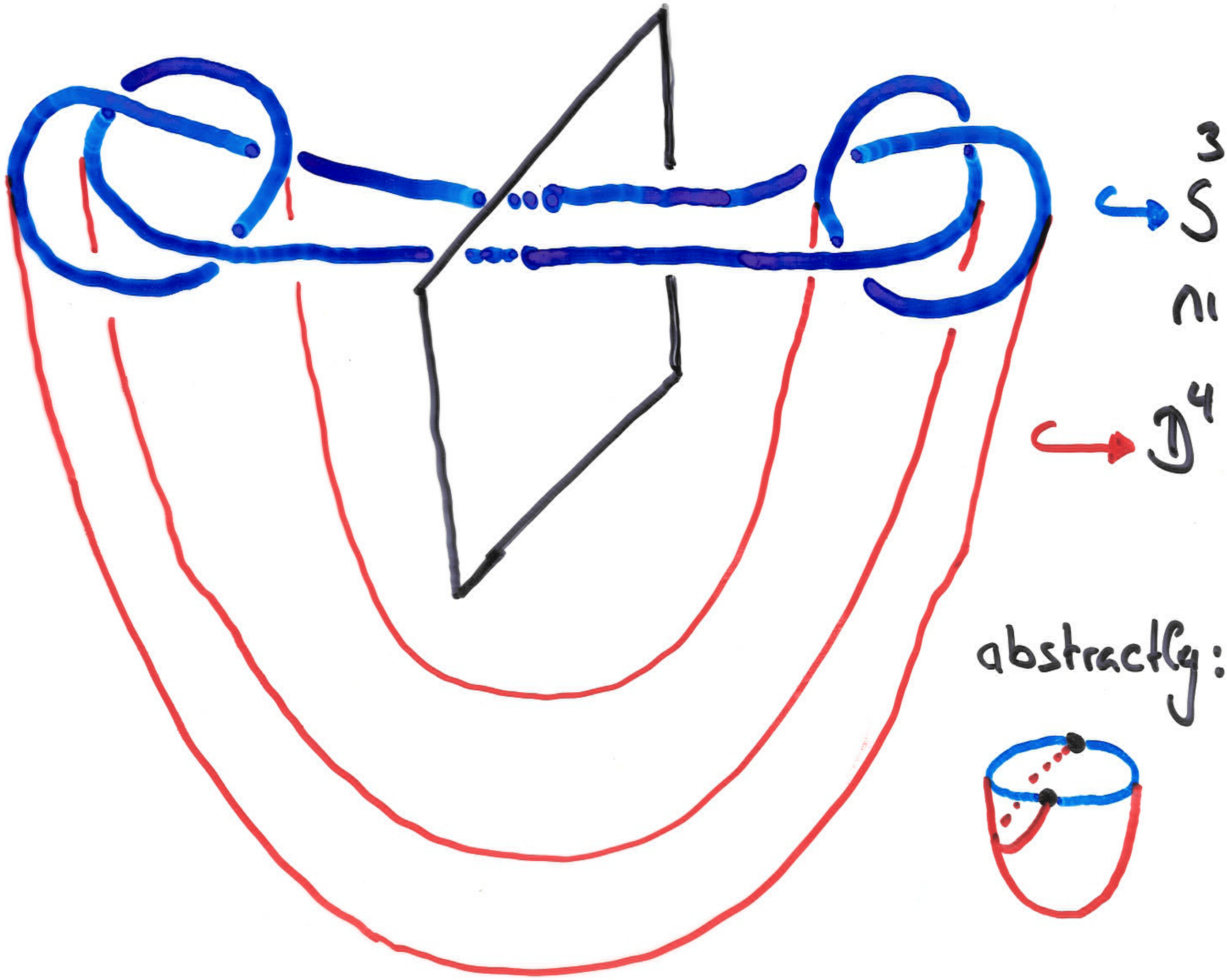
local move, gives inductively a grope cobordism of class = grope degree of C. Proof:



Concordance Group

Lemma: $\dfrac{\{\text{knots}\}}{\{\text{slice knots}\}} =: \mathcal{C}$ is an unknown abelian group

Proof: $K \# (-K)$ is slice:



The concordance group $\mathcal{C}$ is not finitely generated!

In fact, all one needs is the averaged signature

$$\mathbb{R} \ni \sigma_{\mathbb{Z}}(K) := \int_{t \in S^1} \sigma(h_K(t))\,dt$$

Remark: K slice $\Longrightarrow$

K is alg. slice: $S_K \cong \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$

so all signatures $\sigma(h_K(t)) = 0$.

