

# TOPOLOGICAL CONFORMAL FIELD THEORIES AND CALABI-YAU CATEGORIES

KEVIN COSTELLO

ABSTRACT. This is the first of two papers which construct a purely algebraic counterpart to the theory of Gromov-Witten invariants (at all genera). These Gromov-Witten type invariants depend on a Calabi-Yau  $A_\infty$  category, which plays the role of the target in ordinary Gromov-Witten theory. When we use an appropriate  $A_\infty$  version of the derived category of coherent sheaves on a Calabi-Yau variety, this constructs the  $B$  model at all genera. When the Fukaya category of a compact symplectic manifold  $X$  is used, it is shown, under certain assumptions, that the usual Gromov-Witten invariants are recovered. The assumptions are that open-closed Gromov-Witten theory can be constructed for  $X$ , and that the natural map from the Hochschild homology of the Fukaya category of  $X$  to the ordinary homology of  $X$  is an isomorphism.

## CONTENTS

1. Introduction	2
1.1. Topological conformal field theories	3
1.2. Open and open-closed TCFTs	5
1.3. Statement of the main results	7
1.4. Relation with Deligne's conjecture	9
1.5. Relations to the work of Moore-Segal and Lazariou	10
1.6. The non-unital version of the result	11
1.7. Outline of the proof of theorem A	11
2. Examples and applications	12
2.1. String topology	13
2.2. The B model	14
2.3. Gromov-Witten invariants and the Fukaya category	14
2.4. The TCFT associated to Gromov-Witten invariants	15
2.5. Comparing the TCFT associated to Gromov-Witten theory with that from the Fukaya category	15
2.6. Acknowledgements	17
2.7. Notation	17
3. The open-closed moduli spaces in more detail	17
4. Some homological algebra for symmetric monoidal categories	20
4.1. Differential graded symmetric monoidal categories	20
4.2. Notation about exact functors	22
4.3. Derived tensor products	23
4.4. Derived push forwards	26

5. Topological conformal field theories	27
6. Combinatorial models for categories controlling open-closed topological conformal field theory	29
6.1. A cellular model for moduli space	29
6.2. Generators and relations for $\mathcal{D}_{open}^d$	35
6.3. The differential in $\mathcal{D}^d$	38
7. Proof of the main results	41
7.1. $A_\infty$ categories	41
7.2. Calabi-Yau $A_\infty$ categories	41
7.3. Open topological conformal field theories and $A_\infty$ categories	42
7.4. The Hochschild chain complex	44
8. Appendix	47
References	49

## 1. INTRODUCTION

If  $X$  is a Calabi-Yau manifold, Witten [Wit91] describes two different topological twistings of the non-linear sigma model of maps from a Riemann surface to  $X$ , which he calls the  $A$  and  $B$  models. If  $X, X^\vee$  are a mirror pair of Calabi-Yau varieties, then the  $A$  model on  $X$  is equivalent to the  $B$  model on  $X^\vee$ , and vice-versa.

The  $A$  model has been mathematically constructed as the theory of Gromov-Witten invariants. The genus 0 part of the  $B$  model has been constructed by Barannikov-Kontsevich [BK97] and Barannikov [Bar00, Bar99]. They construct a Frobenius manifold from the variations of Hodge structure of a Calabi-Yau. The genus 0 part of mirror symmetry is then the statement that the genus 0 part of the Gromov-Witten theory of a Calabi-Yau variety  $X$  is equivalent to the theory of Barannikov-Kontsevich on a Calabi-Yau  $X^\vee$ .

The higher genus  $B$  model is more mysterious. In the physics literature, it is constructed as a kind of quantisation of the Kodaira-Spencer deformation theory of complex structures on a Calabi-Yau [BCOV94].

However, despite the great deal of interest in mirror symmetry since the subject's inception in the early 1990's, there has been no rigorous construction of the higher-genus part of the  $B$  model. One of the aims of this paper is to construct the  $B$  model rigorously for the first time, and so provide a mirror partner to the entire theory of Gromov-Witten invariants.

Kontsevich [Kon95] formulated mirror symmetry as an equivalence of  $A_\infty$  categories. If  $X, X^\vee$  are a mirror pair, then Kontsevich conjectures that the Fukaya category of a variety  $X$  ( $A$  model) is equivalent to the dg category of complexes of coherent sheaves on  $X^\vee$  ( $B$  model). Kontsevich's homological mirror symmetry conjecture should explain other aspects of mirror symmetry. In particular, the equivalence of the theory of Gromov-Witten invariants on  $X$  with the  $B$  model on  $X^\vee$  should be a corollary of Kontsevich's conjecture.

Both of the  $A_\infty$  categories appearing in Kontsevich's conjecture are of Calabi-Yau type. This means, roughly, that there is a non-degenerate invariant pairing on the space of morphisms.

This immediately suggests the following picture. From each Calabi-Yau  $A_\infty$  category, one should construct something like the theory of Gromov-Witten invariants. If the input Calabi-Yau  $A_\infty$  category is the Fukaya category of a compact symplectic manifold, then this theory should recover the usual theory of Gromov-Witten invariants. If the input Calabi-Yau  $A_\infty$  category is the category of sheaves on a smooth projective variety, the resulting theory will, by definition, be the  $B$  model at all genera.

In this paper, we prove results along these lines. These results are derived from a study of a kind of abstract topological string theory, called a topological conformal field theory (TCFT). We study open, closed and open-closed TCFTs.

Closed TCFTs behave like the Gromov-Witten invariants of a projective variety : a closed TCFT can be described as a collection of cochains on moduli space of Riemann surfaces, with values in tensor powers of an auxiliary chain complex (the complex of "closed states"), and which satisfy certain gluing constraints.

The main results of this paper are as follows. Firstly, we show that open TCFTs are the same as Calabi-Yau  $A_\infty$  categories. Thus to each Calabi-Yau variety we have two open TCFTs : that associated to the Fukaya category ( $A$  model) and that coming from coherent sheaves ( $B$  model).

Then, we show how one can associate to each open TCFT an open-closed TCFT, and in particular a closed TCFT. This is a formal, categorical construction. We observe that to each open TCFT one can associate the homotopy universal open-closed TCFT (this is an example of a homotopy Kan extension). Then we calculate the homology of the space of closed states of this universal closed TCFT : it is the Hochschild homology of the  $A_\infty$  category associated to the open TCFT.

Also, we show, under certain assumptions, how to relate the closed TCFT constructed here from the Fukaya category of a compact symplectic manifold to the ordinary Gromov-Witten invariants of the manifold.

These results are proved using a combination of homotopical algebra, and some results about the topology of the moduli spaces of Riemann surfaces. In particular, the dual version of the ribbon graph decomposition of moduli space [Cos06] plays an essential role.

Let us now turn to describing these results in more detail.

**1.1. Topological conformal field theories.** Let  $\mathcal{M}$  be Segal's category of Riemann surfaces. The objects of  $\mathcal{M}$  are finite sets; for sets  $I, J$ , a morphism from  $I$  to  $J$  is a Riemann surface with  $I$  incoming and  $J$  outgoing parameterised boundary components. (We require that there is at least one incoming boundary on each component). Composition of morphisms is given by gluing of Riemann surfaces. Disjoint union of sets and of surfaces gives  $\mathcal{M}$  the structure of symmetric monoidal category. According to Segal [Seg04], a conformal field theory is a symmetric monoidal functor from this category to the category of vector spaces.

This definition can be modified in several ways. For example, we could look for functors from  $\mathcal{M}$  to the symmetric monoidal category of topological spaces, or of spectra. There is a natural linearised version of these topological functors, obtained by passing from the category of topological spaces to the category of chain complexes. Let  $C_*$  be a symmetric monoidal functor from the category of topological spaces to that of complexes of  $\mathbb{K}$  vector spaces, which computes homology groups. (Here  $\mathbb{K}$  is a base field of characteristic zero). The category  $\mathcal{M}$  has discrete set of objects, but the spaces of morphisms are topological spaces. Applying  $C_*$  to the topological category  $\mathcal{M}$  yields a differential-graded category  $C_*(\mathcal{M})$ . The objects of  $C_*(\mathcal{M})$  are, as before finite sets; the morphisms of  $C_*(\mathcal{M})$  are defined by

$$\mathrm{Mor}_{C_*(\mathcal{M})}(a, b) = C_*(\mathrm{Mor}_{\mathcal{M}}(a, b))$$

Define

$$\mathcal{C} \stackrel{\mathrm{def}}{=} C_*(\mathcal{M})$$

$\mathcal{C}$ , like  $\mathcal{M}$ , is a symmetric monoidal category. The following definition is due independently to Getzler [Get94] and Segal [Seg99].

**Definition.** *A topological conformal field theory is a symmetric monoidal functor  $F$  from the differential graded category  $\mathcal{C}$  to the category of chain complexes.*

What this means is the following. For any finite set,  $F(I)$  is a chain complex. Since  $F$  is a symmetric monoidal functor, there is a map

$$F(I) \otimes F(J) \rightarrow F(I \amalg J)$$

Usually these maps are required to be isomorphisms; if this was the case, the functor  $F$  would be called *split*. We relax this to the condition that these maps are quasi-isomorphisms; we say the functor is *h-split* (homologically split). Each chain  $\alpha$  in the moduli space of Riemann surfaces with  $I$  labelled incoming and  $J$  labelled outgoing boundary components gives a map

$$F(\alpha) : F(I) \rightarrow F(J)$$

which is of the same degree as  $\alpha$ . This map respects the differential :  $F(d\alpha) = dF(\alpha)$ , where  $F(\alpha)$  is considered as an element of the chain complex  $\mathrm{Hom}(F(I), F(J))$ . Gluing Riemann surfaces together must correspond to composition of maps, and disjoint union corresponds to tensor product.

We need to twist the definition of TCFT by a local system. Let  $\det$  be the locally constant sheaf of  $\mathbb{K}$  lines on the morphism spaces of the category  $\mathcal{M}$  whose fibre at a surface  $\Sigma$  is

$$\det(\Sigma) = (\det H^*(\Sigma))[-\chi(\Sigma)]$$

This is situated in degree  $\chi(\Sigma)$ . If  $\Sigma_1, \Sigma_2$  are two surfaces with the incoming boundaries of  $\Sigma_2$  identified with the outgoing boundaries of  $\Sigma_1$ , then there is a natural isomorphism

$$\det(\Sigma_2 \circ \Sigma_1) \cong \det(\Sigma_2) \otimes \det(\Sigma_1)$$

This shows that if we take chains with local coefficients,  $C_*(\mathcal{M}, \det)$ , then we still get a category. Let

$$\mathcal{C}^d = C_*(\mathcal{M}, \det^d)$$

where we use the notation  $\det^d$  for  $\det^{\otimes d}$ .

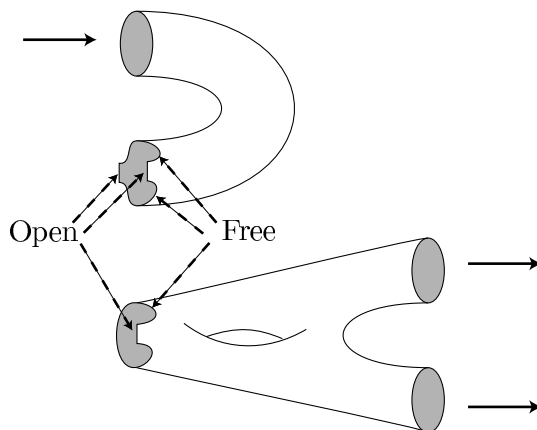


FIGURE 1. A Riemann surface with open-closed boundary. The open boundaries can be either incoming or outgoing boundaries, but this is not illustrated.

**Definition.** A  $d$ -dimensional topological conformal field theory is a symmetric monoidal functor from the category  $\mathcal{C}^d$  to the category of complexes.

It turns out that the local system  $\det$  is trivial (up to a shift). However, it is still important to keep track of it, especially when we consider open-closed conformal field theory; although the local system is trivial, in the open-closed case it can not be trivialised in a way compatible with the category structure. In the closed case, this local system is not so important; however, it is convenient to use it to keep track of the grading.

One apparent disadvantage of the definition of TCFT is that it seems to depend on an arbitrary choice, that of a chain model for the category  $\mathcal{M}$ . However, we show that quasi-isomorphic categories have homotopy equivalent (in a precise sense) categories of functors, so that up to homotopy there is no ambiguity.

**1.2. Open and open-closed TCFTs.** Open-closed conformal field theory was first axiomatised by Moore and Segal [Moo01, Seg99]. A Riemann surface with open-closed boundary is a Riemann surface  $\Sigma$ , some of whose boundary components are parameterised, and labelled as closed (incoming or outgoing); and with some intervals (the open boundaries) embedded in the remaining boundary components. These are also parameterised and labelled as incoming and outgoing. The boundary of such a surface is partitioned into three types: the closed boundaries, the open boundary intervals, and the free boundaries. The free boundaries are the complement of the closed boundaries and the open boundary intervals, and are either circles or intervals. We require that each connected component of  $\Sigma$  has at least one free or incoming closed boundary.

To define an open closed conformal field theory, we need a set  $\Lambda$  of D-branes. Define a category  $\mathcal{M}_\Lambda$ , whose objects are pairs  $O, C$  of finite sets and maps  $s, t : O \rightarrow \Lambda$ . The morphisms in this category are Riemann surfaces with open-closed boundary, whose free boundaries are labelled by D-branes. To each open boundary  $o$  of  $\Sigma$  is associated an ordered pair  $s(o), t(o)$  of D-branes, where it starts and where it ends. Composition

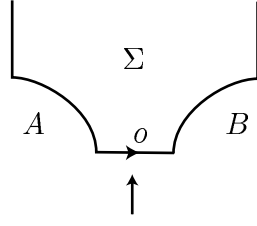


FIGURE 2.  $A, B$  are D-branes, labelling free boundaries.  $o$  is an incoming open boundary with  $s(o) = A$ ,  $t(o) = B$ .

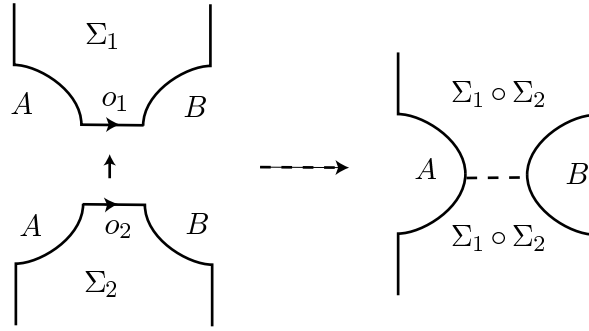


FIGURE 3. Open gluing, corresponding to composition.  $o_1$  on  $\Sigma_1$  is incoming,  $o_2$  on  $\Sigma_2$  is outgoing, and  $s(o_1) = s(o_2) = A$ ,  $t(o_1) = t(o_2) = B$ . Note incoming and outgoing boundaries are parameterised in the opposite sense.

is given by gluing of surfaces; we glue all the outgoing open boundaries of  $\Sigma_1$  to the incoming open boundaries of  $\Sigma_2$ , and similarly for the closed boundaries, to get  $\Sigma_2 \circ \Sigma_1$ . Open boundaries can only be glued when their D-brane labels are compatible, as in figure 1.2. Disjoint union makes  $\mathcal{M}_\Lambda$  into a symmetric monoidal category.

Define an open-closed conformal field theory to be a symmetric monoidal functor from  $\mathcal{M}_\Lambda$  to the category of vector spaces. Let us assume, for simplicity, that this is split, so that the morphisms  $F(\alpha) \otimes F(\beta) \rightarrow F(\alpha \amalg \beta)$ , for  $\alpha, \beta \in \text{Ob } \mathcal{M}_\Lambda$ , are isomorphisms.

Then an open-closed CFT consists of vector spaces  $\mathcal{H}$ , of closed states; and for each pair of D-branes  $\lambda, \lambda'$ , a vector space  $\text{Hom}(\lambda, \lambda')$ .

Let  $\Sigma$  be a Riemann surface with open-closed boundary, each of whose free boundaries is labelled by a D-brane. Suppose the sets of incoming and outgoing closed and open boundaries of  $\Sigma$  are  $C_+, C_-, O_+, O_-$  respectively. Then  $\Sigma$  must give a morphism

$$\mathcal{H}(C_+) \otimes \bigotimes_{o \in O_+} \text{Hom}(s(o), t(o)) \rightarrow \mathcal{H}(C_-) \otimes \bigotimes_{o \in O_-} \text{Hom}(s(o), t(o))$$

As before, disjoint union of surfaces corresponds to tensor products of morphisms, and gluing of surfaces – composition in the category  $\mathcal{M}_\Lambda$  – corresponds to composition of linear maps.

An open CFT is like this, except the surfaces have no closed boundaries, and there is no space of closed states.

The definition of open-closed (or just open) topological CFT is obtained from this definition in the same way the definition of topological CFT is obtained from the definition of CFT. So we replace the category  $\mathcal{M}_\Lambda$  by its associated category of chains,  $C_*\mathcal{M}_\Lambda$ . We can also take chains with twisted coefficients; define

$$\mathcal{O}\mathcal{C}_\Lambda^d = C_*(\mathcal{M}_\Lambda, \det^d)$$

Here  $\det$  is a certain local system on the moduli spaces of Riemann surfaces with open-closed boundary.

An open-closed TCFT of dimension  $d$  is a symmetric monoidal functor from  $\mathcal{O}\mathcal{C}_\Lambda^d$  to complexes, which is h-split, so that the maps  $\Phi(\alpha) \otimes \Phi(\beta) \rightarrow \Phi(\alpha \amalg \beta)$  are quasi-isomorphisms.

Let  $\mathcal{O}_\Lambda^d$  be the full subcategory whose objects are purely open; so they are of the form  $(C, O)$  where  $C = \emptyset$ . Morphisms in  $\mathcal{O}_\Lambda^d$  are chains on moduli of surfaces with no closed boundary. An open TCFT is a h-split symmetric monoidal functor from  $\mathcal{O}_\Lambda^d$  to complexes.

**1.3. Statement of the main results.** There are functors

$$i : \mathcal{O}_\Lambda^d \rightarrow \mathcal{O}\mathcal{C}_\Lambda^d \leftarrow \mathcal{C}^d : j$$

Let  $\Phi$  be an open TCFT, so that  $\Phi : \mathcal{O}_\Lambda^d \rightarrow \text{Comp}_\mathbb{K}$  is a symmetric monoidal functor. Then we can push forward to get  $i_*\Phi : \mathcal{O}\mathcal{C}_\Lambda^d \rightarrow \text{Comp}_\mathbb{K}$ . Here  $i_*$  is the left adjoint to the pull-back functor

$$i^* : \text{Fun}(\mathcal{O}\mathcal{C}_\Lambda^d, \text{Comp}_\mathbb{K}) \rightarrow \text{Fun}(\mathcal{O}_\Lambda^d, \text{Comp}_\mathbb{K})$$

(here  $\text{Comp}_\mathbb{K}$  is the category of complexes of  $\mathbb{K}$  vector spaces).

If we think of a category as like an algebra, then a functor from a category to complexes is like a (left) module; and we can write this as

$$i_*\Phi = \mathcal{O}\mathcal{C}_\Lambda^d \otimes_{\mathcal{O}_\Lambda^d} \Phi$$

The functor  $i_*$  is not exact; it doesn't take quasi-isomorphisms to quasi-isomorphisms. Instead, we use the left derived version

$$\mathbb{L}i_*\Phi = \mathcal{O}\mathcal{C}_\Lambda^d \otimes_{\mathcal{O}_\Lambda^d}^{\mathbb{L}} \Phi$$

which is exact. This is obtained by first replacing  $\Phi$  by a flat resolution, and then applying  $i_*$ .

It turns out that  $\mathbb{L}i_*\Phi$  is an open-closed TCFT (that is, it is h-split).  $\mathbb{L}i_*\Phi$  is the homotopy universal open-closed TCFT associated to  $\Phi$ .

We can pull back along  $j$ , to get a closed TCFT  $j^*\mathbb{L}i_*\Phi$ . This defines a functor from open to closed TCFTs. We can think of this functor as taking an open TCFT  $\Phi$ , and tensoring with the  $\mathcal{C}^d - \mathcal{O}_\Lambda^d$  bimodule,  $\mathcal{O}\mathcal{C}_\Lambda^d$ ; that is,

$$j^*\mathbb{L}i_*\Phi = \mathcal{O}\mathcal{C}_\Lambda^d \otimes_{\mathcal{O}_\Lambda^d}^{\mathbb{L}} \Phi$$

considered as a left  $\mathcal{C}^d$  module.

In this paper the following theorem is proved.

