DERIVED ALGEBRAIC GEOMETRY AND TOPOLOGY

HOT TOPICS COURSE IN BERKELEY, FALL 2006

Contents

The following is an attempt to summarize the main definitions and theorems of the talks given in the course.

1. Simplicial objects and homotopy categories

Let Δ be the category of finite ordered sets, or more precisely, we want the following small representative: the objects of Δ are the ordered sets

$$[n] := \{0, 1, \dots, n\}$$

and the morphisms are (non-strictly) order preserving maps. Every such morphism is a unique composition of a surjection followed by an injection. Any injection can in turn be decomposed into injections $d_i : [n] \to [n+1]$ that *skip the index i*, and any surjection can be decomposed into surjections $s_i : [n+1] \to [n]$ that *repeat the index i*. These are the standard *face and degeneracy maps* and if one orders the indices i increasingly then the above two decompositions are actually unique. All the relations among the d_i, s_j follow from this discussion and need not be remembered!

For any category $\mathbf{C}$, a *simplicial object in* $\mathbf{C}$ is a functor $\Delta^{op} \to \mathbf{C}$, i.e. a presheaf on Δ with values in $\mathbf{C}$. We let $\mathbf{sC} := \mathrm{Fun}(\Delta^{op}, \mathbf{C})$ be the *category of simplicial objects in* $\mathbf{C}$, where the morphisms are natural transformations of functors. To emphasize the values of such a funtor on the sets $[n]$, we sometimes write it as $X_\bullet$ where the bullet is a placeholder, i.e. $X_n := X_\bullet([n])$ are the n-simplices of $X_\bullet$. We shall often simply write $\sigma : X_m \to X_n$ for the value of the morphism given by $\sigma : [n] \to [m]$.

Definition 1.1. A simplex $x \in X_n$ is called *non-degenerate* if it is not of the form $x = s_i(y)$. It is not hard to see that any x has a unique representation $x = \sigma(x')$ with x' non-degenerate and σ surjective.

An important example is the simplicial set $\Delta^k_\bullet \in \mathbf{sSet}$, the *k-simplex* (with exactly one non-degenerate simplex, lying in dimension k) given by

$$(\Delta^k)_n := \mathrm{Hom}_\Delta([n],[k])$$

with the simplicial maps induced by composition. These are exactly the *representable* sets of presheafs on Δ and so Yoneda's lemma implies that for any $X_\bullet \in \mathbf{sSet}$ there is a canonical bijection given by the image of the non-degenerate n-simplex:

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n_\bullet, X_\bullet) \cong X_n. \tag{1}$$

Remark 1.2. There is a closely related notion of *Delta sets*, or *strictly* simplicial sets that are defined by replacing the category Δ by its subcategory $\hat{\Delta}$ whose morphisms are *strictly* order preserving. The forgetful functor from simplicial sets to Delta sets has a left adjoint L given by

$$L(\hat{X}_\bullet)_n = \{\, (x,\sigma) \;\mid\; x \in \hat{X}_m \;,\; \sigma : [n] \twoheadrightarrow [m] \text{ for some } m\}$$

with simplical maps $\tau^*(x,\sigma) = (\delta^*(x),\sigma')$ given by rewriting $\sigma \circ \tau$ as the composition of a surjection σ' followed by an injection δ. For example, if $\hat{\Delta}^k_\bullet$ is a representable presheaf on $\hat{\Delta}$ (the 'Delta k-simplex') then $L(\hat{\Delta}^k_\bullet) = \Delta^k_\bullet$.

The left adjointness of L follows from the remark in Definition 1.1 and it is clear that the non-degenerate simplices in $L(\hat{X}_\bullet)$ are exactly the simplices in the Delta set $\hat{X}_\bullet$. This left adjoint exists for Delta objects in any category $\mathbf{C}$ with finite sums.

Let **Top** be the closed monoidal category of compactly generated topological spaces (with product and mapping spaces given the compactly generated topology) and **CW** the full subcategory of CW-complexes.

Theorem 1.3. *The category* **sSet** *of simplicial sets is a closed monoidal category defined by*

$$(X_\bullet \times Y_\bullet)_n := X_n \times Y_n \quad \textit{and} \quad (Y_\bullet^{X_\bullet})_n := \mathrm{Hom}_{\mathbf{sSet}}(X_\bullet \times \Delta^n_\bullet, Y_\bullet)$$

with the obvious simplicial maps. There are adjoint monoidal functors

$$\begin{array}{ccc} \mathbf{sSet} & \xrightarrow{\;|\cdot|\;} & \mathbf{CW} \\ \uparrow{\scriptstyle inc} & & \downarrow{\scriptstyle inc} \\ \mathbf{Kan} & \xleftarrow{\;S_\bullet\;} & \mathbf{Top} \end{array}$$

Here the *geometric realization* $|X_\bullet|$ is the quotient space of $\amalg_n X_n \times \Delta^n$, by the equivalence relation $(\sigma(x),t) \sim (x,\sigma_*(t))$ for any simplicial map $\sigma : [n] \to [m]$. Note that σ induces an affine linear map σ_* between the corresponding topological standard n-simplices Δ^n and Δ^k.