4D-Theorem:

joint with Tim Cochran, Kent Orr.

Two knots are grope concordant of height k if $\exists\ G \hookrightarrow S^3 \times [0,1]$....

$G_k \leq \mathcal{C}$ are knots bounding such gropes.

$\mathcal{C} = G_1$

1940's $G_1 \geq G_2$: $\mathbb{Z}/2$: Arf invariant ...

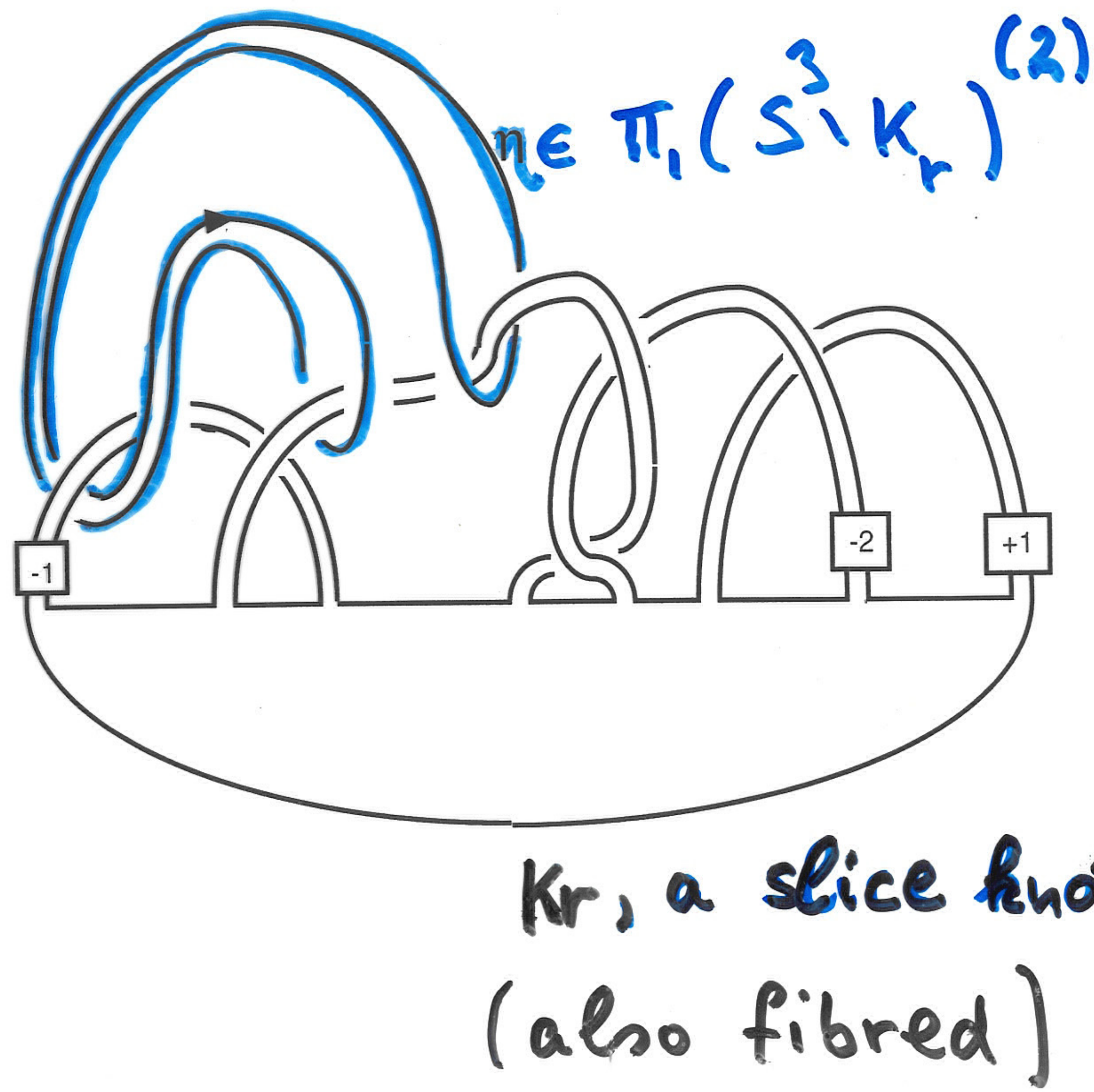
1972 $G_2 \geq G_3$: ∞ rank : twisted signatures & Levine, Cappell-Shaneson

1978 $G_3 \geq G_4$: ∞ rank : dihedral signatures Casson-Gordon, ...