**Theorem A.** (1) *The category of open TCFTs of dimension  $d$ , with fixed set of  $D$ -branes  $\Lambda$ , is homotopy equivalent to the category of (unital) extended Calabi-Yau  $A_\infty$  categories of dimension  $d$ , with set of objects  $\Lambda$ .*

(2) *For any open TCFT  $\Phi$ , the homotopy-universal functor  $\mathbb{L}i_*\Phi : \mathcal{OC}_\Lambda^d \rightarrow \text{Comp}_\mathbb{K}$  is  $h$ -split, and so defines an open-closed TCFT.*

(3) *Let  $HH_*(\Phi)$  denote the Hochschild homology of the  $A_\infty$  category associated to  $\Phi$  by part (1). Then the homology of the closed states of the open-closed TCFT  $\mathbb{L}i_*\Phi$  is  $HH_*(\Phi)$ . The homology of the open states is just that of  $\Phi$ .*

*More precisely, for an object  $(O, C) \in \text{Ob } \mathcal{OC}_\Lambda^d$ , where  $O \in \text{Ob } \mathcal{OC}_\Lambda^d$ ,  $C \in \text{Ob } \mathcal{C}^d$  (so that  $C$  is a finite set), we have*

$$H_*((\mathbb{L}i_*\Phi)(O, C)) = H_*(\Phi(O)) \otimes HH_*(\Phi)^{\otimes C}$$

*In particular, the closed TCFT  $j^*\mathbb{L}i_*\Phi$  has homology*

$$H_*((j^*\mathbb{L}i_*\Phi)(C)) = HH_*(\Phi)^{\otimes C}$$

**Corollary.** *The homology of moduli spaces acts on the Hochschild homology of any Calabi-Yau  $A_\infty$  category  $\mathcal{D}$ . That is there are operations*

$$H_*(\mathcal{M}(I, J), \det^d) \otimes HH_*(\mathcal{D})^{\otimes I} \rightarrow HH_*(\mathcal{D})^{\otimes J}$$

Part (1) can be viewed as a categorification of the ribbon graph decomposition of moduli spaces. The proof relies on the dual version of the ribbon graph decomposition proved by the author in [Cos04, Cos06]. The statement that the categories are homotopy equivalent has a precise meaning. It means that there are functors from open TCFTs to extended CY  $A_\infty$  categories, and from extended CY  $A_\infty$  categories to open TCFTs, which are inverse to each other, up to quasi-isomorphism. A Calabi-Yau category is the categorical generalisation of a Frobenius algebra. In a Calabi-Yau  $A_\infty$  category, the product is only associative up to homotopy, and there is a cyclic symmetry condition on the inner product with the higher multiplications  $m_n$ . The adjective “extended” refers to a small technical generalisation of this definition which will be explained in section 7.

The homotopy universal closed TCFT  $\mathbb{L}i_*\Phi$  has the property that for every open-closed TCFT  $\Psi$ , with a map  $\Phi \rightarrow i^*\Psi$  in an appropriate homotopy category of TCFTs, there is a map  $\mathbb{L}i_*\Phi \rightarrow \Psi$ . Here  $i^*\Psi$  is the open TCFT associated to  $\Psi$  by forgetting the closed part; the fact that  $\mathbb{L}i_*\Phi \rightarrow \Psi$  is a map of open-closed TCFTs means that the diagrams

$$\begin{array}{ccc} \mathcal{OC}_\Lambda^d(\alpha, \beta) \otimes \mathbb{L}i_*\Phi(\alpha) & \longrightarrow & \mathcal{OC}_\Lambda^d(\alpha, \beta) \otimes \Psi(\alpha) \\ \downarrow & & \downarrow \\ \mathbb{L}i_*\Phi(\beta) & \longrightarrow & \Psi(\beta) \end{array}$$

commute, for all objects  $\alpha, \beta$  of  $\mathcal{OC}_\Lambda^d$ .

Passing to homology of the closed states, we see that in particular, for all finite sets  $I, J$ , the diagram

$$\begin{array}{ccc} H_*(\mathcal{M}(I, J), \det^d) \otimes HH_*(\Phi)^{\otimes I} & \longrightarrow & H_*(\mathcal{M}(I, J), \det^d) \otimes H_*(\Psi)^{\otimes I} \\ \downarrow & & \downarrow \\ HH_*(\Phi)^{\otimes J} & \longrightarrow & H_*(\Psi)^{\otimes J} \end{array}$$

commutes. Here,  $HH_*(\Phi)$  refers to the Hochschild homology of the  $A_\infty$  category associated to  $\Phi$ , under the correspondence between  $A_\infty$  categories and open TCFTs.  $H_*(\Psi)$  means  $H_*(\Psi(1))$ , so that  $H_*(\Psi(I)) = H_*(\Psi)^{\otimes I}$ .  $\mathcal{M}(I, J)$  is the moduli space of Riemann surfaces with  $I$  incoming and  $J$  outgoing boundaries.

One could hope that part (3) of this result should give a natural algebraic characterisation of the category of chains on moduli spaces of curves, as morphisms in some homotopy category between the functors which assign to a Calabi-Yau  $A_\infty$  category the tensor powers of its Hochschild chains.

**1.4. Relation with Deligne's conjecture.** Theorem A implies a higher genus generalisation of Deligne's Hochschild cochains conjecture. Deligne conjectured that there is a homotopy action of the chain operad of the little discs operad on the Hochschild cochain complex of an algebra. This has now been proved by several authors, [Tam98, Kon99, KS00, MS02].

A variant of Deligne's conjecture states that the framed little discs operad acts on the Hochschild cochains of a Frobenius algebra. This has been proved by Kaufmann [Kau04] and Tradler-Zeinalian [TZ04].

The framed little discs operad is the operad of genus zero Riemann surfaces with boundary. What is shown here is that there is a homotopy action of chains on all-genus moduli spaces of Riemann surfaces on the Hochschild chains of a Calabi-Yau  $A_\infty$  category, or in particular, of a Frobenius algebra. Restricting to Riemann surfaces of genus zero with precisely one input, we find a homotopy co-action of the framed little discs operad on the Hochschild chain complex. The Hochschild cochain complex of a Calabi-Yau  $A_\infty$  category is dual to the Hochschild chain complex. Therefore we can dualise the coaction on Hochschild chains to find that the Hochschild cochain complex has a homotopy action of the framed little discs operad, recovering the result of Kaufman and Tradler-Zeinalian.

It is not difficult to check that the coproduct on Hochschild homology constructed here, which comes from the class of a point in the moduli space of genus 0 surfaces with one incoming and two outgoing boundaries, coincides with the dual of the standard cup product on Hochschild cohomology. Also, the operator on Hochschild homology which comes from the generator of  $H_1$  of the moduli space of annuli with one incoming and one outgoing boundary coincides with the B operator of Connes.

On the other hand, very few of other operations we construct on Hochschild homology admit such a simple description. In particular, the *product* we construct on Hochschild homology, which can be described explicitly, seems not to have been considered in the literature before.

Statements close to the higher-genus analog of Deligne’s conjecture proved here have been conjectured by Kontsevich as far back as 1994 [Kon95], and have also been conjectured by Segal, Getzler, Kapustin and Rozansky [KR04].... A different approach to constructing an action of chains on moduli space on Hochschild chains has been outlined by Kontsevich [Kon03] using the standard ribbon graph decomposition of moduli spaces of curves, in a lecture at the Hodge centenary conference in 2003.

**1.5. Relations to the work of Moore-Segal and Lazaroiu.** Moore and Segal [Moo01] and independently Lazaroiu [Laz01] have obtained descriptions of open-closed topological field theories. Topological field theory (TFT) is a greatly simplified version of the topological conformal field theory considered in this paper. Instead of taking the singular chains on moduli spaces, in TFT we only use  $H_0$ , or equivalently only consider topological surfaces (with no conformal structure).

These authors show that an open-closed TFT consists of a not necessarily commutative Frobenius algebra  $A$ , a commutative Frobenius algebra  $B$ , with a homomorphism from

$$(1.5.1) \quad \iota_* : B \rightarrow Z(A)$$

(where  $Z(A)$  is the centre of  $A$ ), satisfying an additional constraint, called the Cardy condition.

Their result is closely related to ours. We show that an open topological conformal field theory, with one D-brane, is the same as a Frobenius  $A_\infty$  algebra. This is obviously the derived, or homotopy, version of their result.

In our situation, the closed states are not just a Frobenius algebra. They have a much richer structure coming from the topology of moduli spaces. Also, the inner product on the space of closed states may be degenerate, even on homology. This is because in this paper we need the restriction that all of our Riemann surfaces have at least one incoming boundary, whereas in Moore and Segal’s work this is not imposed.

Suppose  $(A, V)$  is an open-closed TCFT, for simplicity with one D-brane. Then  $A$  is an  $A_\infty$  Frobenius algebra. Then the map  $HH_*(A) \rightarrow H_*(V)$  we construct is an analog of the map 1.5.1. As, if we dualise we get a map

$$H_*(V)^\vee \rightarrow HH^*(A) = HH_*(A)^\vee$$

This map is compatible with the operations coming from the homology of moduli spaces of curves, so in particular, it is a ring homomorphism. If  $A$  purely of degree zero, and all higher products vanish, then  $HH^0(A)$  is the centre of  $A$ . We can view  $HH^*(A)$  as a derived analog of the centre, and this map corresponds to the one constructed by Moore-Segal and Lazaroiu.

The Cardy condition automatically holds in our setting (as it comes from one of the diagrams of open-closed TFT). However it holds in a slightly different form to that used by Moore-Segal and Lazaroiu. For us, the Cardy condition is expressed in terms of the relation between the inner product on  $A$  and a natural inner product on  $HH_*(A)$  (and in particular on  $HH_0(A) = A/[A, A]$ ). For Moore-Segal and Lazaroiu, the Cardy condition expresses the relation between an inner product on  $B$ , which maps to  $HH^0(A)$ , and that on  $A$ . If the inner product on  $HH_*(A)$  was non-degenerate, then

the dual inner product on  $HH^*(A)$  would satisfy Moore-Segal's and Lazaroiu's form of the Cardy condition. However, the inner product on  $HH_*(A)$  is often degenerate.

**1.6. The non-unital version of the result.** There is a variant of the main result that deals with non-unital Calabi-Yau  $A_\infty$  categories. This version is perhaps more suited to applications, as non-unital Calabi-Yau  $A_\infty$  categories are easier to construct and have a better-behaved deformation theory.

As the proof of the non-unital version is essentially the same, I will just indicate how the statement differs. The identity morphisms on an object of a Calabi-Yau  $A_\infty$  category corresponds to the disc with a single open boundary, and free boundary labelled by a D-brane. Therefore, if we want to work with non-unital Calabi-Yau  $A_\infty$  categories, we must remove these morphisms from the category  $\mathcal{O}_\Lambda^d$ . Thus let  $\tilde{\mathcal{O}}_\Lambda^d \subset \mathcal{O}_\Lambda^d$  be the subcategory such that each connected component of the morphism surfaces is not a disc with  $\leq 1$  open boundary.

It turns out we have to perform a similar modification in the closed case. That is, let  $\tilde{\mathcal{C}}^d \subset \mathcal{C}^d$  be the subcategory such that each connected component of the morphism surface is not a disc with one incoming closed boundary. We also have  $\widetilde{\mathcal{O}}_\Lambda^d$ , where we disallow surfaces with a connected component which is a disc with either  $\leq 1$  open or 1 closed boundaries, or an annulus with one closed, one free, and no open boundaries.

Then the analog of theorem A holds. That is, open TCFTs using  $\tilde{\mathcal{O}}_\Lambda^d$  are homotopy equivalent to non-unital extended Calabi-Yau  $A_\infty$  categories of dimension  $d$ . For each such variant open TCFT, there is a homotopy-universal open closed (using  $\widetilde{\mathcal{O}}_\Lambda^d$ ). The homology of the closed states of this is the Hochschild homology of the Calabi-Yau  $A_\infty$  category associated to the open TCFT. Here we have to be careful with the definition of Hochschild homology; for a non-unital category, the correct definition is to formally augment the category by adding on unit morphisms, and then quotient out by the subcomplex of the Hochschild chain complex spanned by these identity morphisms (considered as Hochschild zero chains). It is this version of Hochschild homology we find.

**1.7. Outline of the proof of theorem A.** There are two parts to the proof of the main theorem : a homological algebraic part, and a geometrical part.

The algebraic part consists of constructing some very general homotopy theory for functors from differential graded symmetric monoidal categories. If  $A$  is a differential graded symmetric monoidal (dgs) category, we consider a dg symmetric monoidal functor  $F : A \rightarrow \text{Comp}_{\mathbb{K}}$  as a left  $A$  module. We define the notion of tensor product and homotopy tensor product of an  $A - B$  bimodule with a  $B - C$  bimodule. The main technical point here is the result that in certain situations flat resolutions of modules exist. These results allow us to show that if  $A \rightarrow B$  is a quasi-isomorphism of dgs categories, then the categories of  $A$  modules and  $B$  modules are homotopy equivalent.

The geometric part of the proof amounts to giving an explicit generators-and-relations description for a category quasi-isomorphic to the category  $\mathcal{O}_\Lambda$ , and for the category  $\mathcal{OC}_\Lambda$ , considered as a right  $\mathcal{O}_\Lambda$  module. These explicit models are derived from certain cell complexes weakly equivalent to moduli spaces of Riemann surfaces, constructed in [Cos04, Cos06]. The cell complexes are compatible with the open gluing maps, but *not*

with the closed gluing maps. At no point do we construct a cellular model for Segal's category. I believe that such a model cannot be constructed using the standard ribbon graph decomposition.

Let me describe briefly how these cellular models for moduli space are constructed. A detailed account is contained in [Cos06]. Consider the moduli space  $\mathcal{N}_{g,h,r,s}$  of Riemann surfaces of genus  $g$ , with  $h$  boundary components,  $r$  marked points on the boundary, and  $s$  marked points in the interior. The boundary marked points will play the role of open boundary components, and the marked points in the interior (after we add the data corresponding to the parameterisation) will play the role of closed boundary components. We use a partial compactification  $\overline{\mathcal{N}}_{g,h,r,s}$  into an orbifold with corners, whose interior is  $\mathcal{N}_{g,h,r,s}$ . This partial compactification is modular; it parameterises Riemann surfaces possibly with nodes on the boundary. These nodes appear when we glue together two boundary marked points. This operation is homotopic to the operation of gluing two parameterised intervals on the boundary of a surface together, which gives the composition in the category of Riemann surfaces with open boundaries.

Inside  $\overline{\mathcal{N}}_{g,h,r,s}$  is an orbi-cell complex  $D_{g,h,r,s}$ , which parameterises Riemann surfaces glued together from discs, each of which has at most one internal marked point. This cell complex is compatible with the open gluing maps; if we take a Riemann surface built from discs, and glue two of the marked points, the surface is still built from discs.

If we pass to cellular chains, and restrict to the surfaces with no internal marked points, we can construct a chain model for the category  $\mathcal{O}_\Lambda$ . It turns out the generators are discs, and there are only some very simple relations. The compactified moduli space of marked points on the boundary of a disc is a Stasheff polytope. From this we deduce that open TCFTs are homotopy equivalent to Calabi-Yau  $A_\infty$  categories.

From considering the moduli spaces  $D_{g,h,r,s}$  where  $s \geq 0$ , we can find a model for  $\mathcal{OC}_\Lambda$  as a right  $\mathcal{O}_\Lambda$  module. This again has a very simple generators and relations description. The generators are annuli, one of whose boundaries is a closed (parameterised) boundary, and the other has some open marked points on it. (We get annuli from discs with a single internal marked point, by fattening this marked point into a (parameterised) closed boundary. Up to homotopy there is an  $S^1$  of ways of doing this). There is only one relation, which tells us about forgetting marked points on the boundary of the annulus.

This model allows us to compute the homology of  $\mathcal{OC}_\Lambda \otimes_{\mathcal{O}_\Lambda}^{\mathbb{L}} F$ , for any open TCFT  $F$ . We find this is the Hochschild homology of the  $A_\infty$  category associated to  $F$ . This turns out to follow from simple facts about the geometry of the compactified moduli space of marked points on the boundary of an annulus.

## 2. EXAMPLES AND APPLICATIONS

A Calabi-Yau category is the categorical generalisation of a Frobenius algebra. A CY category  $\mathcal{C}$  of dimension  $d$  (over our base field  $\mathbb{K}$ ) is a linear category with a trace map

$$\mathrm{Tr}_A : \mathrm{Hom}(A, A) \rightarrow \mathbb{K}[d]$$

for each object  $A$  of  $\mathcal{C}$ . The associated pairing

$$\langle \rangle_{A,B} : \mathrm{Hom}(A, B) \otimes \mathrm{Hom}(B, A) \rightarrow \mathbb{K}[d]$$

given by  $\text{Tr}(\alpha\beta)$  is required to be symmetric and non-degenerate. A Calabi-Yau category with one object is then the same as a Frobenius algebra. The grading convention is slightly funny; note that  $\text{Hom}_i(A, B)$  is dual to  $\text{Hom}_{-d-i}(B, A)$ . This is forced on us by using homological grading conventions, so the differential is of degree  $-1$ .

A Calabi-Yau  $A_\infty$  category is an  $A_\infty$  category with a trace map as above, whose associated pairing is symmetric and non-degenerate. If  $\alpha_i : A_i \rightarrow A_{i+1 \bmod n}$  are morphisms, then  $\langle m_{n-1}(\alpha_0 \otimes \dots \otimes \alpha_{n-2}), \alpha_{n-1} \rangle$  is required to be cyclically symmetric.

The notion of extended CY  $A_\infty$  category is a small technical generalisation of this definition, and will be explained later.

One special property enjoyed by Calabi-Yau  $A_\infty$  categories is a duality between Hochschild homology and cohomology;

$$HH_i(D) \cong HH^{d+i}(D)^\vee$$

where  $d$  is the dimension of the category.

Our main result implies that the homology of moduli space acts on the Hochschild homology groups of an Calabi-Yau  $A_\infty$  category. Next we will discuss in detail what happens for some naturally arising classes of Calabi-Yau  $A_\infty$  categories, associated to a compact oriented manifold, a smooth projective Calabi-Yau variety, or a symplectic manifold.

**2.1. String topology.** Let  $M$  be a compact, simply connected, oriented manifold. The cohomology of  $M$  has the structure of  $C_\infty$  (homotopy commutative) algebra, encoding the rational homotopy type of the manifold. Hamilton and Lazarev [HL04] have shown how this enriches naturally to a Frobenius  $C_\infty$  algebra, that is a  $C_\infty$  algebra with a non-degenerate invariant pairing<sup>1</sup>. The pairing is simply the Poincaré pairing.

Thus,  $H^{-*}(M)$ <sup>2</sup> is, in a natural way, a Calabi-Yau  $A_\infty$  category with one object.

Since  $H^{-*}(M)$  is quasi-isomorphic, as an  $A_\infty$  algebra, to  $\Omega^{-*}(M)$ , a well-known theorem of Adams-Chen implies that

$$HH_*(H^{-*}(M)) = H^{-*}(\mathcal{L}M)$$

is the cohomology of the free loop space  $\mathcal{L}M$  of  $M$ .

Theorem A now implies that the homology of the moduli spaces of Riemann surfaces acts on  $H^{-*}(\mathcal{L}M)$ . That is, there are maps

$$H_*(\mathcal{M}(I, J), \det^d) \otimes H^{-*}(\mathcal{L}M)^{\otimes I} \rightarrow H^{-*}(\mathcal{L}M)^{\otimes J}$$

compatible with composition and disjoint union. These operations should correspond to the higher-genus version of the string topology operations of Chas-Sullivan [CS99, CS04, Coh04]. This would follow, using the universality statement in theorem A, from the existence of a theory of open-closed string topology whose associated Calabi-Yau  $A_\infty$  category was equivalent to  $H^{-*}(M)$ .

Note that the degree shift in Chas-Sullivan's theory is incorporated here in to the local system  $\det^d$ .

---

<sup>1</sup>Hamilton and Lazarev's main result is that the deformation theory for Frobenius  $C_\infty$  and  $C_\infty$  algebras coincide; they deduce the existence of the Frobenius  $C_\infty$  structure as an immediate corollary. Note that in the associative world, Frobenius  $A_\infty$  and  $A_\infty$  algebras have different deformation theory.

<sup>2</sup>All our complexes are homological, so we reverse the usual grading

**2.2. The B model.** Let  $X$  be a smooth projective Calabi-Yau variety of dimension  $d$  over  $\mathbb{C}$ . Pick a holomorphic volume form on  $X$ . Consider the dg category  $\text{Perf}(X)$ , whose objects are bounded complexes of holomorphic vector bundles on  $X$ , and whose morphisms are

$$\text{Hom}_{\text{Perf}(X)}(E, F) = \Omega^{0, -*}(E^\vee \otimes F)$$

(we reverse the grading, as all our differentials are homological). The holomorphic volume form gives us a pairing

$$\text{Hom}_{\text{Perf}(X)}(E, F) \otimes_{\mathbb{C}} \text{Hom}_{\text{Perf}(X)}(F, E) \rightarrow \mathbb{C}$$

of degree  $d$ , which is non-degenerate on homology. Using the homological perturbation lemma, we can transfer the  $A_\infty$  structure on  $\text{Perf}(X)$  to homology category. We should be able to ensure that the resulting  $A_\infty$  category is Calabi-Yau for the natural pairing, using Hodge theory and the explicit form of the homological perturbation lemma [Mer99, KS01, Mar04]. Denote by  $\mathcal{D}_\infty^b(X)$  this Calabi-Yau  $A_\infty$  category.

The closed TCFT  $j^* \mathbb{L} i_* \mathcal{D}_\infty^b(X)$  is the  $B$  model mirror to a TCFT constructed from Gromov-Witten invariants of a compact symplectic manifold. We have seen that the homology of  $j^* \mathbb{L} i_* \mathcal{D}_\infty^b(X)$  is the Hochschild homology of  $\mathcal{D}_\infty^b(X)$ .