Lemma 1.4. *The geometric realization is completely determined by two properties: it preserves colimits and* $|\Delta^n_\bullet| \cong \Delta^n$. *Moreover,* $|X_\bullet|$ *is a CW-complex with one n-cell for every non-degenerate n-simplex in* X_n. *(This explains how* $|\cdot|$ *can preserve products, a surprising property since the product of simplices is not a simplex.)*

Proof. It is clear that geometric realization satisfies the two properties. The uniqueness follows from the description of X_n in (1) above which implies that $X_\bullet$ is a colimit over its simplices. □

Note that as a proposed left adjoint, geometric realization must preserve colimits. Its right adjoint $S_\bullet$ needs to induce natural bijections

$$\mathrm{Hom}_{\mathbf{Top}}(|X_\bullet|, Y) \cong \mathrm{Hom}_{\mathbf{sSet}}(X_\bullet, S_\bullet(Y)).$$

Plugging in $X_\bullet = \Delta^n_\bullet$ and using equation (1) above, we see that the *singular simplicial set* $S_\bullet(Y)$ must be defined by

$$S_n(Y) = \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n_\bullet, S_\bullet(Y)) \cong \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n_\bullet|, Y) = Y^{\Delta^n}$$

and that this definition indeed gives a right adjoint. Thus $S_\bullet$ preserves limits (in particular, products) and in addition the simplicial sets $S_\bullet(Y)$ satisfy the essential Kan condition: every horn is fillable.

Definition 1.5. Let Λ^n_k be the subcomplex of $\Delta^n_\bullet$ consisting of those codimension 1 faces that contain the vertex k (or equivalently, all faces except the k-th). This is called the k-th horn in $\Delta^n_\bullet$. A simplicial set $X_\bullet$ satisfies the *Kan condition* if for all $n \geq 0$ and $0 \leq k \leq n$ any map $\Lambda^n_k \to X_\bullet$ can be extended to a map $\Delta^n_\bullet \to X_\bullet$.

This condition is equivalent to the following condition (which makes sense for arbitrary simplicial objects): For every collection of simplices

$$x_0, \ldots, \hat{x}_k, \ldots x_n \in X_{n-1} \quad \text{with} \quad d_i x_j = d_{j-1} x_i \quad i < j, i \neq k, j \neq k \tag{2}$$

there is a simplex $x \in X_n$ such that $d_i x = x_i$ for all $i \neq k$. Besides the singular simplicial set $S_\bullet(X)$ there is another large class of simplicial objects that always satisfy the Kan condition: simplicial groups.

Remark 1.6. There is also a geometric realization $\|\cdot\|$ for Delta sets, given by the same exact formulas, and Lemma 1.4 applies. Therefore, the singular functor $S_\bullet$ into Delta sets is again a right adjoint and one gets a homeomorphism

$$|L(\hat{X}_\bullet)| \cong \|\hat{X}_\bullet\|$$

If $\hat{X}_\bullet$ comes from a simplicial set then the right hand side is called the *fat* realization and the quotient map to the usual realization is a homotopy equivalence. What goes completely wrong in this discussion is that $\|\cdot\|$ cannot preserve products: For example, $\hat{\Delta}^n_\bullet$ is finite with its top simplex in dimension n, therefore the product of two copies of this 'Delta n-simplex' still has its top simplex in dimension n, a fact that's not true geometrically.

By composing the above two functors and using their adjointness, we see that a topological space X comes canonically equipped with a map from a CW-complex $|S_\bullet(X)| \to X$. Similarly, any simplicial set $X_\bullet$ comes with a canonical map from a Kan simplicial set $S_\bullet(|X_\bullet|) \to X_\bullet$. The discussion below will imply that these are *weak equivalences* in the appropriate sense, the cofibrant (respectively fibrant) replacements of X (respectively $X_\bullet$) in Quillen model category language.