1998 $G_4 \geq G_5$: ∞ rank : von Neumann signatures of 3-solvable covers C. O. T.

2001 $G_5 \geq$ ⋮ ?

Constructing new knots

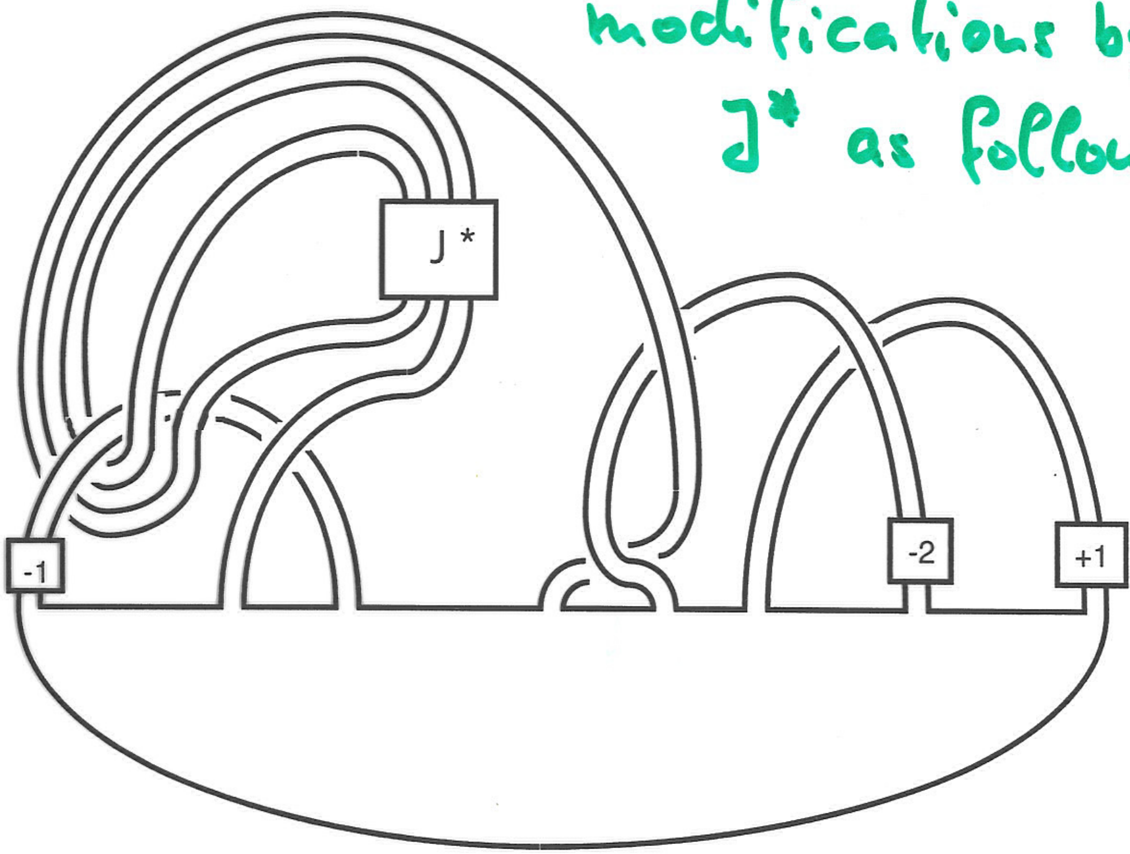


K_r, a slice knot

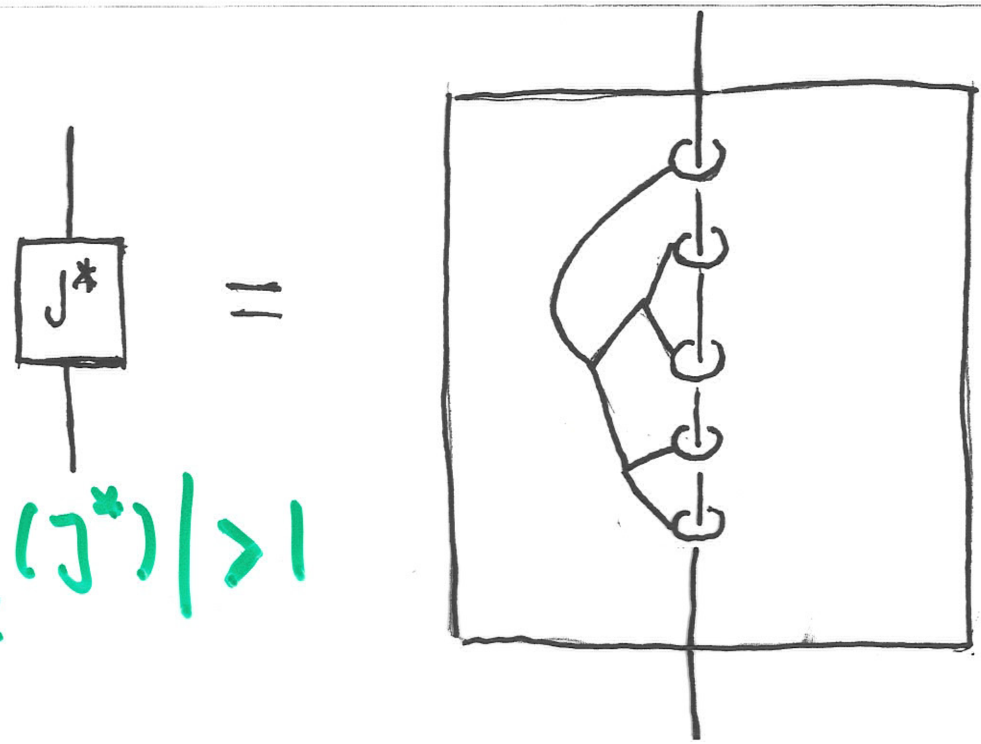
(also fibred)

Example: $K \in G_4 \setminus G_5$

is obtained by a finite sequence of modifications by J^* as follows:



K



$|\sigma_{\mathbb{Z}}(J^*)| > 1$

$\Rightarrow$ bounds a grope of height 2

Proof of 4D-Theorem

A height k grope in D^4 defines

- a $(k-1)$-solvable group Γ_k
- a hermitian form h_k over $\mathbb{Z}\Gamma_k$

Key: $h_k \cong \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ over $\mathcal{N}\Gamma_k \supseteq \mathbb{C}\Gamma_k$, with $\underbrace{}_{P_+}\ \underbrace{}_{P_-}$ and $\mathbb{Z}\Gamma_k \subseteq \mathbb{C}\Gamma_k$

$$\begin{tikzcd} \mathcal{N}\Gamma_k \arrow[d, "\mathrm{tr}"] & \supseteq & \mathbb{C}\Gamma_k \arrow[dll, "\text{coeff. at } e"] \\ \mathbb{C} & & \end{tikzcd}$$

Lemma: The von Neumann signature $\sigma^{(2)}_{\Gamma_k}(h_k) = \dim_{\Gamma_k} P_+ - \dim_{\Gamma_k} P_- \in \mathbb{R}$ vanishes if the grope extends to height $(k+1)$.