As the  $A_\infty$  categories  $\text{Perf}(X)$ ,  $\mathcal{D}_\infty^b(X)$  are quasi-isomorphic, they have the same Hochschild homology. One should be able to show that

$$(2.2.1) \quad HH_i(\mathcal{D}_\infty^b(X)) = HH_i(\text{Perf}(X)) = \bigoplus_{q-p=i} H^p(X, \Omega_X^q)$$

Theorem A, applied to  $\mathcal{D}_\infty^b(X)$ , implies there are operations on  $HH_*(\mathcal{D}_\infty^b(X))$  indexed by homology classes on the moduli spaces of curves. That is, if as before  $\mathcal{M}(I, J)$  is the moduli space of Riemann surfaces with  $I$  incoming and  $J$  outgoing boundaries, there is a map

$$H_*(\mathcal{M}(I, J), \det^d) \rightarrow \text{Hom}(HH_*(\mathcal{D}_\infty^b(X))^{\otimes I}, HH_*(\mathcal{D}_\infty^b(X))^{\otimes J})$$

compatible with gluing and disjoint union. These operations should be the  $B$ -model mirror to corresponding operations on the homology of a symplectic manifold coming from Gromov-Witten invariants.

Note that the usual derived category (without the  $A_\infty$  enrichment) is a Calabi-Yau  $A_\infty$  category. However, as usual, passing to homology loses too much information. This category cannot encode the  $B$  model.

**2.3. Gromov-Witten invariants and the Fukaya category.** The Fukaya category [FOOO00] of a symplectic manifold should be an example of a unital Calabi-Yau  $A_\infty$  category. Thus, associated to the Fukaya category one has a closed TCFT, whose homology is the Hochschild homology of the Fukaya category.

Also, the Floer chains of the loop space of a symplectic manifold should have a natural structure of closed TCFT, where the TCFT operations come from counting pseudo-holomorphic maps. Thus to each symplectic manifold we can associate two TCFTs, and it is natural to conjecture that these are homotopy equivalent. We will see that the universality statement of theorem A allows us to relate these two TCFTs, thus providing evidence for this conjecture.

**2.4. The TCFT associated to Gromov-Witten invariants.** First, let me explain a little about this second construction of a TCFT, in the special case of a compact symplectic manifold  $X$ . In this case, the TCFT arises from Gromov-Witten invariants. Let  $\overline{\mathcal{M}}$  be the Deligne-Mumford analog of Segal's category, that is the category with objects finite sets, and morphisms stable algebraic curves with incoming and outgoing marked points. One can find a homotopy equivalent model  $\mathcal{M}'$  for Segal's category  $\mathcal{M}$  with a natural functor  $\mathcal{M}' \rightarrow \overline{\mathcal{M}}$ . A chain-level theory of Gromov-Witten invariants should give a functor from  $C_*(\overline{\mathcal{M}}) \rightarrow \text{Comp}_{\mathbb{K}}$ ; pulling back via the functor  $C_*(\mathcal{M}') \rightarrow C_*(\overline{\mathcal{M}})$  will give the required TCFT. The model  $\mathcal{M}'$  for  $\mathcal{M}$  we need was first constructed by Kimura, Stasheff and Voronov in [KSV95]. It can be constructed by performing a real blow up of the Deligne-Mumford spaces along their boundary. More precisely, we can take for  $\mathcal{M}'$  the moduli space of curves  $\Sigma \in \overline{\mathcal{M}}$ , together with at each marked point a section of the tautological  $S^1$  bundle, and at each node a section of the tensor product of the two tautological  $S^1$  bundles corresponding to either side of the node.

Suppose for simplicity that  $c_1(X) = 0$ , and let  $\Sigma \in \mathcal{M}(I, J)$ . Then the real virtual dimension of the space of pseudo-holomorphic maps from the fixed surface  $\Sigma$  to  $X$  is  $d(\chi(\Sigma) + \#I + \#J)$ . Thus, each such curve  $\Sigma$  should give an operation

$$C_*(X)^{\otimes I} \rightarrow C_*(X)^{\otimes J}$$

of degree  $d\chi + d\#J - d\#I$ . We want to construct a  $d$  dimensional TCFT from a  $2d$  dimensional symplectic manifold<sup>3</sup>. Therefore there should be a shift in grading, and we should work with  $C_{*+d}(X)$ .

One can check easily that if we work with this shift in grading, we find a  $d$  dimensional TCFT. The point is that the extra signs arising from this shift in grading correspond to working with chains on moduli space with coefficients in the local system  $\det^d$ .

At the level of homology, this TCFT structure follows from the existence of Gromov-Witten invariants; the chain level version we need is, I believe, still conjectural.

**2.5. Comparing the TCFT associated to Gromov-Witten theory with that from the Fukaya category.** Given a compact symplectic manifold, there should therefore be two associated closed TCFTs: that coming from Gromov-Witten invariants, and that constructed from the Fukaya category. We now provide some evidence for the conjecture that these are homotopy equivalent. Let  $X$  be a compact symplectic manifold of dimension  $2d$ , with Fukaya category  $\text{Fuk}(X)$ .

**Conjecture 1.** *There is a natural structure of  $d$ -dimensional open-closed TCFT, whose  $D$ -branes are certain Lagrangian branes<sup>4</sup> in  $X$ , whose morphism spaces between  $D$ -branes  $L_1, L_2$  are the Lagrangian Floer chain groups*

$$\text{Hom}_i(L_1, L_2) = CF^{-i}(L_1, L_2)$$

*and whose complex of closed states is the shifted singular chain complex  $C_{*+d}(X)$  of  $X$ .*

This conjecture is I'm sure obvious to many people. It is simply asserting that the work of Fukaya-Oh-Ohta-Ono [FOOO00] can be generalised to the case of Riemann

<sup>3</sup>If  $X$  does not satisfy  $c_1(X) = 0$ , we can work with only a  $\mathbb{Z}/2$  grading

<sup>4</sup>Lagrangians with the extra structure which makes them into an object of the Fukaya category

surfaces of all genus with open-closed boundary conditions, in a way which takes into account families of surfaces.

Parts of this conjectural open-closed Gromov-Witten theory have previously been constructed by P. Seidel [Sei01b, Sei01a] and C.-C. Liu [Liu02]. Seidel constructs the “topological field theory” version with fixed complex structure on the source Riemann surface. This corresponds to working with  $H_0$  of moduli spaces. The part dealing with only one Lagrangian, and varying source Riemann surface, has been constructed by C.-C. Liu [Liu02].

A corollary of conjecture 1 and theorem A is

**Corollary.** *There is a map of closed TCFTs  $j^* \mathbb{L}i_*(\mathrm{Fuk} X) \rightarrow C_{*+d}(X)$  from the universal closed TCFT to the singular chains of  $X$ . On homology this gives a map of homological TCFTs  $HH_*(\mathrm{Fuk} X) \rightarrow H_{*+d}(X)$  from the Hochschild homology of the Fukaya category to the homology of  $X$ .*

A homological TCFT is like a TCFT except we replace the complex of chains on moduli space by its homology. The fact that the map  $HH_*(\mathrm{Fuk}(X)) \rightarrow H_{*+d}(X)$  is a map of homological TCFTs means that it intertwines all operations coming from the homology of moduli spaces of curves; that is the diagram

$$\begin{array}{ccc} H_*(\mathcal{M}(I, J), \det^d) \otimes HH_*(\mathrm{Fuk}(X))^{\otimes I} & \longrightarrow & H_*(\mathcal{M}(I, J), \det^d) \otimes H_{*+d}(X)^{\otimes I} \\ \downarrow & & \downarrow \\ HH_*(\mathrm{Fuk}(X))^{\otimes J} & \longrightarrow & H_{*+d}(X)^{\otimes J} \end{array}$$

commutes.

The map from Hochschild to (Floer) homology is the same as that constructed by Seidel in [Sei02]. The homology of a TCFT has the structure of cocommutative coalgebra, coming from the pair-of-pants coproduct. Note that as the pair of pants has Euler characteristic  $-1$ , this is a map of degree  $-d$ . This coproduct structure on  $HH_*(\mathrm{Fuk}(X))$  is dual to the standard cup product on Hochschild cohomology, using the isomorphism  $HH_i(\mathrm{Fuk}(X))^\vee \cong HH^{d+i}(\mathrm{Fuk}(X))$ . The coproduct on  $H_{*+d}(X)$  is dual to the quantum cup product on  $H^*(X)$ . Thus, the dual map  $H^*(X) \rightarrow HH^*(\mathrm{Fuk}(X))$  is in particular a ring homomorphism from quantum to Hochschild cohomology. Note that this dual map is of degree 0.

Open-closed Gromov-Witten theory would give a map from the closed TCFT associated to  $\mathrm{Fuk}(X)$  to that coming from the Gromov-Witten theory of  $X$ . It is natural to conjecture that this is a quasi-isomorphism, that is

**Conjecture 2.** *In good circumstances, the map  $HH_*(\mathrm{Fuk}(X)) \rightarrow H_{*+d}(X)$  is an isomorphism.*

This conjecture, which was first proposed by Kontsevich [Kon95], seems to be an integral part of the homological mirror symmetry picture. Unfortunately, however, I really don’t know of much evidence. Kontsevich presents a geometric motivation for this conjecture in [Kon95], which I will reproduce here. We can identify the Hochschild cohomology of the Fukaya category with the endomorphisms of the identity functor, in the  $A_\infty$  category of  $A_\infty$  functors from  $\mathrm{Fuk}(X)$  to itself. If we could identify this  $A_\infty$

category with  $\mathrm{Fuk}(X \times X, \omega \oplus -\omega)$ , as seems natural, we would see that the Hochschild cohomology of  $\mathrm{Fuk}(X)$  would be the Lagrangian Floer cohomology of the diagonal in  $(X \times X, \omega \oplus -\omega)$ , which is known to coincide with the ordinary cohomology of  $X$  with the quantum product.

This conjecture implies that the homotopy Lie algebra controlling deformations of  $\mathrm{Fuk}(X)$  is formal, and quasi-isomorphic to  $H^*(X)$  with the trivial Lie bracket. So that the formal neighbourhood of  $\mathrm{Fuk}(X)$  in the moduli space of  $A_\infty$  categories is isomorphic to the formal neighbourhood of the symplectic form in  $H^*(X)$ . The homotopy Lie algebra structure arises from an action of chains on moduli spaces of genus 0 Riemann surfaces. The homotopy Lie algebra structure on  $C^*(X)$  should be trivial, as the circle action is trivial.

**2.6. Acknowledgements.** I'd like to thank Ezra Getzler, Mike Hopkins, Andrey Lazarev, Tim Perutz, Graeme Segal, Paul Seidel, Jim Stasheff, Dennis Sullivan, Constantin Teleman, and Richard Thomas for helpful conversations and correspondence. I benefitted greatly from an inspiring talk of M. Kontsevich at the Hodge centenary conference in 2003. In this talk he described several results related to those in this paper; in particular, he sketched a different construction of a TCFT structure on the Hochschild chains of an  $A_\infty$  algebra, and also an extension of this to the Deligne-Mumford spaces when the Hodge to de Rham spectral sequence degenerates.

**2.7. Notation.**  $\mathbb{K}$  will denote a field of characteristic zero. All homology and cohomology will be with coefficients in  $\mathbb{K}$ , and all algebras and linear categories will be defined over  $\mathbb{K}$ .  $\mathrm{Comp}_{\mathbb{K}}$  will denote the category of complexes of  $\mathbb{K}$  vector spaces, with differential of degree  $-1$ , and with its standard structure of symmetric monoidal category. For  $r \in \mathbb{Z}$  we denote by  $\mathbb{K}[r]$  the complex in degree  $-r$ , and for  $V \in \mathrm{Comp}_{\mathbb{K}}$  we write  $V[r]$  for  $V \otimes \mathbb{K}[r]$ .  $\mathrm{Vect}^*$  will denote the category of  $\mathbb{Z}$  graded  $\mathbb{K}$  vector spaces.

Instead of working with a field  $\mathbb{K}$  and complexes of  $\mathbb{K}$  vector spaces, the main result remains true if instead we work with a commutative differential graded algebra  $R$  containing  $\mathbb{Q}$ , and flat dg  $R$  modules. (A dg  $R$  module  $M$  is flat if the functor  $M \otimes_R -$  is exact, that is takes quasi-isomorphisms to quasi-isomorphisms).

### 3. THE OPEN-CLOSED MODULI SPACES IN MORE DETAIL

Fix a set  $\Lambda$  of D-branes.

A Riemann surface with open-closed boundary is a possibly disconnected Riemann surface  $\Sigma$ , with boundary, some of whose boundary components are parameterised in a way compatible with the orientation on  $\Sigma$ ; these are the incoming closed boundaries. Other boundary components are parameterised in the opposite sense; these are the outgoing closed boundaries. There are some disjoint intervals embedded in the remaining boundary components; these are the open boundaries. Some of these intervals are embedded in a way compatible with the orientation on  $\Sigma$ ; these are incoming open, the remainder are outgoing open.

If we remove from  $\partial\Sigma$  the open and closed boundaries, what is left is a one-manifold, whose connected components are the free boundaries. Suppose the free boundaries of  $\Sigma$  are labelled by D-branes. Then each open boundary  $o$  of  $\Sigma$  has associated to it an

ordered pair  $(s(o), t(o))$  of D-branes, associated to the free boundaries where it starts and where it ends.

We require that each connected component of  $\Sigma$  has at least one incoming closed boundary or at least one free boundary. We do not impose a stability condition; note that no connected component of  $\Sigma$  can be a sphere or torus with no boundaries. However, it is possible that a connected component of  $\Sigma$  could be a disc or an annulus with no open or closed boundaries, and only free boundaries. This would introduce an infinite automorphism group; to remedy this, we replace the moduli space (stack) of discs or annuli with no open or closed boundaries by a point. One can think of this as either taking the coarse moduli space, or rigidifying in some way.

Define a topological category  $\mathcal{M}_\Lambda$ . The objects of  $\mathcal{M}_\Lambda$  are quadruples  $(C, O, s, t)$  where  $C, O \in \mathbb{Z}_{\geq 0}$ , and  $s, t : O \rightarrow \Lambda$  are two maps. (We use notation which identifies the integer  $O$  with the set  $\{0, 1, \dots, O - 1\}$ ). The space of morphisms  $\mathcal{M}_\Lambda((C_+, O_+, s_+, t_+), (C_-, O_-, s_-, t_-))$  is the moduli space of Riemann surfaces  $\Sigma$  with open-closed boundary, with free boundaries labelled by D-branes, with open incoming (respectively outgoing) boundaries labelled by  $O_+$  (respectively  $O_-$ ), with closed incoming (respectively outgoing) boundaries labelled by  $C_+$  (respectively  $C_-$ ), such that the maps  $s_\pm, t_\pm : O_\pm \rightarrow \Lambda$  coincide with those coming from the D-brane labelling on  $\Sigma$ . Composition in this category is given by gluing incoming and outgoing open and closed boundaries to each other.

As defined,  $\mathcal{M}_\Lambda$  is a non-unital category; it does not have identity maps. To remedy this, we modify it a little. We replace the moduli space of annuli, with one incoming and one outgoing closed boundary, which is  $\text{Diff}_+ S^1 \times_{S^1} \text{Diff}_+ S^1 \times \mathbb{R}_{>0}$ , by the homotopy equivalent space  $\text{Diff}_+ S^1$ , acting by reparameterisation. This should be thought of as the moduli space of infinitely thin annuli. Similarly, we replace the moduli space of discs with one incoming and one outgoing open boundary by a point, which acts as the identity on the open boundaries. We should perform this procedure also for any surfaces which have connected components of one of these forms.

Disjoint union of surfaces and addition of integers  $(C, O)$  makes  $\mathcal{M}_\Lambda$  into a symmetric monoidal topological category, in the sense of [ML98]. Note that this is a *strict* monoidal category; the monoidal structure is strictly associative. It is not, however, strictly symmetric.

Let  $C_*$  be the chain complex functor defined in the appendix, from spaces to complexes of  $\mathbb{K}$  vector spaces.  $C_*$  is a symmetric monoidal functor, in the sense of [ML98]. This means that there is a natural transformation  $C_*(X) \otimes C_*(Y) \rightarrow C_*(X \times Y)$ , satisfying some coherence axioms. Define the category  $C_*(\mathcal{M}_\Lambda)$  to have the same objects as  $\mathcal{M}_\Lambda$ , but with  $C_*(\mathcal{M}_\Lambda)(l_1, l_2) = C_*(\mathcal{M}_\Lambda(l_1, l_2))$  for  $l_i \in \text{Ob } C_*(\mathcal{M}_\Lambda)$ . Since  $C_*$  is a symmetric monoidal functor,  $C_*(\mathcal{M}_\Lambda)$  is again a symmetric monoidal category, but this time enriched over the category of complexes. That is,  $C_*(\mathcal{M}_\Lambda)$  is a differential graded symmetric monoidal category.

As the set of D-branes will be fixed throughout the paper, we will occasionally omit the subscript  $\Lambda$  from the notation.

**Definition 3.0.1.** Let  $\mathcal{O}\mathcal{C}_\Lambda = C_*(\mathcal{M}_\Lambda)$ . Let  $\mathcal{O}_\Lambda$  be the full subcategory whose objects are  $(0, O, s, t)$ , that is have no closed part. Let  $\mathcal{C}$  be the subcategory whose objects have

no open part, and whose morphisms are Riemann surfaces with only closed boundaries.  $\mathcal{C}$  is independent of  $\Lambda$ . These categories are differential graded symmetric monoidal categories.

Note that if  $\Lambda \rightarrow \Lambda'$  is a map of sets, there are corresponding functors  $\mathcal{O}\mathcal{C}_\Lambda \rightarrow \mathcal{O}\mathcal{C}_{\Lambda'}$  and  $\mathcal{O}_\Lambda \rightarrow \mathcal{O}_{\Lambda'}$ . We could think of  $\mathcal{O}$  and  $\mathcal{O}\mathcal{C}$  as categories fibred over the category of sets.

We need twisted versions of these categories. Consider the graded  $\mathbb{K}$  local system  $\det$  on the spaces of morphisms in  $\mathcal{M}_\Lambda$ , whose fibre at a surface  $\Sigma$  is

$$\begin{aligned} \det(\Sigma) &= \det(H^0(\Sigma) - H^1(\Sigma) + \mathbb{K}^{O_-})[O_- - \chi(\Sigma)] \\ &= \det(H^0(\Sigma) \oplus \mathbb{K}^{O_-}) \otimes (\det H^1(\Sigma))^{-1}[O_- - \chi(\Sigma)] \end{aligned}$$

Here  $O_-$  is the number of open outgoing boundary components of  $\Sigma$ . The number in square brackets refers to a shift of degree; so this is a graded local system situated in degree  $\chi(\Sigma) - O_-$ .

Suppose  $\Sigma_1, \Sigma_2$  are composable morphisms in  $\mathcal{M}_\Lambda$ . Then there is a natural isomorphism

$$\det \Sigma_2 \otimes \det \Sigma_1 \rightarrow \det(\Sigma_2 \circ \Sigma_1)$$

This follows from the Mayer-Vietoris exact sequence obtained from writing  $\Sigma_2 \circ \Sigma_1$  as a union of the  $\Sigma_i$ 's. Let  $C_-^i$  and  $O_-^i$  be the open and closed outgoing boundaries of  $\Sigma^i$ . We have

$$\begin{aligned} 0 \rightarrow H^0(\Sigma_2 \circ \Sigma_1) \rightarrow H^0(\Sigma_2) \oplus H^0(\Sigma_1) \rightarrow \mathbb{K}^{C_-^1 + O_-^1} \\ \rightarrow H^1(\Sigma_2 \circ \Sigma_1) \rightarrow H^1(\Sigma_2) \oplus H^1(\Sigma_1) \rightarrow \mathbb{K}^{C_-^1} \rightarrow 0 \end{aligned}$$

Here  $\mathbb{K}^{C_-^1 + O_-^1}$  arises as  $H^0(\Sigma_2 \cap \Sigma_1)$  and  $\mathbb{K}^{C_-^1}$  arises as  $H^1(\Sigma_2 \cap \Sigma_1)$ . Note that the orientation on the outgoing boundary of  $\Sigma_1$  gives a natural isomorphism  $H^1(\Sigma_2 \cap \Sigma_1) \cong \mathbb{K}^{C_-^1}$ .

We will see that  $(\mathcal{M}_\Lambda, \det)$  again forms a kind of category. Consider the symmetric monoidal category whose objects are pairs  $(X, E)$  where  $X$  is a topological space and  $E$  is a graded  $\mathbb{K}$  local system on  $X$ , such that a map  $(X, E) \rightarrow (Y, F)$  is a map  $f : X \rightarrow Y$  and a map  $E \rightarrow f^*F$ , and such that

$$(X, E) \otimes (Y, F) = (X \times Y, \pi_1^*E \otimes \pi_1^*F)$$

The symmetrisation isomorphism  $(X, E) \otimes (Y, F) \cong (Y, F) \otimes (X, E)$  as usual picks up signs from the grading on  $E$  and  $F$ .

