Projects in QFT, 276, Fall 2007

Dan, Theo

1. Let $\mathcal{A}$ be a closed monoidal category. Formulate and prove the Yoneda embedding theorem for $\mathcal{A}$-enriched categories.

Dan, Matthias

2. Find an algebraic way to determine whether a commutative super algebra is isomorphic to $C^\infty(M)$ for some super manifold M.

Manuel

3. For a super manifold M, define the K-theory $K(M)$ as the abelian group given by isomorphism classes of finite dimensional super vector bundles over M, modulo the relations
$$E \oplus \pi E = 0.$$
Show that the inclusion $M_{red} \subset M$ induces an isomorphism $K(M) \cong K(M_{red})$.

Andre, Jonah

4. Describe super Lie groups that lead to all simple super Lie algebras over $\mathbb{C}$. These were given by Kac as explained in class.

Santiago

5. Use Frobenius' Theorem to prove that there is the following bijection for any super Lie group G and super manifold M: One one side one has right G-actions on M, $\rho : M \times G \to M$, and on the other side one has a pair of an ordinary right action and a super Lie homomorphism
$$\rho_{red} : M_{red} \times G_{red} \quad \text{and} \quad r : \mathfrak{g} \to \mathrm{Vect}(M)$$
satisfying the following compatability condition: Differentiating ρ_{red} at the identity leads to a Lie homomorphism $\mathfrak{g}_{red} \to \mathrm{Vect}(M_{red})$ and this should equal the composition
$$\mathfrak{g}_{red} = \mathfrak{g}^{ev} \xrightarrow{r^{ev}} \mathrm{Vect}(M)^{ev} \to \mathrm{Vect}(M_{red}).$$
The map on the right hand side is given by observing that *even* derivations preserve the ideal generated by odd elements.

Alan, Dan B.

6. Formulate the theory of super principal bundles (and their connections) on super manifolds. For structure group $GL(m|n)$ show that a connection on a principal bundle is the same data as a connection (in the sense of class) on the associated vector bundle.

Kevin

7. Describe the structure on the *differential gorms* of a super manifold M. This is the action of $\underline{\mathrm{Aut}}(\mathbb{R}^{0|2})$ on $\underline{\mathrm{SMAN}}(\mathbb{R}^{0|2}, M)$. There is a paper in the literature.

Dmitry

8. Let (M, ω) be a symplectic vector space and $\mathfrak{h}(M, \omega)$ the corresponding Heisenberg Lie algebra. Show that irreducible representations of $\mathfrak{h}(M, \omega)$ are *not* determined by the (nontrivial) action of the centre.

   Is there a way to fix that and get an analogue of the Stone-von Neumann theorem on the uniqueness of irreducible *unitary* representations of the Heisenberg group?

William

9. Research the theory of geometric quantization for symplectic super manifolds.

10. Let $(W = N \oplus N^*, b)$ be a symmetric hyperbolic form on a finite dimensional vector space W. Geometric quantization gives an irreducible representation of the Heisenberg super Lie algebra $\mathfrak{h}(W, b)$ on $C^\infty(\pi N) = \Lambda^*(N^*)$ where elements $n \in N$ act as differentiation and elements in N^* act as multiplication operators.

    We showed in class how to extend the representation to quadratic functions and hence get a representation of the sub-Lie algebra

$$\mathfrak{h}(W, b) \ltimes \mathfrak{o}(b) \cong \Lambda^{\leq 2}(W^*) \subset \Lambda^*(W^*) = C^\infty(\pi W)$$

    Can this Lie algebra representation be extended to all of $C^\infty(\pi W)$?

    Note that we do have a representation of the (associative) Clifford algebra $\mathrm{C}\ell(W, b)$ which can be viewed as a super Lie algebra of the same dimension as $C^\infty(\pi W)$. Are these Lie algebras isomorphic?

11. Let W be an inner product space of dimension n. Recall that the Clifford algebra $\mathrm{C}\ell(W)$ has defining relations $[w_1, w_2] + 2\langle w_1, w_2\rangle = 0$, where $[,]$ is the super commutator. It has the structure of a C^*-algebra determined by $w^* = -w$ for all $w \in W$. A left $\mathrm{C}\ell(W)$-module is an inner product space V together with a C^*-algebra homomorphism $\mathrm{C}\ell(W) \to \mathrm{End}(V)$. Denote by $\mathrm{C}\ell_n$ the Clifford algebra corresponding to the standard positive definite inner product on $\mathbb{R}^n$. Prove the following assertions:

    - There is a natural bijection between orientations of W and isomorphism classes of graded irreducible $\mathrm{C}\ell(W) - \mathrm{C}\ell_n$ bimodules. Such a bimodule is called a *spin structure* on W.
    - If two inner product spaces W_i are equipped with spin structures, the definition in class leads to a connected double covering of $SO(W_1, W_2)$.

Arturo

12. Let E be a real vector bundle over a manifold M with fibrewise metric. Define a *spin structure* on E to be a graded bimodule bundle S over the algebra bundles $\mathrm{C}\ell(E)$ (respectively the trivial bundle $M \times \mathrm{C}\ell_n$) such that the fibres S_m are spin structures on E_m for all $m \in M$.

    - Show that a reduction of the structure group of E from $SO(n)$ to its connected double covering $Spin(n)$ gives a spin structure on E.
    - If ∇ is a metric connection on E and S is a spin structure, show that there is a unique connection ∇^S on S so that for each vector field X on M the corresponding operator $\nabla^S_X : \Gamma(S) \to \Gamma(S)$ is (right) $\mathrm{C}\ell_n$-linear and satisfies the following condition on left Clifford multiplication by sections ξ of $E \subset \mathrm{C}\ell(E)$:

$$\nabla^S_X(\xi \cdot s) = \nabla_X(\xi) \cdot s + \xi \cdot \nabla^S_X(s) \quad \forall s \in \Gamma(S).$$