12c

Non-commutative Alexander modules

Main technical problem: Need $\sigma^{(2)}_{\Gamma_k}(h_k) \neq 0$ for all height k gropes.

Common feature: Behave well w.r.t. Blanchfield duality on $\mathcal{A}_k(K) := H_1((S^3 \setminus K)^{(k)})$ order k Alexander module.

Done if $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{k-1}$ have a unique Lagrangian. This can be achieved for $k = 4$ ∎

Von Neumann algebras

For any discrete group Γ, define

$$\begin{array}{ccccccc} \mathbb{C}\Gamma & \subseteq & C^*\Gamma & \subseteq & \mathcal{N}\Gamma & \xrightarrow{tr_\Gamma} & \mathbb{C} \\ \rotatebox{90}{$\in$} & & & & & & \rotatebox{90}{$\ni$} \\ \sum_{\text{finite}} a_g \cdot g & & & \longmapsto & & & a_e \end{array}$$

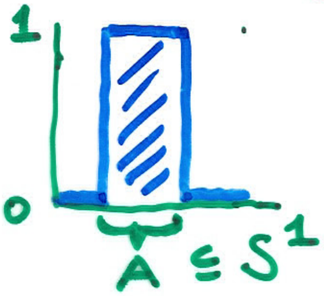
e.g. $\Gamma = \mathbb{Z} \Rightarrow \mathbb{C}\Gamma =$ polynomial functions on S^1

$$\Rightarrow \begin{array}{ccccccc} \mathbb{C}\mathbb{Z} & \subseteq & C^0(S^1) & \subseteq & L^\infty(S^1) & \xrightarrow{tr_\mathbb{Z}} & \mathbb{C} \\ & & & & \rotatebox{90}{$\in$} & & \\ & & & & f & \longmapsto & \int_{S^1} f \end{array}$$