We want to show that  $(\mathcal{M}_\Lambda, \det)$  forms a category enriched over the category of spaces with graded local systems. All that needs to be checked is that for composable surfaces  $\Sigma_1, \Sigma_2, \Sigma_3$  the diagram

$$\begin{array}{ccc} \det(\Sigma_3) \otimes \det(\Sigma_2) \otimes \det(\Sigma_1) & \longrightarrow & \det(\Sigma_3) \otimes \det(\Sigma_2 \circ \Sigma_1) \\ \downarrow & & \downarrow \\ \det(\Sigma_3 \circ \Sigma_2) \otimes \det(\Sigma_1) & \longrightarrow & \det(\Sigma_3 \circ \Sigma_2 \circ \Sigma_1) \end{array}$$

commutes, where  $\det(\Sigma)$  is the fibre of the local system at  $\Sigma$ . This is a fairly straightforward calculation.

There is also a natural isomorphism  $\det(\Sigma_1 \amalg \Sigma_2) \cong \det(\Sigma_1) \otimes \det(\Sigma_2)$ . This gives  $(\mathcal{M}_\Lambda, \det)$  the structure of symmetric monoidal category.

The functor  $C_*$  defined in the appendix is a functor from the category of spaces with graded  $\mathbb{K}$  local systems to complexes, which computes homology with local coefficients. Since  $C_*$  is a symmetric monoidal functor, it follows that  $C_*(\mathcal{M}_\Lambda, \det)$  is again a symmetric monoidal category.

We can think of the chain category  $C_*(\mathcal{M}_\Lambda, \det)$  geometrically as follows. A chain with local coefficients on  $(X, E)$  can be thought of as a singular simplex  $f : \Delta_n \rightarrow X$  together with a section of  $f^*E \otimes \omega$ , where  $\omega$  is the orientation sheaf on  $\Delta_n$ . Thus a chain in  $C_*(\mathcal{M}_\Lambda, \det)$  should be thought of as an oriented  $n$  parameter family of Riemann surfaces  $\Sigma$  with a section of  $\det(\Sigma)$ .

We can also twist  $\mathcal{M}_\Lambda$  by tensor powers  $\det^d = \det^{\otimes d}$ , where  $d \in \mathbb{Z}$ .

**Definition 3.0.2.** *Let  $\mathcal{O}\mathcal{C}_\Lambda^d$  be the category  $C_*(\mathcal{M}_\Lambda, \det^d)$ . As before, let  $\mathcal{O}_\Lambda^d$  be the full subcategory whose objects have no closed part, and let  $\mathcal{C}^d$  be the subcategory whose objects have no open part and whose morphisms have only closed boundaries. These are differential graded symmetric monoidal categories.*

As before, if  $\Lambda \rightarrow \Lambda'$  is a map of sets, there are corresponding functors  $\mathcal{O}_\Lambda^d \rightarrow \mathcal{O}_{\Lambda'}^d$  and  $\mathcal{C}_\Lambda^d \rightarrow \mathcal{C}_{\Lambda'}^d$ .

#### 4. SOME HOMOLOGICAL ALGEBRA FOR SYMMETRIC MONOIDAL CATEGORIES

**4.1. Differential graded symmetric monoidal categories.** We work with differential graded symmetric monoidal categories, over  $\mathbb{K}$ . Symmetric monoidal is in the sense of MacLane [ML98]; differential graded means that the morphism spaces are complexes of  $\mathbb{K}$  vector spaces (with differential of degree  $-1$ ), and the composition maps are bilinear and compatible with the differential. Call these dgsm categories, for short. A good reference for the general theory of dg categories is [Kel94].

The dgsm categories controlling topological conformal field theory are strictly monoidal. On objects,  $(\alpha \amalg \beta) \amalg \gamma = \alpha \amalg (\beta \amalg \gamma)$ , and similarly the diagram

$$\begin{array}{ccc} \mathrm{Hom}(\alpha, \alpha') \otimes \mathrm{Hom}(\beta, \beta') \otimes \mathrm{Hom}(\gamma, \gamma') & \longrightarrow & \mathrm{Hom}(\alpha \amalg \beta, \alpha' \amalg \beta') \otimes \mathrm{Hom}(\gamma, \gamma') \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\alpha, \alpha') \otimes \mathrm{Hom}(\beta \amalg \gamma, \beta' \amalg \gamma') & \longrightarrow & \mathrm{Hom}(\alpha \amalg \beta \amalg \gamma, \alpha' \amalg \beta' \amalg \gamma') \end{array}$$

commutes. (We use  $\amalg$  and  $\otimes$  interchangeably for the tensor product in the categories controlling TCFT). However, the symmetry isomorphism  $\alpha \amalg \beta \rightarrow \beta \amalg \alpha$  is not an identity, nor do we always have  $\alpha \amalg \beta = \beta \amalg \alpha$ . If  $A$  is strictly monoidal, for each  $\sigma \in S_n$  there is an isomorphism  $a_1 \otimes \dots \otimes a_n \cong a_{\sigma(1)} \otimes \dots \otimes a_{\sigma(n)}$ , compatible with composition in the symmetric groups.

Let  $A, B$  be dgsm categories, which for simplicity we assume are strictly monoidal. A monoidal functor  $F : A \rightarrow B$  is a functor  $F$ , compatible with the dg structure, together

with natural transformations  $F(a) \otimes F(a') \rightarrow F(a \otimes a')$ , such that the diagrams

$$\begin{array}{ccc} F(a) \otimes F(a') \otimes F(a'') & \longrightarrow & F(a \otimes a') \otimes F(a'') \\ \downarrow & & \downarrow \\ F(a) \otimes F(a' \otimes a'') & \longrightarrow & F(a \otimes a' \otimes a'') \end{array}$$

and

$$\begin{array}{ccc} F(a) \otimes F(a') & \longrightarrow & F(a \otimes a') \\ \downarrow & & \downarrow \\ F(a') \otimes F(a) & \longrightarrow & F(a' \otimes a) \end{array}$$

commute.

Although our dgsms may (or may not) have an object which is a unit for the tensor product, we do not assume the functor  $F$  takes units to units.

To a dgsms category  $A$  are associated several important auxiliary categories. First there is the homology category  $H_*A$ , whose objects are the same as those of  $A$ , but with

$$\mathrm{Hom}_{H_*A}(a, a') = H_* \mathrm{Hom}_A(a, a')$$

$H_*A$  is a graded symmetric monoidal category; the morphisms are graded vector spaces. Similarly we have the category  $H_0A$ , whose morphisms are  $H_0 \mathrm{Hom}_A(a, a')$ . Also, there is the category  $Z_0A$ , which is a subcategory of  $A$  with the same objects, but whose morphisms are closed maps of degree 0. A map in  $Z_0A$  is called a quasi-isomorphism if it is an isomorphism in  $H_0A$ .

One example of a dgsms category is the category  $\mathrm{Comp}_{\mathbb{K}}$  of complexes of  $\mathbb{K}$  vector spaces. The monoidal structure is given by tensor product.

A left  $A$  module is a (monoidal) functor  $A \rightarrow \mathrm{Comp}_{\mathbb{K}}$ . A right  $A$  module is a (monoidal) functor  $A^{op} \rightarrow \mathrm{Comp}_{\mathbb{K}}$ , where  $A^{op}$  is the opposite category to  $A$ . If  $M, N : A \rightarrow B$  are monoidal functors to a dgsms category  $B$ , a natural transformation  $\phi : M \rightarrow N$  consists of a collection of maps  $\phi(a) \in \mathrm{Hom}_B(M(a), N(a))$  satisfying the following conditions.

- (1)  $\phi(a)$  is natural for morphisms in  $a$ . That is, if  $f : a \rightarrow a'$  then  $\phi(a')M(f) = N(f)\phi(a)$ .
- (2) The morphisms  $\phi(a) \in \mathrm{Hom}_B(M(a), N(a))$  are all closed and of degree 0.
- (3) The diagram

$$\begin{array}{ccc} M(a) \otimes M(a') & \longrightarrow & N(a) \otimes N(a') \\ \downarrow & & \downarrow \\ M(a \otimes a') & \longrightarrow & N(a \otimes a') \end{array}$$

commutes.

Thus for example we have a category  $A - \mathrm{mod}$  of left  $A$  modules and  $\mathrm{mod} - A$  of right  $A$  modules. Note that  $A - \mathrm{mod}$  is just a category, not a dg category; it is not even an additive category.

If  $A, B$  are dgsm categories, we can form their tensor product  $A \otimes B$ . The objects are

$$\text{Ob}(A \otimes B) = \text{Ob } A \times \text{Ob } B$$

and the morphisms are described by

$$\text{Hom}(a \times b, a' \times b') = \text{Hom}(a, a') \otimes_{\mathbb{K}} \text{Hom}(b, b')$$

$A \otimes B$  is again a dgsm category. An  $A - B$  bimodule is a monoidal functor  $A \otimes B^{op} \rightarrow \text{Comp}_{\mathbb{K}}$ .

We will often use the notation

$$A(a, a') = \text{Hom}_A(a, a')$$

$A$  defines an  $A - A$  bimodule over itself, by the functor  $A \otimes A^{op} \rightarrow \text{Comp}_{\mathbb{K}}$  which sends

$$(a_1, a_2) \mapsto A(a_2, a_1)$$

However, if  $a \in \text{Ob } A$  is an object, then the functor  $A \rightarrow \text{Comp}_{\mathbb{K}}$  defined by  $\text{Hom}(a, -)$  is not in general monoidal, and so does not give an  $A$ -module in our sense.

**4.2. Notation about exact functors.** Suppose a category  $C$  has a notion of quasi-isomorphism. That is, suppose we are given a subset of the set of morphisms of  $C$ , which is closed under composition and which contains all isomorphisms. We say objects in  $C$  are quasi-isomorphic if they can be connected by a chain of quasi-isomorphisms. We write  $c \simeq c'$  to indicate that  $c, c'$  are quasi-isomorphic.

If  $D$  also has a class of quasi-isomorphisms, a functor  $F : C \rightarrow D$  is called exact if it takes quasi-isomorphisms to quasi-isomorphisms.

A natural transformation between exact functors  $F, G : C \rightarrow D$  is called a quasi-isomorphism if for each  $c \in C$  the maps  $F(c) \rightarrow G(c)$  are quasi-isomorphisms.

**Definition 4.2.1.** *A quasi equivalence between  $C, D$  is a pair of functors  $F : C \rightarrow D$  and  $G : D \rightarrow C$  such that  $F \circ G$  is quasi-isomorphic to  $\text{Id}_D$ , and  $G \circ F$  is quasi-isomorphic to  $\text{Id}_C$ . That is,*

$$F \circ G \simeq \text{Id}_D \qquad G \circ F \simeq \text{Id}_C$$

For example, let  $A$  be a dgsm category. Recall  $Z_0 A$  is the category with the same objects as  $A$  but whose morphisms are closed of degree 0. A map  $a \rightarrow a'$  in  $Z_0 A$  is a quasi-isomorphism if it is an isomorphism in  $H_0 A$ .

Any functor  $F : A \rightarrow B$  between dgsm categories restricts to an exact functor  $Z_0 A \rightarrow Z_0 B$ . Thus the category of functors  $A \rightarrow B$  acquires a notion of quasi-isomorphism. In particular we can talk about quasi-isomorphisms in  $A - \text{mod}$ ; these are just morphisms which are quasi-isomorphisms of the underlying complexes.

We would like to do some kind of homotopy theory with categories  $A - \text{mod}$  for various  $A$ . I am going to do this in a slightly ad hoc fashion. Probably one should put some extra structure on the categories  $A - \text{mod}$  which would allow a more canonical notion of derived functor. For example, one could try to make  $A - \text{mod}$  into a closed model category whose weak equivalences are quasi-isomorphisms. However, closed model structures are difficult to construct. One alternative structure which

seems more natural in this situation would be to consider categories fibred over the category of differential graded commutative algebras. The fibre over  $R$  should be the category of  $R$  linear functors from  $A \otimes R$  to flat complexes of  $R$  modules. One could use this structure to define notions of homotopy between maps, and eventually to define derived functors in a more canonical way.

Instead of attempting to construct any such general theory, or give a closed model structure on  $A - \text{mod}$ , I will perform homotopic constructions in a slightly more ad hoc fashion. We only need to derive one kind of functor; if  $f : A \rightarrow B$  is a functor, there is a pull back functor  $f^* : B - \text{mod} \rightarrow A - \text{mod}$ , which is exact, and a left adjoint  $f_*$  which is not. We will construct the derived functor of  $f_*$ ; it will be clear from the construction that there is a unique left derived functor  $\mathbb{L}f_*$  up to quasi-isomorphism.

**4.3. Derived tensor products.** Let  $M$  be a  $B - A$  bimodule. Let  $N$  be a left  $A$  module. Then we can form a left  $B$  module  $M \otimes_A N$ . For each  $b \in B$ ,  $M \otimes_A N(b)$  is defined to be the universal complex with maps  $M(b, a) \otimes_{\mathbb{K}} N(a) \rightarrow (M \otimes_A N)(b)$ , such that the diagram

$$\begin{array}{ccc} M(b, a) \otimes_{\mathbb{K}} A(a', a) \otimes_{\mathbb{K}} N(a') & \longrightarrow & M(b, a) \otimes_{\mathbb{K}} N(a) \\ \downarrow & & \downarrow \\ M(b, a') \otimes_{\mathbb{K}} N(a') & \longrightarrow & (M \otimes_A N)(b) \end{array}$$

commutes. One can check that  $M \otimes_A N$  is again a monoidal functor from  $B$  to complexes. Thus  $M \otimes_A -$  defines a functor  $A - \text{mod} \rightarrow B - \text{mod}$ .

Let  $f : A \rightarrow B$  be a functor between dgs categories. Then  $B$  is a  $B - B$  bimodule, and so becomes an  $A - B$  bimodule and a  $B - A$  bimodule via the functors  $A \otimes B^{op} \rightarrow B \otimes B^{op}$  and  $B \otimes A^{op} \rightarrow B \otimes B^{op}$ . We can define functors  $f_* : A - \text{mod} \rightarrow B - \text{mod}$  by

$$f_* M = B \otimes_A M$$

and  $f^* : B - \text{mod} \rightarrow A - \text{mod}$  by defining  $f^* N$  to be  $N$  with the induced  $A$  action. So as a functor  $A \rightarrow \text{Comp}_{\mathbb{K}}$ ,  $f^* N$  is the composition of  $N : B \rightarrow \text{Comp}_{\mathbb{K}}$  with the functor  $f : A \rightarrow B$ . The push forward functor  $f_*$  is the left adjoint to  $f^*$ .

Note that  $f^*$  is exact. In general  $f_*$  is not exact. However, we can construct a derived version of  $f_*$  which is exact.

We say an  $A$  module  $M$  is flat if the functor  $- \otimes_A M$  from right  $A$  modules to complexes is exact. Let  $A - \text{flat}$  be the subcategory of flat  $A$  modules, and let  $i : A - \text{flat} \hookrightarrow A - \text{mod}$  be the inclusion.

**Definition 4.3.1.** Let  $\text{SymOb } A \subset A$  be the subcategory with the same objects, but whose morphisms are the identity maps and the symmetry isomorphisms  $a_1 \otimes \dots \otimes a_n \cong a_{\sigma(1)} \otimes \dots \otimes a_{\sigma(n)}$ , for  $\sigma \in S_n$ .  $\text{SymOb } A$  is again a symmetric monoidal category with a monoidal functor  $\text{SymOb } A \rightarrow A$ . Also  $\text{SymOb } A$  is a groupoid.

Let  $\text{SymOb}_{\mathbb{K}} A \subset A$  be the sub linear category whose morphisms are spanned by those of  $\text{SymOb } A$ .

**Theorem 4.3.2.** Suppose  $A$  is strictly monoidal,  $A$  has a unit for the tensor product, and suppose the groupoid  $\text{SymOb } A$  has finite automorphism groups for any object.

Then there is a functor  $F : A - \text{mod} \rightarrow A - \text{flat}$  such that  $F \circ i$  and  $i \circ F$  are quasi-isomorphic to the identity functors.

The conditions of the theorem hold for the dgsm categories controlling topological conformal field theory.

This result is false except in characteristic zero. We will assume the conditions of the theorem for the dgsm categories  $A, B$  we use for the rest of this section.

An  $A - B$  bimodule  $M$  is called  $A$ -flat if the functor  $- \otimes_A M$  is exact, as a functor from right  $A$  modules to right  $B$  modules. The proof of this result will also show that there exists functorial  $A$ -flat resolutions of  $A - B$  bimodules.

*Proof.* Let  $\text{Ord}$  be the simplicial category, whose objects are the non-empty totally ordered finite sets, and whose morphisms are non-decreasing maps. We will refer to the object  $\{1, \dots, n\}$  of  $\text{Ord}$  by  $\{n\}$ . For a category  $\mathcal{C}$ , a simplicial object of  $\mathcal{C}$  is a functor  $\text{Ord}^{\text{op}} \rightarrow \mathcal{C}$ . If  $M$  is a simplicial object of  $\mathcal{C}$ , we will write  $M\{n\} \in \text{Ob } \mathcal{C}$  for the  $n$  simplices of  $M$ .

For each  $A$  module  $M$ , define a simplicial  $A$  module  $\text{Bar}_A^\Delta M$  to have for  $n$  simplices the  $A$  module

$$\text{Bar}_A^\Delta M\{n\} = A \otimes_{\text{SymOb}_{\mathbb{K}} A} A \otimes \dots \otimes_{\text{SymOb}_{\mathbb{K}} A} A \otimes M = A^{\otimes_{\text{SymOb}_{\mathbb{K}} A} n} \otimes_{\text{SymOb}_{\mathbb{K}} A} M$$

The face maps come from the product maps of  $\text{SymOb}_{\mathbb{K}} A$  bimodules  $A \otimes_{\text{SymOb}_{\mathbb{K}} A} A \rightarrow A$ , and the map  $A \otimes_{\text{SymOb}_{\mathbb{K}} A} AM \rightarrow M$  of left  $\text{SymOb}_{\mathbb{K}} A$  modules. The degeneracy maps come from the map  $\text{SymOb}_{\mathbb{K}} A \rightarrow A$  of  $\text{SymOb}_{\mathbb{K}} A$  bimodules.

Denote by  $\text{Comp}_{\mathbb{K}}^\Delta$  the category of simplicial chain complexes, that is functors  $\text{Ord}^{\text{op}} \rightarrow \text{Comp}_{\mathbb{K}}$ . This is a symmetric monoidal category. The tensor product is pointwise; if  $C, D$  are simplicial chain complexes, then  $(C \otimes D)\{n\} = C\{n\} \otimes D\{n\}$ . A simplicial  $A$  module is the same as a symmetric monoidal functor  $A \rightarrow \text{Comp}_{\mathbb{K}}^\Delta$ .

The normalised realisation functor  $|\cdot| : \text{Comp}_{\mathbb{K}}^\Delta \rightarrow \text{Comp}_{\mathbb{K}}$  is defined by sending a simplicial chain complex  $C$  to

$$|C| = \bigoplus_{n>0} C\{n\} / C^{\text{degenerate}}\{n\}[-n]$$

Here  $C^{\text{degenerate}}\{n\}$  is the image of the degeneracy maps. The symbol  $[-n]$  refers to a shift in degree. The differential on  $|C|$  is composed of the differential on the summands  $C\{n\} / C^{\text{degenerate}}\{n\}$  and the alternating sum of the face maps.

A map  $C \rightarrow D$  of simplicial chain complexes is a quasi-isomorphism if the maps  $C\{n\} \rightarrow D\{n\}$  are quasi-isomorphisms. The realisation functor  $|\cdot|$  is exact, that is it takes quasi-isomorphisms to quasi-isomorphisms.

The shuffle product maps

$$|C| \otimes |D| \rightarrow |C \otimes D|$$

make  $|\cdot|$  into a symmetric monoidal functor.

Thus, in particular,

$$\text{Bar}_A M \stackrel{\text{def}}{=} \left| \text{Bar}_A^\Delta M \right|$$

is a symmetric monoidal functor  $A \rightarrow \text{Comp}_{\mathbb{K}}$ , in other words a left  $A$  module.

We can consider  $M$  as a constant simplicial  $A$  module. There is a natural map  $\text{Bar}_A^\Delta M \rightarrow M$ , which on  $n$  simplices comes from the product map  $A^{\otimes_{\text{SymOb}_{\mathbb{K}} A} n} \otimes_{\text{SymOb}_{\mathbb{K}} A} M \rightarrow M$ . This induces a map of realisations  $\text{Bar}_A M \rightarrow M$ .

**Lemma 4.3.3.** *The map  $\text{Bar}_A M \rightarrow M$  is a quasi-isomorphism.*

*Proof.* This is essentially standard.  $\square$

**Lemma 4.3.4.** *For any  $A$  module  $M$ ,  $\text{Bar}_A M$  is a flat  $A$  module.*

*Proof.* Let  $N, N'$  be right  $A$  modules, with a quasi-isomorphism  $N \rightarrow N'$ . We need to show that the map

$$N \otimes \text{Bar}_A M \rightarrow N' \otimes \text{Bar}_A M$$

is a quasi-isomorphism.