The Kan condition is crucial since it implies that the obvious definition of *homotopy* of simplicial maps is an equivalence relation. This homotopy relation is given as in topology:

Definition 1.7. Let $K_\bullet$ be Kan. Then two maps $f_i : X_\bullet \to K_\bullet$ are called *homotopic* if they extend to a simplicial map $F : X_\bullet \times \Delta^1_\bullet \to K_\bullet$. If these are simplicial objects in a category **C** then one requires that for all $t \in \Delta^1_n$, the map $F_t = F(-,t) : X_n \to K_n$ is a morphisms in **C**. Note that these maps F_t satisfy the simplicial identities

$$\sigma \circ F_t = F_{\sigma(t)} \circ \sigma \quad \forall\, t \in \Delta^1_n\,,\ \sigma : [m] \to [n]$$

and they together determine F. Therefore, one doesn't really need to make sense out of $X_\bullet \times \Delta^1_\bullet$ as a simplicial object in **C** in order to define homotopy.

For example, one gets the *homotopy groups* of a simplicial Kan set as homotopy classes of maps:

$$\pi_n(X_\bullet, *) := [(\Delta^n_\bullet, \partial\Delta^n_\bullet), (X_\bullet, *)]$$

Definition 1.8. The homotopy category **hCW** is defined to have the same objects as **CW** and morphisms which are homotopy classes of continuous maps between CW-complexes (similarly for **Kan**). In other words, we do not need to invert weak equivalences (as we should if we started with **Top** respectively **sSet** instead). In the language of Quillen model categories, this comes from the fact that CW-complexes (and Kan sets) are fibrant and cofibrant objects in the relevant model structures.

Theorem 1.9. *The homotopy categories of* **Kan** *and* **CW** *are equivalent.*

Proof (Quillen). The fact that the realization and singular functors are monoidal implies that they induce maps between the homotopy categories of **Kan** and **CW**. It is also formal to prove that the right adjoint $S_\bullet$ preserves homotopy groups. With more work one can show that for Kan simplicial sets, the geometric realization functor also preserves homotopy groups and by the Whitehead theorems in both categories, the result follows. □

Definition 1.8 also applies to projective complexes in the usual model structure on the category $\mathbf{Chain}^+(\mathbf{C})$ of positive chain complexes (of objects in an abelian category $\mathbf{C}$). There the homotopy category of projective chain complexes is equivalent to the *derived category* of $\mathbf{C}$, i.e. one does not need to invert weak equivalence between projective complexes, they already are invertible up to homotopy (by a Whitehead theorem in this setting).

Let $\mathbf{C}$ be an abelian category and $X_\bullet$ a simplicial object in $\mathbf{C}$. Then we can form a chain complex $NX_\bullet$ whose n-th chain object and n-th differential are given by

$$(NX_\bullet)_n := \bigcap_{i=0}^{n-1} \ker(d_i) \quad \text{and} \quad \partial_n := d_n$$

This chain complex is chain homotopy equivalent to the chain complex given by using all of X_n as chain groups and with differentials ∂ given by the alternating sums of the d_i. It is also homotopy equivalent to a normalized version, where X_n is quotiented out by degenerate simplices. Neither of these two latter constructions can be used in the following result, known as the *Dold-Kan correspondence*.

Theorem 1.10. *The functor N induces an equivalence of categories*

$$\mathbf{sC} \simeq \mathbf{Chain}^+(\mathbf{C})$$

that preserves homotopy. In particular, it takes homotopy groups (of simplicial objects in $\mathbf{C}$*) to homology groups (of chain complexes). If* $\mathbf{C}$ *has a monoidal structure (like the tensor product of modules over a commutative ring) then the usual formulas define products on both categories and N preserves these products up to weak equivalence.*

Note that N factors through the category of Delta objects in $\mathbf{C}$. For the inverse map, one takes a chain complex (C_*, ∂) and first makes it into a Delta object $\hat{C}_\bullet$ by $\hat{C}_n := C_n$ and by setting all $d_i = 0$, except for $d_n := \partial_n$. Then the inverse of N sends C_* to $L(\hat{C}_\bullet)$, where L is the left adjoint of the forgetful map from Remark 1.2.

As an application of the theorem, take $\mathbf{C}$ to be the category of abelian groups and take the chain complex with exactly one group A in dimension n and all other groups vanishing. Then we can geometrically realize the corresponding simplicial abelian group and the above theorems say that we get an Eilenberg-MacLane space of type $K(A, n)$, depending functiorially on A.

2. Infinity categories

Summarizing the previous talk, we saw that the language of simplicial objects can describe derived categories as well as the homotopy theory of topological spaces. In this lecture we'll show that it can also express many notions in higher category theory. There are many notions of n-categories but for this course, we are going to use a very particular one, suited best to our applications.