Continuous Dimension: $p = p^2 \in \mathcal{N}\Gamma$

projection operator

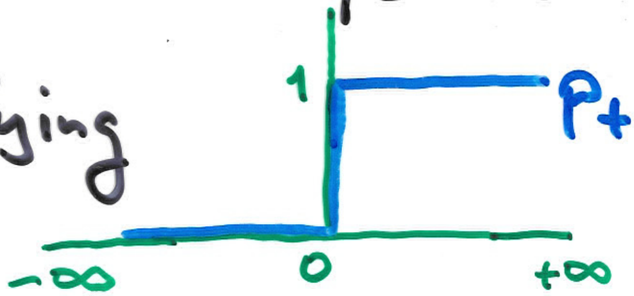
$$\Rightarrow \quad tr_\Gamma(p) \in [0,1].$$

e.g. $\Gamma = \mathbb{Z}$, $p =$  $= \chi_A$

$tr_\mathbb{Z}(p) = vol(A)$

Functional Calculus

Given a hermitian form $h = h^*$ over $\mathcal{N}\Gamma$, get idempotents $p_+(h)$ and $p_-(h) \in \mathcal{N}\Gamma$ by applying



to $\mathrm{spec}(h) \subset \mathbb{R}$

Definition: $\sigma^{(2)}_{\Gamma}(h) := \mathrm{tr}_{\Gamma}(p_+) - \mathrm{tr}_{\Gamma}(p_-)$

von Neumann signature of h.

e.g. $\Gamma = \mathbb{Z} \Rightarrow$

$$\sigma^{(2)}_{\mathbb{Z}}(h) = \int_{w \in S^1} \sigma(h(w)) \in \mathbb{R}$$

complex hermitian form

Main Theorem : joint with Tim Cochran

$$G_k / G_{k+1} \text{ is infinite } \forall k \geq 2$$

Key New Ingredient :

Cheeger - Gromov Estimate for the von Neumann ρ-invariant :

For each closed oriented M^3

$$\exists\, C_M : |\rho_\Gamma(M)| < C_M \quad \forall\, \pi_1 M \to \Gamma$$

Def.: $\rho_\Gamma(M) := \eta_\Gamma(M,g) - \eta(M,g)$

$$\eta_\Gamma(M) := \int_0^\infty tr_\Gamma\left(\tilde{\mathcal{D}} \cdot e^{-t^2 \tilde{\mathcal{D}}^2}\right) dt$$

$\tilde{\mathcal{D}} :=$ signature operator on Γ-cover of M.

(L^2)- Index Theorem implies :

$$\rho_\Gamma(M) = \sigma_\Gamma^{(2)}(W) - \sigma(W) \rightsquigarrow \text{indep. of metric } g$$

if $M = \partial W^4$ (over Γ) .

Proof of Main Theorem:

- If $K \in \mathcal{G}_k$ then $M := \overset{\circ}{S}(K)$ bounds a 4-mfd. W over a k-solvable group Γ_k. $\left(\Gamma^{(k+1)} = 0 \right)$

- $\rho_{\Gamma_k}(M) = \sigma^{(2)}_{\Gamma_k}(W) - \sigma(W)$

  vanishes if the grope extends to height $k+1$.

  "generators"

- Let $K :=$ genetic infection of ribbon R by J along $\eta_i \in \pi_1(S^3 \setminus R)^{(k)}$

  where $\sigma^{(2)}_{\mathbb{Z}}(J) > C_R$ Cheeger-Gromov constant

- Blanchfield duality in $\pi^{(n)}$-covers $(1 \le n \le k)$ $\Rightarrow$ $\left(\frac{1}{2}\right)^k$ of the η_i survive any map

$$\pi_1(S^3 \setminus R)^{(k)} \longrightarrow \Gamma_k^{(k)}$$

All together :

- Choosing isotopy classes of η_i carefully gives $K \in \mathcal{F}_k$.

- If $K \in \mathcal{F}_{k+1}$ then $\exists\, \Gamma_k$ s.t.

$$0 = \rho_{\Gamma_k}(M) = \rho_{\Gamma_k}(R) + \rho_{\mathbb{Z}}(J)$$

contradicting

$$|\rho_{\mathbb{Z}}(J)| = |\rho_{\Gamma_k}(R)| < C_R$$