We can consider  $N, N'$  as constant simplicial right  $A$  modules, and form the tensor product simplicial chain complex  $N \otimes_A \text{Bar}_A^\Delta M$ . This has for  $n$  simplices

$$N \otimes_A \text{Bar}_A^\Delta M\{n\} = N \otimes_{\text{SymOb}_{\mathbb{K}} A} A^{\otimes_{\text{SymOb}_{\mathbb{K}} A} n-1} \otimes_{\text{SymOb}_{\mathbb{K}} A} M$$

It is easy to see that  $N \otimes \text{Bar}_A M$  is the realisation of this simplicial chain complex. Since the realisation functor is exact, it suffices to show that the map

$$N \otimes \text{Bar}_A^\Delta M \rightarrow N' \otimes \text{Bar}_A^\Delta M$$

of simplicial chain complexes is a quasi-isomorphism. To show this, it suffices to show that the map

$$N \otimes_{\text{SymOb}_{\mathbb{K}} A} A^{\otimes_{\text{SymOb}_{\mathbb{K}} A} n-1} \otimes_{\text{SymOb}_{\mathbb{K}} A} M \rightarrow N' \otimes_{\text{SymOb}_{\mathbb{K}} A} A^{\otimes_{\text{SymOb}_{\mathbb{K}} A} n-1} \otimes_{\text{SymOb}_{\mathbb{K}} A} M$$

is a quasi-isomorphism. More generally, if  $P$  is any left  $\text{SymOb}_{\mathbb{K}} A$  module, consider the map

$$N \otimes_{\text{SymOb}_{\mathbb{K}} A} P \rightarrow N' \otimes_{\text{SymOb}_{\mathbb{K}} A} P$$

This is always a quasi-isomorphism. As, tensor product over  $\text{SymOb}_{\mathbb{K}} A$  only involves taking coinvariants for finite group actions (using the assumption that all isomorphism groups in the groupoid  $\text{SymOb} A$  are finite). As we are working in characteristic zero, the functor of coinvariants for a finite group action is exact.  $\square$

This completes the proof of theorem 4.3.2. A similar argument shows that there exists functorial  $A$ -flat resolutions for  $A - B$  bimodules.  $\square$

If  $M$  is an  $A - B$  bimodule, and  $N$  is a left  $B$  module, define a left  $A$  module by

$$M \otimes_B^{\mathbb{L}} N = M \otimes_B \text{Bar}_B N$$

Any other flat resolution of  $N$  will give a quasi-isomorphic answer; as, suppose  $N', N''$  are flat resolutions of  $N$ , and  $M'$  is a  $B$  flat resolution of  $M$ . Then

$$M \otimes_B N' \simeq M' \otimes_B N' \simeq M' \otimes_B N'' \simeq M \otimes_B N''$$

**4.4. Derived push forwards.** Let  $f : A \rightarrow B$  be a functor, between dg symmetric monoidal categories. Let  $N$  be a left  $A$  module. Define

$$\mathbb{L}f_*N = B \otimes_A^{\mathbb{L}} N$$

Note that if we define  $\mathbb{L}'f_*N = B' \otimes_A N$ , where  $B'$  is an  $A$ -flat resolution of  $B$ , then  $\mathbb{L}'f_*$  and  $\mathbb{L}f_*$  are quasi-isomorphic functors. Also, if we took any other choice of functorial flat resolution of  $N$  we would get a quasi-isomorphic left derived functor.

Recall that  $f^*$  is defined by considering a left  $B$  module as a left  $A$  module.

**Lemma 4.4.1.** *Suppose that the homology functor  $H_*(f) : H_*(A) \rightarrow H_*(B)$  is fully faithful. Then the functor  $f^*\mathbb{L}f_*$  is quasi-isomorphic to the identity functor on  $A\text{-mod}$ .*

*Proof.* Note that

$$f^*\mathbb{L}f_*N = B \otimes_A \text{Bar}_A N$$

considered as a left  $A$  module. There is a map  $A \rightarrow B$  of  $A$  bimodules, and so a map

$$\text{Bar}_A N = A \otimes_A \text{Bar}_A N \rightarrow B \otimes_A \text{Bar}_A N$$

We need to show this is a quasi-isomorphism. This is clear, as  $\text{Bar}_A N$  is flat as an  $A$  module, and the statement that  $H_*(f)$  is full and faithful means that the map  $A \rightarrow B$  of  $A - A$  bimodules is a quasi-isomorphism.  $\square$

**Definition 4.4.2.** *A quasi-isomorphism between dgsm categories is a functor  $f : A \rightarrow B$  such that  $H_*(f)$  is full and faithful and  $f$  induces an isomorphism on the set of objects.*

**Theorem 4.4.3.** *If  $f : A \rightarrow B$  is a quasi-isomorphism, then the functors  $\mathbb{L}f_*$  and  $f^*$  are inverse quasi-equivalences between  $A\text{-mod}$  and  $B\text{-mod}$ , and between  $\text{mod}-A$  and  $\text{mod}-B$ .*

*Proof.* We have seen that the functor  $f^*\mathbb{L}f_*$  is quasi-isomorphic to the identity functor on  $A\text{-mod}$ . We need to show that  $\mathbb{L}f_*f^*$  is quasi-isomorphic to the identity functor on  $B\text{-mod}$ .

Note that

$$\mathbb{L}f_*f^*N = B \otimes_A^{\mathbb{L}} N = B \otimes_A^{\mathbb{L}} B \otimes_B N$$

Therefore it suffices to write down a weak equivalence  $B \otimes_A^{\mathbb{L}} B \rightarrow B$  of  $B - B$  bialgebras.

The  $B - B$  bialgebra  $B \otimes_A^{\mathbb{L}} B$  is the realisation of the simplicial  $B - B$  bialgebra  $B \otimes_A \text{Bar}_A^{\Delta} B$ , using the notation of the proof of theorem 4.3.2.

By assumption, the functor  $A \rightarrow B$  induces an isomorphism on the set of objects. Thus the categories  $\text{SymOb}_{\mathbb{K}} A$  and  $\text{SymOb}_{\mathbb{K}} B$  are isomorphic; let us use the notation

$$C \stackrel{\text{def}}{=} \text{SymOb}_{\mathbb{K}} A = \text{SymOb}_{\mathbb{K}} B$$

The  $n$  simplices of  $B \otimes_A \text{Bar}_A^{\Delta} B$

$$B \otimes_C A^{\otimes_C n-1} \otimes_C B$$

The map  $A \rightarrow B$  of  $C$  bimodules induces a map

$$B \otimes_C A^{\otimes_C n-1} \otimes_C B \rightarrow B \otimes_C B^{\otimes_C n-1} \otimes_C B$$

which is a quasi-isomorphism, because tensor product of  $C$  bimodules is an exact functor. It is easy to see that this map is the  $n$ th component of a quasi-isomorphism of simplicial chain complexes

$$B \otimes_A \text{Bar}_A^\Delta B \rightarrow B \otimes_B \text{Bar}_B^\Delta B$$

The associated map on realisations is a quasi-isomorphism. There is a natural quasi-isomorphism of  $B$ - $B$  bimodules  $B \otimes_B \text{Bar}_B B \rightarrow B$ . Putting these quasi-isomorphisms together we get a quasi-isomorphism  $B \otimes_A^\mathbb{L} B \rightarrow B$ .  $\square$

**Lemma 4.4.4.** *Denote also by  $f^*$  and  $\mathbb{L}f_*$  the induced quasi-equivalences  $\text{mod } -A \times A - \text{mod} \xrightarrow{\simeq} \text{mod } -B \times B - \text{mod}$ . Both triangles in the following diagram commute up to quasi-isomorphism.*

$$\begin{array}{ccc}
 \text{mod } -A \times A - \text{mod} & & \\
 \uparrow f^* & \searrow^{\otimes^{\mathbb{L}}} & \\
 \text{mod } -B \times B - \text{mod} & \xrightarrow{\mathbb{L}f_*} & \text{Comp}_{\mathbb{K}} \\
 & \nearrow_{\otimes^{\mathbb{L}}} & 
 \end{array}$$

The diagonal arrows are the tensor product maps which take a pair  $(M, N)$  where  $M \in \text{mod } -A$  and  $N \in A - \text{mod}$  to  $M \otimes_A^\mathbb{L} N$ .

*Proof.* It is sufficient to show that one of the triangles commutes up to quasi-isomorphism. So we need to show that

$$\mathbb{L}f_*(M) \otimes_B^\mathbb{L} \mathbb{L}f_*(N) \simeq M \otimes_A^\mathbb{L} N$$

This follows from the chain of quasi-isomorphisms

$$\begin{aligned}
 \mathbb{L}f_* M \otimes_B^\mathbb{L} \mathbb{L}f_* N &\simeq M \otimes_A^\mathbb{L} B \otimes_B^\mathbb{L} B \otimes_A^\mathbb{L} N \\
 &\simeq M \otimes_A^\mathbb{L} (B \otimes_A^\mathbb{L} N) \\
 &\simeq M \otimes_A^\mathbb{L} N
 \end{aligned}$$

where the last quasi-isomorphism comes from the quasi-isomorphism  $N \simeq f^* \mathbb{L}f_* N = B \otimes_A^\mathbb{L} N$  as an  $A$ -module.  $\square$

## 5. TOPOLOGICAL CONFORMAL FIELD THEORIES

A symmetric monoidal functor  $F : A \rightarrow B$  between dgsm categories is called *split* if the maps  $F(a) \otimes F(a') \rightarrow F(a \otimes a')$  are all isomorphisms. This is what MacLane [ML98] calls strong.  $F$  is called h-split, or homologically split, if  $H_*(F) : H_*(A) \rightarrow H_*(B)$  is split. Note that being h-split is an exact condition : if  $F \simeq F'$  then  $F$  is h-split if and only if  $F'$  is.

**Definition 5.0.5.** (1) An open topological conformal field theory of dimension  $d$  is a pair  $(\Lambda, \Phi)$  where  $\Lambda$  is a set of  $D$ -branes, and  $\Phi \in \mathcal{O}_\Lambda^d - \text{mod}$  is a symmetric monoidal functor

$$\Phi : \mathcal{O}_\Lambda^d \rightarrow \text{Comp}_{\mathbb{K}}$$

which is  $h$ -split.

A morphism of open TCFTs  $(\Lambda, \Phi) \rightarrow (\Lambda', \Phi')$  is a map  $\Lambda \rightarrow \Lambda'$  of sets, and a morphism  $\Phi \rightarrow f^*\Phi'$  in  $\mathcal{O}_\Lambda^d - \text{mod}$ . Here  $f : \mathcal{O}_\Lambda^d \rightarrow \mathcal{O}_{\Lambda'}^d$  is the functor induced by the map of sets  $\Lambda \rightarrow \Lambda'$ .

(2) A closed topological conformal field theory of dimension  $d$  is a  $h$ -split symmetric monoidal functor  $\mathcal{C}^d \rightarrow \text{Comp}_{\mathbb{K}}$ . A morphism of closed TCFTs is a morphism in  $\mathcal{C}^d - \text{mod}$ .

(3) An open-closed topological conformal field theory of dimension  $d$  is a pair  $(\Lambda, \Phi)$  where  $\Lambda$  is a set of  $D$ -branes and  $\Phi$  is a symmetric monoidal functor

$$\Phi : \mathcal{OC}_\Lambda^d \rightarrow \text{Comp}_{\mathbb{K}}$$

which is  $h$ -split.

A morphism of open-closed TCFTs  $(\Lambda, \Phi) \rightarrow (\Lambda', \Phi')$  is a map  $\Lambda \rightarrow \Lambda'$  of sets, and a morphism  $\Phi \rightarrow f^*\Phi'$  in  $\mathcal{OC}_\Lambda^d - \text{mod}$ . Here  $f : \mathcal{OC}_\Lambda^d \rightarrow \mathcal{OC}_{\Lambda'}^d$  is the functor induced by the map of sets  $\Lambda \rightarrow \Lambda'$ .

The condition that the functors are  $h$ -split is important. For example, if  $\Psi$  is a closed TCFT, then this means that

$$H_*(\Psi(C)) = H_*(\Psi(1))^{\otimes C}$$

where  $C$  is the number of closed boundaries. Thus, if  $\Psi$  is a closed TCFT we can talk about its homology, which is just a graded vector space; we will use the notation  $H_*(\Psi)$  for  $H_*(\Psi(1))$ . Then  $H_*(\Psi)$  carries operations from the homology of moduli spaces of curves. That is, there are maps

$$H_*(\mathcal{C}^d(I, J)) \rightarrow \text{Hom}(H_*(\Psi)^{\otimes I}, H_*(\Psi)^{\otimes J})$$

A pair  $\lambda_1, \lambda_2$  of  $D$ -branes gives an object  $\{\lambda_1, \lambda_2\}$  of  $\mathcal{O}_\Lambda^d$ , corresponding to one open boundary from  $\lambda_1$  to  $\lambda_2$ . For an open TCFT  $(\Lambda, \Phi)$  we have a space  $H_*(\Phi(\{\lambda_1, \lambda_2\}))$ . Any other object of  $\mathcal{O}_\Lambda^d$  can be written as a union of objects of the form  $\{\lambda_1, \lambda_2\}$ . Since  $\Phi$  is  $h$ -split, for any object  $(O, s, t)$  of  $\mathcal{O}_\Lambda^d$ , where  $O$  is a non-negative integer, and  $s, t : O \rightarrow \Lambda$  are maps,

$$H_*(\Phi(O, s, t)) = \otimes_{i=0}^{O-1} H_*(\Phi(\{s(i), t(i)\}))$$

Let  $i : \mathcal{O}_\Lambda^d \rightarrow \mathcal{OC}_\Lambda^d$  and  $j : \mathcal{C}^d \rightarrow \mathcal{OC}_\Lambda^d$  denote the natural functors. If  $\Phi$  is an open-closed TCFT, then  $j^*\Phi$  is a closed TCFT and  $i^*\Phi$  is an open TCFT.

Recall the objects of  $\mathcal{OC}_\Lambda^d$  are of the form  $(C, O, s, t)$  where  $C, O$  are integers and  $s, t : O \rightarrow \Lambda$  are maps. If  $(\Lambda, \Phi)$  is an open-closed TCFT, then  $H_*(j^*\Phi)$  is the homology of the associated closed TCFT, or equivalently the homology of  $\Phi$  applied to the object where  $C = 1$  and  $O = 0$ . Then,

$$H_*(\Phi(C, O, s, t)) = \otimes_{o=0}^{O-1} H_*(\Phi(\{s(o), t(o)\})) \otimes H_*(j^*\Phi)^{\otimes C}$$

Now we can state the main results of this paper.

- Theorem A.** (1) *The category of open TCFTs is quasi-equivalent to the category of (unital) Calabi-Yau extended  $A_\infty$  categories.*
- (2) *Given any open TCFT,  $(\Lambda, \Phi)$ , we can push forward the functor  $\Phi : \mathcal{O}_\Lambda^d \rightarrow \text{Comp}_{\mathbb{K}}$  to  $\mathbb{L}i_*\Phi : \mathcal{O}\mathcal{C}_\Lambda^d \rightarrow \text{Comp}_{\mathbb{K}}$ . This functor is again h-split, so that  $(\Lambda, \mathbb{L}i_*\Phi)$  is an open-closed TCFT. This is the homotopy universal TCFT.*
- (3) *We have a natural isomorphism*

$$H_*(j^*\mathbb{L}i_*\Phi) \cong HH_*(A)$$

where  $A$  is the  $A_\infty$  category corresponding to  $(\Lambda, \Phi)$ , and  $HH_*(A)$  is the Hochschild homology group.

The notion of unital Calabi-Yau extended  $A_\infty$  category will be explained later.

## 6. COMBINATORIAL MODELS FOR CATEGORIES CONTROLLING OPEN-CLOSED TOPOLOGICAL CONFORMAL FIELD THEORY

In this section, an explicit dgsm category  $\mathcal{D}_{\Lambda, \text{open}}^d$  is constructed which is quasi-isomorphic to  $\mathcal{O}_\Lambda^d$ . This uses the cellular models for moduli spaces which I introduced in [Cos04], and which are discussed in detail in [Cos06].

The categories of modules for  $\mathcal{D}_{\Lambda, \text{open}}^d$  and  $\mathcal{O}_\Lambda^d$  are quasi-equivalent. We have an  $\text{SymOb } \mathcal{O}\mathcal{C}_\Lambda - \mathcal{O}_\Lambda^d$  bimodule,  $\mathcal{O}\mathcal{C}_\Lambda^d$ . An explicit model  $\mathcal{D}_\Lambda^d$  for the corresponding  $\text{SymOb } \mathcal{O}\mathcal{C}_\Lambda - \mathcal{D}_{\Lambda, \text{open}}^d$  bimodule is constructed.

These results are enough to prove Theorem A. We will show later that a h-split  $\mathcal{D}_{\Lambda, \text{open}}^d$  module is a unital extended Calabi-Yau  $A_\infty$  category with set of objects  $\Lambda$ . For each such, say  $\Phi$ , we will calculate  $H_*(\mathcal{D}_\Lambda^d \otimes_{\mathcal{D}_{\Lambda, \text{open}}^d} \Phi)$ , and find it is a tensor product of Hochschild homology groups of  $\Phi$  and homology of morphism complexes of  $\Phi$ . This will show that for any open TCFT, the corresponding functor  $\mathcal{O}\mathcal{C}_\Lambda^d \rightarrow \text{Comp}_{\mathbb{K}}$  is h-split and has for homology of the closed states the Hochschild homology of the corresponding  $A_\infty$  category.

We do this by constructing cellular models for certain of our moduli spaces of Riemann surfaces with open closed boundary. Let  $\alpha, \beta \in \text{Ob } \mathcal{M}_\Lambda$  be such that  $\alpha$  has no closed part (so  $\alpha = (0, O, s, t)$ ). We will construct combinatorial models for the spaces  $\mathcal{M}_\Lambda(\alpha, \beta)$ . The cell complex  $\mathcal{G}(\alpha, \beta)$  we will construct will live in a moduli space of Riemann surface with nodes along the boundary; the surfaces in  $\mathcal{G}(\alpha, \beta)$  will be those which are built up from discs and annuli.

**6.1. A cellular model for moduli space.** The first step is to describe the moduli space of Riemann surfaces with possibly nodal boundary. Let  $\alpha, \beta \in \text{Ob } \mathcal{M}_\Lambda$  as before, and assume  $\alpha$  has no closed part. Write  $C(\beta), O(\beta), O(\alpha)$  for the closed and open boundaries in  $\alpha$  and  $\beta$ . Note that  $C(\alpha) = 0$ .

To keep the notation simple, I will omit the references to the category of D-branes  $\Lambda$ , so that  $\mathcal{M}$  will be synonymous with  $\mathcal{M}_\Lambda$ .

**Definition 6.1.1.** *Let  $\overline{\mathcal{N}}(\alpha, \beta)$  be the moduli space of Riemann surfaces  $\Sigma$  with boundary, with (outgoing) closed boundary components labelled by  $0, \dots, C(\beta) - 1$ . These boundary components each have exactly one marked point on them (this replaces the*

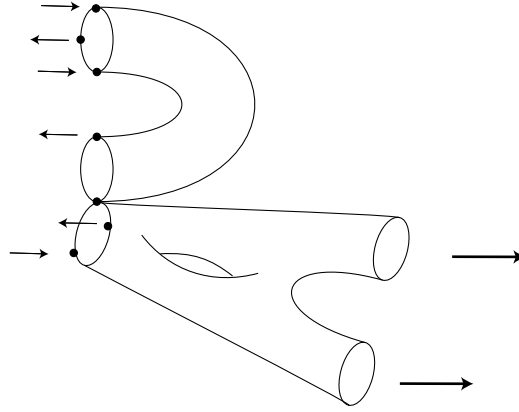


FIGURE 4. A surface in  $\overline{\mathcal{N}}(l)$ . The dots represent incoming or outgoing open boundaries. The boundaries with no dots are closed and outgoing.

parameterisation on the boundary components of the surfaces in  $\mathcal{M}$ ). There are further marked points labelled by  $O(\alpha)$  and  $O(\beta)$  distributed along the remaining boundary components of  $\Sigma$ ; these correspond to the open boundaries. The free boundaries are the intervals which lie between open boundaries, and the boundary components with no marked points on them; these are labelled by  $D$ -branes in  $\Lambda$  in a way compatible with the maps  $s, t : O(\alpha) \rightarrow \Lambda$  and  $s, t : O(\beta) \rightarrow \Lambda$ . Each connected component of the surface must have at least one free boundary.

The surface  $\Sigma$  may have nodes along the boundary, as in [Liu02], [Cos04, Cos06]. However, unlike in Liu's work, there can be no nodes on the interior of  $\Sigma$ , nor are there marked points on the interior. Marked points are not allowed to collide with nodes. Each closed boundary component of  $\Sigma$  must be smooth (that is contain no nodes). Another difference from Liu's work is that each boundary must be of positive length; boundaries cannot shrink to punctures. The surface  $\Sigma$  must be stable, that is have a finite automorphism group. This corresponds to the requirement that no irreducible component of  $\Sigma$  can be a disc with  $\leq 2$  open marked points.

There are four exceptional kinds of surface; we allow surfaces with connected components of this form. The disc with zero, one or two open marked points and the annulus with no open or closed marked points are unstable; we declare the moduli space of any of these types of surfaces to be a point.

It is important to put in these exceptional cases. Part of  $\overline{\mathcal{N}}$  will be made into a category, and the disc with one incoming and one outgoing open point will be the identity. The disc with one open point will give the unit in an  $A_\infty$  category.