Remark 2.1. We should point out from the beginning that this notion is only suited to describe what might be called $(n, 1)$-categories. Namely, n-categories where all k-morphisms are invertible for $k > 1$. For example, the theory of n-categories with no invertability hypothesis should have n different notions of ”opposite category”, applied to the morphisms at each level. However, the formalism of simplicial sets that we are about to explain, has just one notion of ”opposite” (induced by the involution on the category Δ given by reversing the order).

Definition 2.2. An *infinity category* is a simplicial sets $C_\bullet$ where all *inner horns* are fillable. More precisely, for all $n \geq 0$ and $0 < k < n$ any map $\Lambda^n_k \to C_\bullet$ can be extended to a map $\Delta^n_\bullet \to C_\bullet$.

This condition on a simplicial set was introduced by Boardman and Vogt who called it the *weak Kan condition*. Compared to Definition 1.5 we are just leaving out those horns that contain the first or last face. Note that the horns Λ^2_k are not isomorphic for $k = 0, 1, 2$ and the reason we want to prefer $k = 1$ comes from the following example.

Example 2.3. Let C be a small category. Then the nerve $N_\bullet(C)$ is an infinity category. Note that $N_\bullet(C)$ is Kan if and only if C is a groupoid.

If one looks at the nerve construction carefully, one sees that it actually satisfies the case $n = 1$ of the following definition, in fact, small categories are the same thing as 1-categories.

Definition 2.4. An *n-category* is an infinity category $C_\bullet$ where for $m > n$ all inner horns are *uniquely* fillable. More precisely, for all $m > n$ and $0 < k < m$ any map $\Lambda^m_k \to C_\bullet$ can be extended to a *unique* map $\Delta^m_\bullet \to C_\bullet$.

The forgetful functor from infinity categories to n-categories has a left adjoint h_n. We shall describe only the functor $h := h_1$, known as ‘taking the homotopy category of an infinity category’. For an infinity category $C_\bullet$, we define its *homotopy category* $hC = h(C_\bullet)$ as follows. The objects of hC are the vertices C_0 of C, and the morphisms of hC are given by homotopy classes of edges in C_1. Here two edges e, f (from x to y) are *homotopic* if there is a 2-simplex in C_2 whose 3 boundary faces are given by $s_0(x)$, f and e (in that order). The identity morphism id_x is given by $s_0(x)$ and composition comes from the one Kan condition that we have for 2-simplices: Given $e : x \to y$ and $e' : y \to z$, choose a 2-simplex $\sigma \in C_2$ with $d_2(\sigma) = e$ and $d_0(\sigma) = e'$. Then $e' \circ e$ is defined to be $d_1(\sigma)$.

One can check that in the presence of the filling condition for an infinity category, the above notions are well defined and satisfy all axioms of a (small) category. It is also clear that if $C_\bullet = N_\bullet(C)$ is the nerve of a category C then the homotopy relation is trivial (only equal edges=morphisms are identified) and one gets $C = h(N_\bullet(C))$.

Definition 2.5. An *infinity groupoid* is an infinity category $\mathbf{C}_\bullet$ such that hC is a groupoid.

A result by Joyal shows that an infinity groupoid is the same thing as a Kan complex. It is clear that the filling conditions for 2-horns are sufficient to invert all arrows (and vice versa) but invertible arrows also garantee all higher dimensional filling conditions!

Remark 2.6. The left adjoint functors h_n have a particularly appealing interpretation when restricted to infinity groupoids, i.e. Kan complexes. Then they give the Postnikov tower in the sense that for any Kan complex $K_\bullet$, the natural projection

$$K_\bullet \longrightarrow h_n(K_\bullet)$$

induces an isomorphism of homotopy groups π_i for all $i \leq n$ and the higher homotopy groups of the right hand side vanish. In the homotopy category, this actually determines the functors h_n. Note that h_1 can be thought of as the fundamental groupoid.

The above gives an interpretation of all of homotopy theory in terms of higher category theory. More precisely, the singular functor $S_\bullet$ can be thought of as going from **Top** to the category $\infty-$**Groupoids** of small infinity groupoids, thought of as full subcategory of $\infty-$**Cat**, the category of infinity categories. In fact, it is much better to think of these not as categories but as infinity categories: For $\infty-$**Cat** this is easy enough, since we can use the internal hom of simplicial sets $D_\bullet^{C_\bullet}$. It turns out that if $D_\bullet$ is an infinity category (i.e. all inner horns are fillable) then so is this mapping space.