The moduli spaces  $\overline{\mathcal{N}}$  are orbifolds with corners. This follows from the work of Liu [Liu02]. One can see this by comparing the moduli spaces  $\overline{\mathcal{N}}$  to the real points of the Deligne-Mumford moduli spaces of curves. The interiors,  $\mathcal{N}$ , are therefore smooth orbifolds. The spaces  $\mathcal{N}$  parameterise non-singular surfaces in  $\overline{\mathcal{N}}$ . The inclusion

$$\mathcal{N} \hookrightarrow \overline{\mathcal{N}}$$

is a  $\mathbb{Q}$  homotopy equivalence.

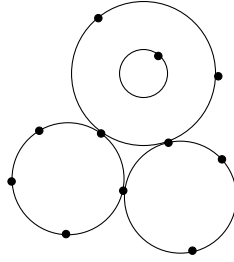


FIGURE 5. A surface in  $\mathcal{G}$ , with 7 open boundaries and one closed boundary. The inside of the annulus is the outgoing closed boundary component, the open boundaries may be incoming or outgoing.

The next step is to write down a subspace of the boundary of  $\overline{\mathcal{N}}$  which is  $\mathbb{Q}$  homotopy equivalent to  $\overline{\mathcal{N}}$ . Recall that the space of (isomorphism classes of) annuli can be identified with  $\mathbb{R}_{>0}$ . Every annulus is isomorphic to an annulus of the form  $\{z \mid 1 < |z| < 1 + R\}$  for some unique  $R \in \mathbb{R}_{>0}$ , which we call the modulus of the annulus.

**Definition 6.1.2.** Define  $\mathcal{G}(\alpha, \beta) \subset \overline{\mathcal{N}}(\alpha, \beta)$  to be the subspace consisting of surfaces  $\Sigma \in \overline{\mathcal{N}}(\alpha, \beta)$ , each of whose irreducible components is either a disc, or an annulus of modulus 1. We require that one side of each annulus is an outgoing boundary component. Recall that in  $\overline{\mathcal{N}}(\alpha, \beta)$  the outgoing closed boundary components are required to be smooth; this implies that the annuli are in one to one correspondence with the outgoing closed boundary components  $C(\beta)$ .

$\mathcal{G}(\alpha, \beta)$  also contains the exceptional surfaces; we allow surfaces with connected components which are discs with  $\leq 2$  marked points or annuli with no open or closed marked points.

**Proposition 6.1.3.** The inclusion  $\mathcal{G}(\alpha, \beta) \hookrightarrow \overline{\mathcal{N}}(\alpha, \beta)$  is a weak homotopy equivalence of orbispaces (and therefore a  $\mathbb{Q}$  homotopy equivalence of coarse moduli spaces).

*Proof.* This follows immediately from the results of [Cos04, Cos06]. For integers  $g, h, r, s$  with  $g, r, s \geq 0$ ,  $h > 0$ , define the orbi-space  $\overline{\mathcal{N}}_{g,h,r,s}$  to be the moduli space of stable Riemann surfaces with possibly nodal boundary as above, with  $r$  boundary (open) marked points and  $s$  internal marked points, of genus  $g$  with  $h$  boundary components. As I discuss in detail in [Cos06], we have an orbi-cell complex  $D_{g,h,r,s} \subset \overline{\mathcal{N}}_{g,h,r,s}$  consisting of Riemann surfaces built up from discs, each of which has at most one internal marked points. The inclusion  $D_{g,h,r,s} \hookrightarrow \overline{\mathcal{N}}_{g,h,r,s}$  is a weak homotopy equivalence.

We can replace the  $s$  internal marked points by unparameterised boundary components, in the moduli spaces  $D_{g,h,r,s}$  and  $\overline{\mathcal{N}}_{g,h,r,s}$ . Evidently, all the corresponding moduli spaces are homotopy equivalent, so the inclusion of these new spaces is also a homotopy equivalence. We can also add on to each of these  $s$  boundary components a marked point, and the result continues to hold, as we are simply passing to the total space of a torus bundle.

It follows immediately that the inclusion  $\mathcal{G}(\alpha, \beta) \hookrightarrow \overline{\mathcal{N}}(\alpha, \beta)$  is a weak homotopy equivalence of orbispaces. □

Suppose  $\alpha, \beta$  both satisfy  $C(\alpha) = C(\beta) = 0$ . Then there are gluing maps

$$\overline{\mathcal{N}}(\alpha, \beta) \times \overline{\mathcal{N}}(\beta, \gamma) \rightarrow \overline{\mathcal{N}}(\alpha, \gamma)$$

These maps glue the outgoing open marked points of a surface in  $\overline{\mathcal{N}}(\alpha, \beta)$  to the corresponding incoming marked points of a surface in  $\overline{\mathcal{N}}(\beta, \gamma)$ . We need to describe how to glue the exceptional surfaces; the discs with one or two marked points. Gluing the disc with two open marked points, one incoming and one outgoing, is the identity operation. Gluing the disc with two outgoing marked points onto two incoming marked points of a surface  $\Sigma$  corresponds to gluing the two marked points of  $\Sigma$  together; similarly for the disc with two incoming. Gluing the disc with one marked point onto a marked point of a surface  $\Sigma$  causes us to forget that marked point.

**Lemma 6.1.4.** *There is a category whose objects are the objects  $\alpha$  of  $\mathcal{M}_\Lambda$  with  $C(\alpha) = 0$  (i.e. no incoming closed boundaries), whose morphisms are the spaces  $\overline{\mathcal{N}}(\alpha, \beta)$  and whose composition maps are the gluing described above.*

Recall that  $\mathcal{M}_\Lambda$  is the topological version of  $\mathcal{OC}_\Lambda$ ;  $\mathcal{OC}_\Lambda$  is chains on  $\mathcal{M}_\Lambda$ . We defined  $\text{Ob } \mathcal{M}_\Lambda$  to be the symmetric monoidal category with the same objects as  $\mathcal{M}_\Lambda$  but whose morphisms are the symmetry maps  $a_1 \otimes \dots \otimes a_n \cong a_{\sigma(1)} \otimes \dots \otimes a_{\sigma(n)}$ , for  $\sigma \in S_n$ .

This lemma is clear. Call this category  $\overline{\mathcal{N}}_{open}$ . The spaces  $\mathcal{G}(\alpha, \beta) \subset \overline{\mathcal{N}}(\alpha, \beta)$  define a subcategory  $\mathcal{G}_{open} \subset \overline{\mathcal{N}}_{open}$ . There is also the structure of symmetric monoidal category on  $\mathcal{G}_{open}$  and  $\overline{\mathcal{N}}_{open}$  given by disjoint union.

$\overline{\mathcal{N}}$  defines a monoidal functor  $\text{Ob } \mathcal{M}_\Lambda \times \overline{\mathcal{N}}_{open}^{op} \rightarrow \text{Top}$ , given by  $(\beta, \alpha) \mapsto \overline{\mathcal{N}}(\alpha, \beta)$ . Similarly  $\mathcal{G}$  defines a functor  $\text{Ob } \mathcal{M}_\Lambda \times \mathcal{G}_{open}^{op} \rightarrow \text{Top}$ .

Let us take chain complexes  $C_*(\overline{\mathcal{N}}, \det^d)$  where  $\det^d$  is the local system defined before;  $C_*(\overline{\mathcal{N}}_{open}, \det^d)$  is a differential graded symmetric monoidal category, and  $C_*(\overline{\mathcal{N}}, \det^d)$  defines an  $\text{Ob } \mathcal{OC}^d - C_*(\overline{\mathcal{N}}_{open}, \det^d)$  bimodule.

**Proposition 6.1.5.** *The dgsm category  $C_*(\overline{\mathcal{N}}_{open}, \det^d)$  is quasi-isomorphic to the dgsm category  $\mathcal{O}^d$ .*

*Under the induced quasi-equivalence of categories between  $\text{Ob } \mathcal{OC}^d - C_*(\overline{\mathcal{N}}_{open}, \det^d)$  bimodules and  $\text{Ob } \mathcal{OC}^d - \mathcal{O}^d$  bimodules,  $C_*(\overline{\mathcal{N}}, \det^d)$  corresponds to  $\mathcal{OC}^d$ .*

We are suppressing the set  $\Lambda$  of D-branes from the notation here.

*Proof.* I will sketch the proof of the statement about categories, in the case  $d = 0$ ; the remaining statements are proved in a similar way. We will do the topological version, and find a topological category  $\widetilde{\mathcal{M}}_{open}$ , with the same objects as  $\mathcal{M}_{open}$ , and with functors  $\overline{\mathcal{N}}_{open} \rightarrow \widetilde{\mathcal{M}}_{open} \leftarrow \mathcal{M}_{open}$  which are  $\mathbb{Q}$  homotopy equivalences on the spaces of morphisms.

For  $\alpha, \beta \in \text{Ob } \mathcal{M}$ , let  $\widetilde{\mathcal{M}}_{open}(\alpha, \beta)$  be the moduli space of surfaces with nodal boundary, as in  $\overline{\mathcal{N}}(\alpha, \beta)$ , but now the open boundaries are parameterised embedded intervals, like in  $\mathcal{M}(\alpha, \beta)$ . These intervals do not intersect the nodes or each other. Each outgoing open boundary has a number  $t \in [0, 1/2]$  attached to it.

The gluing which defines the maps  $\widetilde{\mathcal{M}}_{open}(\alpha, \beta) \times \widetilde{\mathcal{M}}_{open}(\beta, \gamma) \rightarrow \widetilde{\mathcal{M}}_{open}(\alpha, \gamma)$  is defined as follows. Let  $\Sigma_1 \in \widetilde{\mathcal{M}}_{open}(\alpha, \beta)$  and  $\Sigma_2 \in \widetilde{\mathcal{M}}_{open}(\beta, \gamma)$ , and let  $o \in O(\beta)$ .

This corresponds to an open boundary on each of the  $\Sigma_i$ . Let  $t \in [0, 1/2]$  be number corresponding to  $o$ . Glue the subinterval  $[t, 1 - t] \subset [0, 1]$  of the boundary on  $\Sigma_1$  to the corresponding subinterval  $[t, 1 - t]$  of the corresponding boundary on  $\Sigma_2$ .

This evidently makes  $\widetilde{\mathcal{M}}_{open}$  into a category. The map  $\mathcal{M}_{open} \hookrightarrow \widetilde{\mathcal{M}}_{open}$  assigns the number 0 to the open boundaries, and is a homotopy equivalence on spaces of morphisms. Similarly, the map  $\overline{\mathcal{N}}_{open} \hookrightarrow \widetilde{\mathcal{M}}_{open}$  assigns the number 1/2 to open boundaries, and is a homotopy equivalence on the space of morphisms.

This argument implies the corresponding result at chain level, and extends without difficulty to the case of twisted coefficients and to yield an equivalence of modules.  $\square$

We want to give an orbi-cell decomposition of the spaces  $\mathcal{G}$ . We will do this by writing down a stratification of  $\mathcal{G}$  whose strata are orbi-cells, that is the quotient of a cell by a finite group. There is an obvious stratification of  $\mathcal{G}$ , given by the topological isomorphism type of the corresponding marked nodal surface. This is not quite a cell decomposition, as the moduli space of marked points on the boundary of an annulus, one of whose boundaries is closed, is not contractible, but is homotopic to  $S^1$ . We need to refine this stratification a little.

Let  $\Sigma \in \mathcal{G}(\alpha, \beta)$ . Let us assume for simplicity that no connected component of  $\Sigma$  is an exceptional (unstable) surface. We will give  $\Sigma$  a cell decomposition. Let  $A \subset \Sigma$  be an irreducible component which is an annulus with a closed boundary. In order to get a cell decomposition on  $\Sigma$ , we have to make a cut on the annulus. Let  $A_{closed}, A_{open}$  be the boundary components of  $A$ ; where  $A_{closed}$  has precisely one marked point,  $p$  say, corresponding to an outgoing closed boundary of  $\Sigma$ , and  $A_{open}$  may have several incoming and outgoing open marked points and possibly some nodes. There is a unique holomorphic isomorphism from  $A$  to the cylinder  $S^1 \times [0, 1]$ , such that  $p \in A_{closed}$  goes to  $(1, 0) \in S^1 \times [0, 1]$ . The inverse image of  $1 \times [0, 1]$  in this gives a cut on the annulus, starting at  $p \in A_{closed}$  and ending at some point  $p'$  on  $A_{open}$ . Now give  $\Sigma$  a cell decomposition, by declaring that the 0 skeleton consists of the nodes, marked points, and the places where the cut on an annulus intersect the boundary of the annulus; the one cells are  $\partial\Sigma$ , together with the cuts on the annuli; and the 2 skeleton is  $\Sigma$ . The two cells of  $\Sigma$  are oriented, and  $\Sigma$  is marked by D-branes, incoming/outgoing open marked points, and closed outgoing marked points.

Give  $\mathcal{G}(\alpha, \beta)$  a stratification by saying that two surface  $\Sigma_1, \Sigma_2$  are in the same stratum if and only if the corresponding marked, oriented 2-cell complexes in  $\mathcal{A}(\alpha, \beta)$  are isomorphic.

**Lemma 6.1.6.** *This stratification of  $\mathcal{G}(\alpha, \beta)$  is an orbi-cell decomposition, and further the composition maps  $\mathcal{G}(\alpha, \beta) \times \mathcal{G}(\beta, \gamma) \rightarrow \mathcal{G}(\alpha, \gamma)$  are cellular.*

To show that this stratification is an orbi-cell decomposition, the main point to observe is that the stratification of the space of marked points on the annulus is indeed a cell decomposition.

We are using a strong notion of cellular map : a map  $f : X \rightarrow Y$  between (orbi)-cell complexes is cellular if  $f^{-1}Y_i = X_i$ , where  $X_i$  is the union of cells of dimension  $\leq i$ .

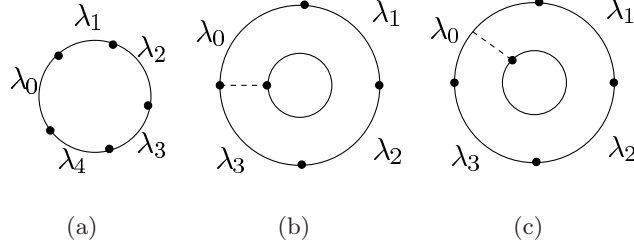


FIGURE 6

These are the three basic types of cell in the moduli spaces  $\mathcal{G}$ , from which all others are built by open gluing.

Figure 6(a) represents the cell in moduli space of points moving on a disc. The marked points are open boundaries, and may be incoming or outgoing; the  $\lambda_i$  are D-brane labellings on free boundaries.

Figures 6(b) and 6(c) are the two kinds of cell in the space of marked points on the annulus. The interior of the annulus is a closed outgoing boundary; the marked point on this represents the start of the closed boundary. The remaining marked points are open, incoming or outgoing.

In figure 6(b) the closed marked point is parallel to an open one, where as in figure 6(c), the closed marked point is parallel to the interior of a free boundary.

Define

$$\mathcal{D}(\alpha, \beta) = C_*^{cell}(\mathcal{G}(\alpha, \beta)) \otimes \mathbb{K}$$

to be the associated complex of  $\mathbb{K}$  cellular chains. Similarly, for an integer  $d \geq 0$ , define

$$\mathcal{D}^d(\alpha, \beta) = C_*^{cell}(\mathcal{G}(\alpha, \beta), \det^d) \otimes \mathbb{K}$$

Here we take cellular chains with local coefficients.

Let us describe informally the chain complexes  $\mathcal{D}(\alpha, \beta)$ . Each Riemann surface in  $\mathcal{G}(\alpha, \beta)$  determines a cell in the moduli space, and so an element of the cellular chain group  $\mathcal{G}(\alpha, \beta)$ . Thus, we can think of a chain in  $\mathcal{D}(\alpha, \beta)$  as being represented by a surface, and similarly for  $\mathcal{D}^d(\alpha, \beta)$ . The boundary maps in  $\mathcal{D}^d(\alpha, \beta)$  correspond to degenerating surfaces to allow more nodes, and also allowing a closed marked point, on the boundary of an annulus, to become parallel to an open marked point or node on the other boundary of an annulus.

There are composition maps  $\mathcal{D}^d(\alpha, \beta) \otimes \mathcal{D}^d(\beta, \gamma) \rightarrow \mathcal{D}^d(\alpha, \gamma)$ , which make  $\mathcal{D}_{open}^d$  (the part where  $\alpha, \beta$  have only open boundaries) into a differential graded symmetric monoidal category, and  $\mathcal{D}^d$  into a  $\text{Ob } \mathcal{O}\mathcal{C}^d - \mathcal{D}_{open}^d$  bimodule.

**Lemma 6.1.7.** *The differential graded symmetric monoidal categories  $\mathcal{D}_{open}^d$  and  $\mathcal{O}^d$  are quasi-isomorphic. Under the induced quasi-equivalence of categories between  $\text{Ob } \mathcal{O}^d - \mathcal{D}_{open}^d$  bimodules and  $\text{Ob } \mathcal{O}^d - \mathcal{O}^d$  bimodules,  $\mathcal{D}^d$  corresponds to  $\mathcal{O}^d$ .*

The point is that the chain complex functor  $C_*$  constructed in the appendix has the property that for each (orbi)-cell complex  $X$ , there is a quasi-isomorphism  $C_*^{cell}(X) \rightarrow C_*(X)$ , compatible with products and natural for cellular maps. The same holds when we take chains with local coefficients. This shows that the functor  $\mathcal{D}_{open}^d \rightarrow C_*(\mathcal{G}_{open}, \det^d)$  is a quasi-isomorphism, and we have already seen that  $C_*(\mathcal{G}, \det^d) \simeq C_*(\overline{\mathcal{N}}_{open}, \det^d) \simeq \mathcal{O}^d$ . Similar remarks prove the statement about  $\mathcal{D}^d$  as an  $\text{Ob } \mathcal{O}^d - \mathcal{D}_{open}^d$  bimodule.

**6.2. Generators and relations for  $\mathcal{D}_{open}^d$ .** If  $\lambda_0, \dots, \lambda_n$  is an ordered set of D-branes, let  $\{\lambda_0, \dots, \lambda_n\}$  be the object in  $\text{Ob } \mathcal{D}_{open}^d$  with  $O = n$  and  $s(i) = \lambda_i$ ,  $t(i) = \lambda_{i+1}$  for  $0 \leq i \leq O - 1$ .

Use the notation

$$\{\lambda_0, \dots, \lambda_{n-1}\}^c = \{\lambda_0, \dots, \lambda_{n-1}, \lambda_0\}$$

The superscript  $c$  stands for cyclic.

Define an element  $D(\lambda_0, \dots, \lambda_{n-1})$  of  $\mathcal{D}_{open}^d(\{\lambda_0, \dots, \lambda_{n-1}\}^c, 0)$ , given by the cellular chain which is the disc with  $n$  marked points on it, all incoming, with the cyclic order  $0, 1, \dots, n - 1$ , labelled in the obvious way by D-branes; as in figure 6(a). (Pick, arbitrarily, some orientation on this cell, and a section of  $\det^d$ , in order to get a cellular chain).

Note that  $D(\lambda_0, \dots, \lambda_{n-1})$  is cyclically symmetric up to sign; so that

$$D(\lambda_0, \dots, \lambda_{n-1}) = \pm D(\lambda_1, \dots, \lambda_{n-1}, \lambda_0)$$

under the permutation isomorphism between  $\{\lambda_0, \dots, \lambda_{n-1}\}^c$  and  $\{\lambda_1, \dots, \lambda_{n-1}, \lambda_0\}^c$ .

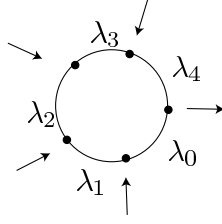
When  $n \geq 3$ ,  $D(\lambda_0, \dots, \lambda_{n-1})$  is an element of degree  $n - 3 + d$ . When  $n = 1, 2$  it is an element of degree  $d$ .

Let  $C \subset \mathcal{D}_{open}^d$  be the subcategory with the same objects, but whose morphism surfaces are not allowed to have connected components which are the disc with  $\leq 1$  open marked points; or the disc with two open marked points, both incoming; or the annulus with neither open or closed marked points. We consider the morphisms in  $C$  not to be complexes, but to be graded vector spaces : we forget the differential.

**Proposition 6.2.1.**  *$C$  is freely generated, as a symmetric monoidal category over the symmetric monoidal category  $\text{Ob } \mathcal{D}_{open}^d$ , by the discs  $D(\lambda_0, \dots, \lambda_{n-1})$ , where  $n \geq 3$ , and the discs with two outgoing marked points, subject to the relation that  $D(\lambda_0, \dots, \lambda_{n-1})$  is cyclically symmetric (up to an appropriate sign).*

The sign in the cyclic symmetry is determined by the choice of orientation on the cell in  $\mathcal{G}$  corresponding to  $D(\lambda_0, \dots, \lambda_{n-1})$ .

Note that it makes sense to talk about generators and relations for a symmetric monoidal category; this is because we have fixed the base category  $\text{Ob } \mathcal{D}_{open}^d$ , and the new symmetric monoidal category we are constructing has the same set of objects. The morphism spaces of a symmetric monoidal category given by generators and relations

FIGURE 7. The chain  $D^+(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ .

will be built up using composition, tensor product, and adding morphisms of the base category  $\text{Ob } \mathcal{D}_{open}^d$ , from the generators.

*Proof of proposition 6.2.1.* Let  $C'$  be the category with these generators and relations. There is a functor  $C' \rightarrow C$ ; firstly we will show this is full. We can take disjoint union of surfaces in  $C'$ , and we can use the disc with two outgoing marked points to change an incoming boundary of a surface to an outgoing boundary. Clearly, any surface in  $C(\alpha, \beta)$  can be built up using disjoint union and gluing from discs. This shows  $C' \rightarrow C$  is full.

Next, we need to show that this functor is faithful. It suffices to write down a functor  $C \rightarrow C'$  which is an inverse. On objects, this is the identity. Suppose we have a surface  $\Sigma$  in  $C(\alpha, \beta)$ . We can write

$$\Sigma = \Sigma' \circ \phi$$

in a unique way, where  $\Sigma'$  is a disjoint union of identity maps and discs with all incoming boundaries, and  $\phi$  is a disjoint union of discs with two outgoing boundaries and identity maps.  $\Sigma'$  is the normalisation of  $\Sigma$  with all of its marked points made incoming.  $\phi$  has the effect of gluing the marked points of  $\Sigma'$  which correspond to nodes of  $\Sigma$  together, and of changing the incoming points of  $\Sigma'$  which correspond to outgoing points of  $\Sigma$  into outgoing.

This decomposition of  $\Sigma$  allows us to write down the inverse map  $C(\alpha, \beta) \rightarrow C'(\alpha, \beta)$ , and it is easy to check this defines a functor.  $\square$

Let  $\mathcal{D}_{open}^+ \subset \mathcal{D}_{open}^d$  be the subcategory with the same objects but whose morphisms are given by disjoint unions of discs, with each connected component having precisely one outgoing boundary. Note that this is indeed a subcategory, and is also independent of  $d$ ; the local system  $\det^d$  can be canonically trivialised in degree 0 on the moduli space of discs with one outgoing boundary.

For each ordered set  $\lambda_0, \dots, \lambda_{n-1}$  of D-branes, where  $n \geq 1$ , let  $D^+(\lambda_0, \dots, \lambda_{n-1})$  be the disc with  $n$  marked points, and D-brane labels by the  $\lambda_i$ , but such that all of the marked points are incoming except that between  $\lambda_{n-1}$  and  $\lambda_0$ , as in figure 7.

$D^+(\lambda_0, \dots, \lambda_{n-1})$  is in  $\text{Hom}(\{\lambda_0, \dots, \lambda_{n-1}\}, \{\lambda_0, \lambda_1\})$ .

**Lemma 6.2.2.**  $\mathcal{D}_{open}^+$  is freely generated, as a symmetric monoidal category over  $\text{Ob } \mathcal{D}_{open}^d$ , by the discs  $D^+(\lambda_0, \dots, \lambda_{n-1})$ , modulo the relation that

$$D^+(\lambda_0, \dots, \lambda_i, \lambda_i, \dots, \lambda_{n-1}) \circ D^+(\lambda_i) = 0$$

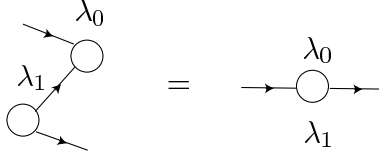


FIGURE 8. Gluing of a disc with two outgoing to a disc with two incoming yields the identity, a disc with an incoming and an outgoing.

whenever  $n \geq 4$ , and when  $n = 3$ ,

$$\begin{aligned} D^+(\lambda_0, \lambda_0, \lambda_1) \circ D^+(\lambda_0) \\ D^+(\lambda_0, \lambda_1, \lambda_1) \circ D^+(\lambda_1) \end{aligned}$$

are both the identity map on the object  $\{\lambda_0, \lambda_1\}$ .

This is basically a corollary of the previous result. Note that the relations stated do indeed hold; composing with  $D^+(\lambda_i)$  has the effect of forgetting the open marked point which lies between the two copies of  $\lambda_i$ . By “composing” we mean of course placing the identity on all other factors.

**Theorem 6.2.3.**  $\mathcal{D}_{open}^d$  is freely generated, as a symmetric monoidal category over  $\text{Ob } \mathcal{D}_{open}^d$ , by  $\mathcal{D}_{open}^+$ , and the discs with two incoming or two outgoing boundaries, modulo the following relations.

The first relation is illustrated in the figure 8 ; it says that an appropriate gluing of the disc with two outgoing boundaries and with two incoming boundaries yields the identity (a disc with one incoming and one outgoing boundary).

Observe that we can change an outgoing boundary to an incoming boundary; let  $D(\lambda_0, \dots, \lambda_{n-1})$  be obtained from  $D^+(\lambda_0, \dots, \lambda_{n-1})$  by making the outgoing boundary incoming. The second relation is that  $D(\lambda_0, \dots, \lambda_{n-1})$  is cyclically symmetric (up to an appropriate sign).

This follows almost immediately from the previous result. This generators and relations description of course refers to the category without the differential. Note that the disc with no marked points and the annulus with no open or closed marked points are included in  $\mathcal{D}_{open}^d$ ; for example, the annulus with no marked points is given by gluing the disc with two outgoing marked points to that with two incoming marked points.

Let  $\lambda_0, \dots, \lambda_{n-1}$  be an ordered set of D-branes. There is an element

$$A(\lambda_0, \dots, \lambda_{n-1}) \in \mathcal{D}^d(\{\lambda_0, \dots, \lambda_{n-1}\}^c, (1, 0))$$

given by given by the annulus with  $n$  marked points, and the intervals between the marked points labelled by the D-branes  $\lambda_i$ , as in figure 6(b). The parameterisation on the closed boundary - on the interior of the annulus - starts at the open marked point 0 between  $\lambda_{n-1}$  and  $\lambda_0$ .

The object  $(1, 0)$  of  $\mathcal{O}\mathcal{C}^d$  has one closed boundary and no open boundaries. Note that  $A(\lambda_0, \dots, \lambda_{n-1})$  is an  $n - 1$  chain in  $\mathcal{D}^d$ .

**Theorem 6.2.4.** The  $\text{Ob } \mathcal{O}\mathcal{C} - \mathcal{D}_{open}^d$  bimodule  $\mathcal{D}^d$  is freely generated, by the  $A(\lambda_0, \dots, \lambda_{n-1})$ , and the identity maps  $1 \in \mathcal{D}_{open}^d(\alpha, \alpha) \subset \mathcal{D}^d(\alpha, \alpha)$ , modulo the following relations.

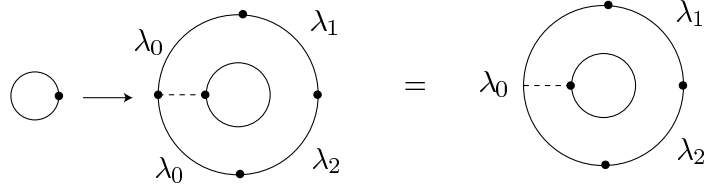


FIGURE 9

Firstly, if we glue the disc with one boundary to any of the open marked points of  $A(\lambda_0, \dots, \lambda_{n-1})$ , except that lying between  $\lambda_{n-1}$  and  $\lambda_0$ , we get 0. Secondly, the disjoint union of the identity element on  $\alpha$  with that on  $\beta$  is the identity on  $\alpha \amalg \beta$ .

This is proved in essentially the same way that the previous results are. The main points are as follows. Since  $\mathcal{D}^d$  is an  $\text{Ob } \mathcal{OC} - \mathcal{D}_{open}^d$  bimodule, we can take disjoint unions, so we get disjoint unions of annuli and identity elements. We also get discs using the action of  $\mathcal{D}_{open}^d$  on the identity elements. For example, we have the identity element for the zero object  $\alpha = 0$ , which gives us discs with all incoming boundaries. The action of  $\mathcal{D}_{open}^d$  allows us to glue discs to annuli. This also lets us glue annuli together, and change incoming marked points on annuli to outgoing, using the disc with two outgoing boundaries. One point to observe is that if we glue a disc with one marked point to the annulus  $A(\lambda_0, \dots, \lambda_{n-2}, \lambda_0)$  at the marked point between  $\lambda_{n-1}$  and  $\lambda_0$  we get an annulus where the starting point for the parameterisation of the closed boundary lies in the free boundary  $\lambda_0$ , as in figure 9.

This ensures that although the moduli space of annuli contains two types of cells, depending on whether the start of the parameterisation on the closed boundary is at an open or a free boundary, we need only take one type as a generator.

**Definition 6.2.5.** Let  $\mathcal{D}^+$  be the  $\text{Ob } \mathcal{OC} - \mathcal{D}_{open}^+$  bimodule with the same generators and relations as  $\mathcal{D}^d$ .

Note that this makes sense, as the relations involve only the disc with one outgoing marked point, which comes from  $\mathcal{D}_{open}^+$ . It is clear that

$$\mathcal{D}^d = \mathcal{D}^+ \otimes_{\mathcal{D}_{open}^+} \mathcal{D}_{open}^d$$

as a  $\text{Ob } \mathcal{OC} - \mathcal{D}_{open}^d$  bimodule. Further, for any left  $\mathcal{D}_{open}^d$  module  $M$ ,

$$\mathcal{D}^d \otimes_{\mathcal{D}_{open}^d} M = \mathcal{D}^+ \otimes_{\mathcal{D}_{open}^+} \mathcal{D}_{open}^d \otimes_{\mathcal{D}_{open}^d} M = \mathcal{D}^+ \otimes_{\mathcal{D}_{open}^+} M$$

**6.3. The differential in  $\mathcal{D}^d$ .** We also want to describe the differential in the complexes  $\mathcal{D}^d$ . This is characterised by the fact that it respects the composition maps  $\mathcal{D}^d(\alpha, \beta) \otimes \mathcal{D}^d(\beta, \gamma) \rightarrow \mathcal{D}^d(\alpha, \gamma)$ , and the way it behaves on discs and annuli, which are the generators. I will only write down the formula up to sign; the precise signs will depend on the orientation chosen for the cells in  $\mathcal{G}$  of marked points on discs and annuli. The precise signs don't matter.

The differential on discs is shown in figure 10. This can be written as

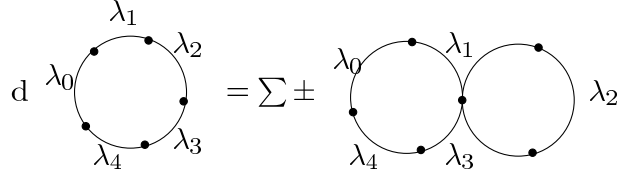


FIGURE 10. The differential of a chain given by marked points on a disc. The marked points may be incoming or outgoing; the  $\lambda_i$  are D-branes.

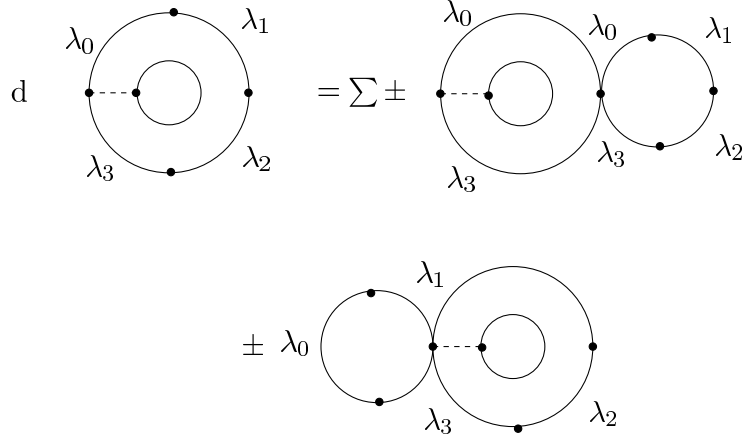


FIGURE 11. The differential of a chain given by marked points on an annulus. The interior circle of the annulus is a closed outgoing boundary, the marked points on the exterior may be incoming or outgoing open, and the  $\lambda_i$  are D-branes.

$$dD(\lambda_0, \dots, \lambda_{n-1}) = \sum_{\substack{0 \leq i < j \leq n-1 \\ j-i \geq 2}} \pm D(\lambda_i, \dots, \lambda_j) * D(\lambda_j, \dots, \lambda_i)$$

where the  $*$  indicates that we glue the open marked points between  $\lambda_i$  and  $\lambda_j$  on each disc together.

On annuli, it is given in figure 11. This can be written as

$$(6.3.1) \quad \begin{aligned} dA(\lambda_0, \dots, \lambda_{n-1}) = & \sum_{\substack{0 \leq i < j \leq n-1 \\ |i-j| \geq 2}} \pm A(\lambda_0, \dots, \lambda_{i-1}, \lambda_i, \lambda_j, \lambda_{j+1}, \dots, \lambda_{n-1}) * D(\lambda_i, \dots, \lambda_j) \\ & + \sum_{\substack{0 \leq j \leq i \leq n-1 \\ (j,i) \neq (0,n-1)}} \pm A(\lambda_j, \dots, \lambda_i) * D(\lambda_i, \dots, 0, 1, \dots, \lambda_j) \end{aligned}$$

where, as before, the symbol  $*$  means we should glue at the open marked points between the D-branes  $\lambda_i$  and  $\lambda_j$ .

**Lemma 6.3.1.** (1) *The  $\text{Ob } \mathcal{O}\mathcal{C}^d - \mathcal{D}_{\text{open}}^d$  bimodule  $\mathcal{D}^d$  is  $\mathcal{D}_{\text{open}}^d$ -flat.*

(2) If  $M$  is a  $h$ -split  $\mathcal{D}_{open}^d$  module, then

$$\mathcal{D}^d \otimes_{\mathcal{D}_{open}^d} M$$

is a  $h$ -split  $\text{Ob } \mathcal{O}\mathcal{C}^d$ -module.

The same is true if we consider  $\mathcal{D}_{open}^+$  and  $\mathcal{D}^+$  instead of  $\mathcal{D}_{open}^d$  and  $\mathcal{D}^d$ .

*Proof.* Recall  $\mathcal{D}^d$  is generated as a  $\mathcal{D}_{open}^d - \text{Ob } \mathcal{O}\mathcal{C}^d$  bimodule by the annuli  $A(\lambda_0, \dots, \lambda_{n-1})$ , and the identity elements in  $\mathcal{D}^d(\alpha, \alpha)$  where  $\alpha \in \text{Ob } \mathcal{D}_{open}^d$ .

Filter  $\mathcal{D}^d$  as a bimodule, by giving a filtration on the generators, defined by saying each identity element in  $\mathcal{D}^d(\alpha, \alpha)$  is in  $F^0$  and each annulus  $A(\lambda_0, \dots, \lambda_{n-1})$  is in  $F^n$ . The formula for the differential of the annuli guarantees that this is a filtration as complexes; indeed,  $dA(\lambda_0, \dots, \lambda_{n-1})$  is in  $F^{n-1}$ .

Let  $M$  be a left  $\mathcal{D}_{open}^d$  module. Suppose  $M \rightarrow M'$  is a quasi-isomorphism; we want to show that the map

$$(6.3.2) \quad \mathcal{D}^d(-, \beta) \otimes_{\mathcal{D}_{open}^d} M \rightarrow \mathcal{D}^d(-, \beta) \otimes_{\mathcal{D}_{open}^d} M'$$

is a quasi-isomorphism. Give both sides the filtration induced from that on  $\mathcal{D}^d(-, \beta)$ ; it suffices to show that the map on the associated graded complexes is a quasi-isomorphism.

This follows immediately from the generators and relations description of  $\mathcal{D}^d$ . Let  $\alpha \in \text{Ob } \mathcal{D}_{open}^d$ ; for an integer  $C$ ,  $C \amalg \alpha \in \text{Ob } \mathcal{O}\mathcal{C}^d$ ; we add on  $C$  closed states. We want to show that the map 6.3.2 is a quasi-isomorphism, with  $\beta = C \amalg \alpha$ . For simplicity I will show this when  $C = 1$ .

Then,  $\text{Gr}^n M \otimes_{\mathcal{D}_{open}^d} \mathcal{D}^d(-, \alpha \amalg 1)$  is spanned by the spaces

$$M(\alpha \amalg \{\lambda_0, \dots, \lambda_{n-1}\}^c)$$

This corresponds to putting the generators of  $\mathcal{D}^d$  which are the identity in  $\mathcal{D}_{open}^d$  on the  $\alpha$  factor and the annulus  $A(\lambda_0, \dots, \lambda_{n-1})$  on the  $\{\lambda_0, \dots, \lambda_{n-1}\}^c$  factor.

The only relation is that the composed map

$$(6.3.3) \quad \begin{aligned} M(\alpha \amalg \{\lambda_0, \dots, \lambda_{i-1}, \lambda_{i+1}, \lambda_{n-1}\}^c) &\rightarrow M(\alpha \amalg \{\lambda_0, \dots, \lambda_{i-1}, \lambda_i, \lambda_i, \lambda_{i+1}, \lambda_{n-1}\}^c) \\ &\rightarrow \text{Gr}^n \mathcal{D}^d(-, \alpha \amalg 1) \otimes_{\mathcal{D}_{open}^d} M \end{aligned}$$

is zero. The first map comes from the element of

$$\mathcal{D}_{open}^d(\alpha \amalg \{\lambda_0, \dots, \lambda_{i-1}, \lambda_{i+1}, \lambda_{n-1}\}^c, \alpha \amalg \{\lambda_0, \dots, \lambda_{i-1}, \lambda_i, \lambda_i, \lambda_{i+1}, \lambda_{n-1}\}^c)$$

which is the tensor product of the identity on  $\alpha$  and  $\lambda_0, \dots, \lambda_{i-1}, \lambda_{i+1}, \lambda_{n-1}^c$  and the map  $0 \rightarrow \{\lambda_i, \lambda_i\}$  given by the disc with one outgoing marked point.

The first map in the diagram 6.3.3 is always injective; we can find a splitting coming from the disc with one incoming marked point. Thus the operation of taking the quotient is exact.

There is a similar description of  $\mathcal{D}^d(-, \beta) \otimes_{\mathcal{D}_{open}^d} M$ , for all  $\beta$ , and the same argument shows that the functor  $\mathcal{D}^d(-, \beta) \otimes_{\mathcal{D}_{open}^d} -$  is exact. This proves the first part of the lemma.

The second part of the lemma is proved in a similar way. Let

$$N = \mathcal{D}^d \otimes_{\mathcal{D}_{open}^d} M$$

Then the filtration on  $\mathcal{D}^d$  induces one on  $N$ . To show the maps  $N(\beta) \otimes N(\beta') \rightarrow N(\beta \amalg \beta')$  are quasi-isomorphism, it suffices to do so on the associated graded. This follows immediately from the description of  $N$  given above.

Exactly the same proof shows the corresponding results for  $\mathcal{D}_{open}^+$  and  $\mathcal{D}^+$ .  $\square$

## 7. PROOF OF THE MAIN RESULTS

**7.1.  $A_\infty$  categories.** Let us recall some details of the definition of an  $A_\infty$  category  $D$ . There is a set  $\text{Ob } D$  of objects, and for each pair  $A, B$  of objects, a finite dimensional complex of  $\mathbb{K}$  vector spaces  $\text{Hom}(A, B)$ . The homological grading convention is used, so that the differential is of degree  $-1$ . For each sequence  $A_0, \dots, A_n$  of objects, where  $n \geq 2$ , there are maps

$$m_n : \text{Hom}(A_0, A_1) \otimes \dots \otimes \text{Hom}(A_{n-1}, A_n) \rightarrow \text{Hom}(A_0, A_n)$$

of degree  $n - 2$ . (Note this is different from the standard convention of  $2 - n$ ). The differential on the complex  $\text{Hom}(A, B)$  is  $m_1$ . These maps must satisfy identities of the form

$$\sum_{0 \leq i \leq j \leq n-1} \pm m_{n-j+i}(\alpha_0 \otimes \dots \otimes \alpha_{i-1} \otimes m_{j-i+1}(\alpha_i \otimes \dots \otimes \alpha_j) \otimes \alpha_{j+1} \otimes \dots \otimes \alpha_{n-1}) = 0$$

All our  $A_\infty$  categories will be unital. A unital  $A_\infty$  category is an  $A_\infty$  category  $D$ , together with for each  $A \in \text{Ob } D$  a closed element  $1_A \in \text{Hom}_0(A, A)$ , with the following properties. Firstly,

$$\begin{aligned} m_2(\alpha \otimes 1_A) &= \alpha \\ m_2(1_A \otimes \beta) &= \beta \end{aligned}$$

for any  $\alpha : B \rightarrow A$  and  $\beta : A \rightarrow B$ . Secondly, if  $\alpha_i : A_i \rightarrow A_{i+1}$  are maps, for  $0 \leq i < n$ , and if  $j = j + 1$ , then

$$m_n(\alpha_0 \otimes \alpha_1 \otimes \dots \otimes 1_{A_j} \otimes \dots \otimes \alpha_{n-1}) = 0$$

**7.2. Calabi-Yau  $A_\infty$  categories.** A *Calabi-Yau*  $A_\infty$  category of dimension  $d$  is an  $A_\infty$  category  $D$ , with for each pair  $A, B \in \text{Ob } D$ , a closed non-degenerate pairing

$$\langle \rangle_{A,B} : \text{Hom}(A, B) \otimes \text{Hom}(B, A) \rightarrow \mathbb{K}[d]$$

which is symmetric (in the sense that  $\langle \rangle_{A,B} = \langle \rangle_{B,A}$  under the natural symmetry isomorphism  $\text{Hom}(A, B) \otimes \text{Hom}(B, A) \cong \text{Hom}(B, A) \otimes \text{Hom}(A, B)$ ), and such that the cyclic symmetry identity

$$\langle m_{n-1}(\alpha_0 \otimes \dots \otimes \alpha_{n-2}), \alpha_{n-1} \rangle = (-1)^{(n+1)+|\alpha_0| \sum_{i=1}^{n-1} |\alpha_i|} \langle m_{n-1}(\alpha_1 \otimes \dots \otimes \alpha_{n-2}), \alpha_0 \rangle$$

holds.