Furthermore, we note that **Top** is a *topological category*, i.e. the mapping space (or inner hom) Y^X of two topological spaces carries a topology such that the composition maps are continuous. It is therefore desirable to be able to turn a topological category into an infinity category. This is indeed possible by a *topological nerve* construction that extends the usual one from categories (thought of as having the discrete topology on morphism spaces) to all small topological categories. This functor

$$N : \textbf{Top-Cat} \longrightarrow \infty-\textbf{Cat}$$

turns out to have a left adjoint $\mathfrak{C}$ that is defined on all simplicial sets, not just infinity categories. As a left adjoint, $\mathfrak{C}$ preserves colimits and by the fact that all simplicial sets are colimits of their simplices, it is sufficient to define $\mathfrak{C}(\Delta^n_\bullet)$ functorially. The objects of this topological category are elements of the set $[n] = \{0, \dots, n\}$ and the morphisms between i and j are empty unless $i \leq j$. In that case, they are given by the $(j-i-1)$-cube of functions

$$p : \{k \in [n] \mid i \leq k \leq j\} \longrightarrow [0,1] \quad \text{with} \quad p(i) = p(j) = 1$$

and composition of morphisms is given by concatenation of functions. Finally, any order preserving map $[n] \to [n']$ clearly induces a functor $\mathfrak{C}(\Delta^n_\bullet) \to \mathfrak{C}(\Delta^{n'}_\bullet)$ so that these categories can be glued together according to the simplices of a simplicial set to get our colimit preserving functor $\mathfrak{C}$. By the adjointness property, we can now define the simplicial set $N(C)$ for any topological category C via

$$N(C)_n = \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n_\bullet, N(C)) := \mathrm{Hom}_{\mathbf{Top-Cat}}(\mathfrak{C}(\Delta^n_\bullet), C)$$

and we need to prove the lemma that all inner horns are fillable in this simplicial set, i.e. the topological nerve $N(C)$ is an infinity category. One can check that the vertices in $N(C)_0$ are just the objects of C and the edges in $N(C)_1$ are the morphisms in C. Moreover, a 2-simplex $\sigma \in N(C)_2$ consists of the following data:

- the 3 vertices of σ, X_0, X_1, X_2, are arbitrary objects of C.
- the 3 edges of σ, $f_{i,j} : X_j \to X_i$, are arbitrary morphisms in C with the given source and target. Here we write the index for the domain second for the usual (bad) reason, namely that we think of forming $f_{i,j}(-)$. This is easiest remembered by drawing all arrows from right to left when writing out diagrams.
- finally, the information about the 'interior of σ'. It is given by a homotopy in the space $\mathrm{Hom}_C(X_0, X_2)$ between the two points $f_{0,2}$ and $f_{2,1} \circ f_{1,0}$.

This process for describing the n-simplices of $N(C)$ continues and is best formalized by the discussion above.

Lemma 2.7. *Given an infinity category $C_\bullet$, its homotopy category is equivalent to the category $\pi_0(\mathfrak{C}(C_\bullet))$.*

Here we have used the left adjoint

$$\pi_0 : \mathbf{Top} - \mathbf{Cat} \longrightarrow \mathbf{Cat}$$

to the functor that considers a category as a discrete topological category. Let $\mathcal{H} := \mathbf{hCW}$ be the homotopy category of CW-complexes discussed in Definition 1.8. Recall that $|S_\bullet(-)|$ is a product preserving functor $\mathbf{Top} \to \mathcal{H}$ and therefore we can take a topological category C and push it forward to a category hC that is enriched over $\mathcal{H}$. This means that we keep the same objects and only remember the homotopy types of the morphism spaces $\mathrm{Hom}_C(X, Y)$.

Definition 2.8. A morphism $f \in \mathrm{Hom}_C(X, Y)$ in a topological category C is called an *equivalence* if f becomes an isomorphism in hC. A functor $F : C \to D$ between topological categories is called an *equivalence* of categories if the induced functor $hF : hC \to hD$ is an equivalence. Finally, a simplicial map $\Phi : X_\bullet \to Y_\bullet$ is called a *categorical equivalence* if the induced functor

$$\mathfrak{C}(\Phi) : \mathfrak{C}(X_\bullet) \longrightarrow \mathfrak{C}(Y_\bullet)$$

is an equivalence. This last definition applies in particular to infinity categories.

Recall that the topological nerve functor N :**Top-Cat**$\to \infty-$**Cat** extends to topological categories the usual nerve construction on categories, as well as the singular functor on topological spaces (thought of as some model for topological groupoids).

Theorem 2.9. *The adjoint functors N and $\mathfrak{C}$ induce a bijection between equivalence classes of (small) topological categories and categorical equivalence classes of infinity categories.*

In this sense, we will be able to go back and forth between infinity categories and topological category (as well as simplicial categories by Quillen's Theorem 1.9).