Let  $X$  be a smooth projective Calabi-Yau variety of dimension  $d$ . Let  $\mathcal{D}^b(X)$  be the bounded derived category of coherent sheaves on  $X$ . Then  $\mathcal{D}^b(X)$  is a unital Calabi-Yau  $A_\infty$  category, of dimension  $d$ . We have to change the grading, so that

$$\mathrm{Hom}_i(A, B) = \mathrm{Ext}^{-i}(A, B)$$

The composition maps  $m_n$  are all zero for  $n \neq 2$ . The pairing

$$\mathrm{Hom}_i(A, B)^\vee \cong \mathrm{Hom}_{-d-i}(B, A)$$

is Serre duality. (We need to trivialise the Serre functor, by picking a non-zero holomorphic top form.)

This should not be regarded as being the correct Calabi-Yau  $A_\infty$  category for the  $B$ -model, as I mentioned in the introduction. We should use an  $A_\infty$  version  $\mathcal{D}_\infty^b(X)$ .

**7.3. Open topological conformal field theories and  $A_\infty$  categories.** Let  $\Lambda$  be a set of D-branes.

Recall a monoidal functor between monoidal categories is called *split* if the maps

$$F(a) \otimes F(b) \rightarrow F(a \otimes b)$$

are isomorphisms.

**Lemma 7.3.1.** *A split functor  $\Phi : \mathcal{D}_{open, \Lambda}^+ \rightarrow \mathrm{Comp}_{\mathbb{K}}$  is the same as a unital  $A_\infty$  category with set of objects  $\Lambda$ .*

*Proof.* Let  $\Phi : \mathcal{D}_{open, \Lambda}^+ \rightarrow \mathrm{Comp}_{\mathbb{K}}$  be a split symmetric monoidal functor. Then for each integer  $O$ , with D-brane labels  $s(i), t(i)$ , for  $0 \leq i \leq O-1$ , we have a natural isomorphism

$$\Phi(O, s, t) \cong \otimes_{i=0}^{O-1} \Phi(\{s(i), t(i)\})$$

For each pair  $\lambda, \lambda'$  of D-branes, write  $\mathrm{Hom}(\lambda, \lambda') = \Phi(\{\lambda, \lambda'\})$ .

Generators and relations for  $\mathcal{D}_{open, \Lambda}^+$  are given in 6.2.2. The discs  $D^+(\lambda_0, \dots, \lambda_{n-1})$  give maps

$$\mathrm{Hom}(\lambda_0, \lambda_1) \otimes \dots \otimes \mathrm{Hom}(\lambda_{n-2}, \lambda_{n-1}) \rightarrow \mathrm{Hom}(\lambda_0, \lambda_{n-1})$$

which are of degree  $n-3$ , when  $n \geq 3$ . These correspond to the  $A_\infty$  multiplications  $m_{n-1}$  when  $n \geq 3$  (when appropriate sign conventions, and orientations on the cells corresponding to  $D^+(\lambda_0, \dots, \lambda_{n-1})$  are chosen). The formula for the differentials  $dD$  gives the  $A_\infty$  relation; indeed this is essentially the original definition of  $A_\infty$  algebra of Stasheff [Sta63].

When  $n=2, 1$  the maps  $D^+(\lambda)$  and  $D^+(\lambda, \lambda')$  are of degree 0.  $D^+(\lambda)$  gives a map  $\mathbb{K} \rightarrow \mathrm{Hom}(\lambda, \lambda)$ , which gives the unit in the  $A_\infty$  category. The axioms for units in an  $A_\infty$  category correspond to the relations in  $\mathcal{D}_{open, \Lambda}^+$  described in 6.2.2.

$D^+(\lambda, \lambda')$  is the identity map  $\mathrm{Hom}(\lambda, \lambda') \rightarrow \mathrm{Hom}(\lambda, \lambda')$ . □

**Lemma 7.3.2.** *A split monoidal functor  $\Phi : \mathcal{D}_{open, \Lambda}^d \rightarrow \mathrm{Comp}_{\mathbb{K}}$  is the same as a unital Calabi-Yau  $A_\infty$  category with set of objects  $\Lambda$ .*

*Proof.* This follows from the generators and relations description for the category  $\mathcal{D}_{open, \Lambda}^d$ . There are two more generators for  $\mathcal{D}_{open, \Lambda}^d$  over  $\mathcal{D}_{open, \Lambda}^+$ , namely the disc with two incoming and two outgoing boundaries. These give the pairing  $\mathrm{Hom}(\lambda_0, \lambda_1) \otimes$

$\text{Hom}(\lambda_1, \lambda_0) \rightarrow \mathbb{K}[d]$ , and its inverse. The extra relations in  $\mathcal{D}_{open}^d$  correspond to the cyclic symmetry condition.  $\square$

**Definition 7.3.3.** *A unital extended Calabi-Yau  $A_\infty$  category, with objects  $\Lambda$ , is a  $h$ -split symmetric monoidal functor  $\Phi : \mathcal{D}_{open, \Lambda}^d \rightarrow \text{Comp}_{\mathbb{K}}$ .*

So a split extended Calabi-Yau  $A_\infty$  category is the same as an ordinary Calabi-Yau  $A_\infty$  category. Let  $\Phi$  be an extended Calabi-Yau  $A_\infty$  category. For each  $\alpha \in \text{Ob } \mathcal{D}_{open, \Lambda}^d$  there is a complex  $\Phi(\alpha)$ , and quasi-isomorphisms  $\Phi(\alpha) \otimes \Phi(\beta) \rightarrow \Phi(\alpha \amalg \beta)$ . There are maps

$$\Phi(\alpha \amalg \{\lambda_0, \dots, \lambda_n\}) \rightarrow \Phi(\alpha \amalg \{\lambda_0, \lambda_n\})$$

coming from the disjoint union of the disc  $D^+(\lambda_0, \dots, \lambda_{n-1})$  and the identity map  $\alpha \rightarrow \alpha$ . These play the role of the  $A_\infty$  operations  $m_n$ , when  $n \geq 2$ . They satisfy relations analogous to the usual  $A_\infty$  relation. There are also maps

$$\begin{aligned} \Phi(\alpha \amalg \{\lambda_0, \lambda_1\}) &\rightarrow \Phi(\alpha) \\ \Phi(\alpha) &\rightarrow \Phi(\alpha \amalg \{\lambda_0, \lambda_1\}) \end{aligned}$$

which play the role of the pairing and its inverse. A cyclic symmetry condition holds for the operation  $\Phi(\alpha \amalg \{\lambda_0, \dots, \lambda_{n-1}, \lambda_0\}) \rightarrow \Phi(\alpha)$  constructed from the  $A_\infty$  operation  $m_n$  and the pairing. Also there are units, in  $\Phi(\{\lambda, \lambda\})$  satisfying the usual constraints.

**Lemma 7.3.4.** *The category of unital extended Calabi-Yau  $A_\infty$  categories of dimension  $d$ , with set of objects  $\Lambda$ , is quasi-equivalent, in the sense of definition 4.2.1 to the category of open TCFTs of dimension  $d$ .*

This is immediate from theorem 4.4.3, and the fact that  $\mathcal{D}_{open, \Lambda}^d$  is quasi-isomorphic to  $\mathcal{O}_\Lambda^d$ . Thus we have proved theorem A, part 1.

**Definition 7.3.5.** *A unital extended  $A_\infty$  category, with set of objects  $\Lambda$ , is a  $h$ -split monoidal functor  $\Phi : \mathcal{D}_{open, \Lambda}^+ \rightarrow \text{Comp}_{\mathbb{K}}$ .*

This makes sense, as we have already seen that such a functor which is split is the same as a unital  $A_\infty$  category.

**Proposition 7.3.6.** *The following categories are quasi-equivalent.*

- (1) *The category of unital extended  $A_\infty$  categories, with set of objects  $\Lambda$ .*
- (2) *The category of unital  $A_\infty$  categories, with set of objects  $\Lambda$ .*
- (3) *The category of unital dg categories, with set of objects  $\Lambda$ .*

*Proof.* Observe that for each pair of objects  $\alpha, \beta \in \text{Ob } \mathcal{D}_{open, \Lambda}^+$ ,  $H_i(\mathcal{D}_{open, \Lambda}^+(\alpha, \beta)) = 0$  if  $i \neq 0$ . As, the morphisms spaces are chains on moduli spaces of marked points on discs, which are contractible. Also the complexes  $\mathcal{D}_{open, \Lambda}^+(\alpha, \beta)$  are concentrated in degrees  $\geq 0$ . This implies that  $\mathcal{D}_{open, \Lambda}^+$  is formal, that is quasi-isomorphic to its homology, and quasi-isomorphic to  $H_0(\mathcal{D}_{open, \Lambda}^+)$ .

It is not difficult to see that a split functor  $\Phi : H_0(\mathcal{D}_{open, \Lambda}^+) \rightarrow \text{Comp}_{\mathbb{K}}$  is the same as a unital dg category with set of objects  $\Lambda$ . Indeed,  $H_0(\mathcal{D}_{open, \Lambda}^+(\{\lambda_0, \lambda_1, \dots, \lambda_n\}, \{\lambda_0, \lambda_n\}))$

is one dimensional, and corresponds to the product map

$$\mathrm{Hom}(\lambda_0, \lambda_1) \otimes \dots \otimes \mathrm{Hom}(\lambda_{n-1}, \lambda_n) \rightarrow \mathrm{Hom}(\lambda_0, \lambda_n)$$

Because it is one-dimensional, associativity holds. Further,  $H_0(\mathcal{D}_{open,\Lambda}^+(\alpha, \beta))$  is given by disjoint unions of morphisms of this type.

Call a h-split functor  $\Phi : H_0(\mathcal{D}_{open,\Lambda}^+) \rightarrow \mathrm{Comp}_{\mathbb{K}}$  a unital extended category. Since there is a quasi-isomorphism

$$\mathcal{D}_{open,\Lambda}^+ \rightarrow H_0(\mathcal{D}_{open,\Lambda}^+)$$

there is a quasi-equivalence between unital extended  $A_\infty$  categories and unital extended categories.

It remains to show how to remove the adjectival extended. The category  $\mathcal{D}_{open,\Lambda}^+$  has the property that the maps

$$\begin{aligned} (\dagger) \quad \mathcal{D}_{open,\Lambda}^+(\alpha_1, \{\lambda_1, \lambda'_1\}) \otimes \mathcal{D}_{open,\Lambda}^+(\alpha_2, \{\lambda_2, \lambda'_2\}) \dots \otimes \mathcal{D}_{open,\Lambda}^+(\alpha_n, \{\lambda_n, \lambda'_n\}) \\ \rightarrow \mathcal{D}_{open,\Lambda}^+(\alpha_1 \amalg \dots \amalg \alpha_n, \{\lambda_1, \lambda'_1\} \amalg \dots \amalg \{\lambda_n, \lambda'_n\}) \end{aligned}$$

are isomorphisms.

Let  $\Phi$  be a unital extended  $A_\infty$  category. Define a unital  $A_\infty$  category  $F(\phi)$ , i.e. a split monoidal functor  $\mathcal{D}_{open,\Lambda}^+ \rightarrow \mathrm{Comp}_{\mathbb{K}}$ , by

$$F(\phi)(O, s, t) = \otimes_{i=0}^{O-1} \Phi(\{s(i), t(i)\})$$

There are maps  $F(\phi)(\alpha) \rightarrow \phi(\alpha)$ . Composing with the action of  $\mathcal{D}_{open,\Lambda}^+$  on  $\phi$  gives maps

$$F(\phi)(\alpha) \otimes \mathcal{D}_{open,\Lambda}^+(\alpha, \{\lambda_0, \lambda_1\}) \rightarrow F(\phi)(\{\lambda_0, \lambda_1\})$$

Because the maps  $\dagger$  are isomorphisms, it follows that these extend to give a unique  $\mathcal{D}_{open,\Lambda}^+$  module structure on  $F(\phi)$ , with a quasi-isomorphism  $F(\phi) \rightarrow \phi$ .

This shows that the category of extended  $A_\infty$  categories is quasi-equivalent to the category of  $A_\infty$  categories. Similarly the category of extended dg categories is quasi-equivalent to the category of dg categories; this finishes the proof.  $\square$

The proof shows something stronger; the obvious map from dg categories to  $A_\infty$  categories is half of a quasi-equivalence. This means that every  $A_\infty$  category is quasi-isomorphic, in a functorial way, to a dg category.

**7.4. The Hochschild chain complex.** For an associative algebra  $A$ , over our ground field  $\mathbb{K}$ , and an  $A$ -bimodule  $M$ , recall the Hochschild complex  $C_*(A, M)$  is defined by

$$C_n(A, M) = M \otimes A^{\otimes n}$$

The differential  $d : C_n(A, M) \rightarrow C_{n-1}(A, M)$  is given by the formula

$$(7.4.1) \quad d(m \otimes a_1 \otimes \dots \otimes a_n) = ma_1 \otimes a_2 \otimes \dots \otimes a_n \\ + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n + (-1)^n a_n m \otimes a_1 \otimes \dots \otimes a_{n-1}$$

When  $M = A$ , we write  $C_*(A)$  for  $C_*(A, A)$ .

The normalised Hochschild chain complex is a quotient of  $C_*(A, M)$  by the contractible complex spanned by elements  $m \otimes a_1 \otimes \dots \otimes a_n$  where at least one of the  $a_i = 1$ . We write  $\overline{C}_*(A, M)$  for the normalised chain complex, and  $\overline{C}_*(A)$  for the normalised chain complex with coefficients in  $A$ .

Similar definitions hold for dg algebras  $A$  and dg modules  $M$ , except with extra terms in the differential coming from the differential on  $A$  and  $M$ , and a change in sign coming from the grading on  $A$  and  $M$ .

Let  $A$  be a dg category. Define the Hochschild chain complex

$$C_*(A) = \bigoplus (\text{Hom}(\alpha_0, \alpha_1) \otimes \dots \otimes \text{Hom}(\alpha_{n-1}, \alpha_0)) [1 - n]$$

where the direct sum is over  $n$  and sequences  $\alpha_0, \dots, \alpha_{n-1}$  of objects in  $A$ .

The differential is given by essentially the same formula as in the algebra case:

$$\begin{aligned} d(\phi_0 \otimes \dots \otimes \phi_{n-1}) &= \sum_{i=0}^{n-1} \pm \phi_0 \dots d\phi_i \dots \phi_{n-1} \\ &\quad + \sum_{i=0}^{n-2} \pm \phi_0 \dots \otimes (\phi_{i+1} \circ \phi_i) \otimes \dots \otimes \phi_{n-1} \pm (\phi_0 \circ \phi_{n-1}) \otimes \dots \otimes \phi_{n-2} \end{aligned}$$

If  $A$  is unital, then we can define the normalised Hochschild chain complex  $\overline{C}_*(A)$  by taking the quotient by the contractible subcomplex spanned by  $\phi_0 \otimes \dots \otimes \phi_{n-1}$  where at least one of the  $\phi_i$ , where  $i > 0$ , is an identity map.

**Lemma 7.4.1.** *The functor  $A \mapsto \overline{C}_*(A)$  is an exact functor from the category of dg categories with fixed set of objects  $\Lambda$  to the category of complexes.*

*Proof.* Give the normalised Hochschild chain complex  $\overline{C}_*(A)$  the obvious filtration, defined by  $F^i(\overline{C}_*(A))$  is the subcomplex spanned by  $\phi_0 \otimes \dots \otimes \phi_{i-1}$ . If  $A \rightarrow B$  is a map of dg categories with fixed set of objects, the induced map  $\overline{C}_*(A) \rightarrow \overline{C}_*(B)$  preserves the filtration. We need to show that if  $A \rightarrow B$  is a quasi-isomorphism then so is  $\overline{C}_*(A) \rightarrow \overline{C}_*(B)$ . It is sufficient to show that the associated graded map is a quasi-isomorphism; but this is obvious.  $\square$

**Definition 7.4.2.** *Let  $A$  be a (possibly extended)  $A_\infty$  category. Define the Hochschild homology  $HH_*(A)$  to be the homology of the dg category associated to it under the quasi-equivalence between (extended)  $A_\infty$  and dg categories.*

*If  $\Phi$  is an extended Calabi-Yau  $A_\infty$  category, define the Hochschild homology of  $\Phi$  to be the homology of the associated extended  $A_\infty$  category.*

We could also use an explicit complex to define the Hochschild homology, but this would involve getting the signs correct.

**Proposition 7.4.3.** *Let  $\Phi$  be a unital extended Calabi-Yau  $A_\infty$  category. Then*

$$H_*(\mathcal{D}^d(-, 1)_\Lambda \otimes_{\mathcal{D}_{open, \Lambda}^d} \Phi) = HH_*(\Phi)$$

*Proof.* Recall the definition of  $\mathcal{D}^+$  in definition 6.2.5. We have a generators and relations description of  $\mathcal{D}^d$ , in theorem 6.2.4, and we defined  $\mathcal{D}^+$  to have the same generators and relations but as a  $\text{Ob } \mathcal{O}\mathcal{C} - \mathcal{D}_{open}^+$  bimodule rather than a  $\text{Ob } \mathcal{O}\mathcal{C} - \mathcal{D}_{open}^d$  bimodule.

We have

$$\mathcal{D}^d(-, 1)_\Lambda \otimes_{\mathcal{D}_{open, \Lambda}^d} \Phi = \mathcal{D}^+(-, 1)_\Lambda \otimes_{\mathcal{D}_{open, \Lambda}^+} \Phi$$

Further, we have shown that the functor  $\mathcal{D}^+(-, 1)_\Lambda \otimes_{\mathcal{D}_{open, \Lambda}^+} -$  is exact (lemma 6.3.1).

The  $\mathcal{D}_{open, \Lambda}^+$  module underlying  $\Phi$  is the extended  $A_\infty$  category associated to  $\Phi$ .

What remains to be shown is that, for an actual dg category  $B$ , considered as a left  $\mathcal{D}_{open, \Lambda}^+$  module,

$$H_*(\mathcal{D}^+(-, 1)_\Lambda \otimes_{\mathcal{D}_{open, \Lambda}^+} B) = HH_*(B)$$

We will show something a bit more; we will show that

$$\mathcal{D}^+(-, 1)_\Lambda \otimes_{\mathcal{D}_{open, \Lambda}^+} B = \overline{C}_*(B)$$

is the normalised Hochschild chain complex.

This follows from the generators and relations description of the right  $\mathcal{D}_{open, \Lambda}^+$  module,  $\mathcal{D}^+(-, 1)_\Lambda$ . Recall it is generated by the annuli  $A(\lambda_0, \dots, \lambda_{n-1})$ , modulo the relation that when we glue the disc with one outgoing marked point onto a marked point of  $A(\lambda_0, \dots, \lambda_{n-1})$ , we get zero, except for the marked point between  $\lambda_{n-1}$  and  $\lambda_0$ . The annulus  $A(\lambda_0, \dots, \lambda_{n-1})$  is in degree  $n - 1$ .

This shows us that  $\mathcal{D}^+(-, 1)_\Lambda \otimes_{\mathcal{D}_{open, \Lambda}^+} B$ , as a vector space, is the quotient of  $\oplus(\text{Hom}(\alpha_0, \alpha_1) \otimes \dots \otimes \text{Hom}(\alpha_{n-1}, \alpha_0))[1 - n]$  by the subspace spanned by elements of the form  $\phi_0 \otimes \dots \otimes \phi_{n-1}$ , where at least one of the  $\phi_i$  with  $i > 0$  is an identity map.

That is, as a vector space, there is a natural isomorphism

$$\mathcal{D}^+(-, 1)_\Lambda \otimes_{\mathcal{D}_{open, \Lambda}^+} B \cong \overline{C}_*(B)$$

It remains to show that this is compatible with the differential. This follows immediately from the formula for the differential of the annulus  $A(\lambda_0, \dots, \lambda_{n-1})$ , see figure 11. Recall that  $m_n = 0$  when  $n > 2$  in our category  $B$ , so that the only thing that contributes is when two marked points on the boundary of the annulus collide. This corresponds to composing the corresponding consecutive morphisms in the formula for the Hochschild differential. □

This completes the proof of theorem A.

We have shown theorem A part 1: the category of unital extended Calabi-Yau  $A_\infty$  categories is quasi-equivalent to the category of open TCFTs.

We have also shown that under the quasi-equivalence between  $\text{Ob } \mathcal{O}\mathcal{C}_\Lambda^d - \mathcal{D}_{open, \Lambda}^d$  bimodules and  $\text{Ob } \mathcal{O}\mathcal{C}_\Lambda^d - \mathcal{O}_\Lambda^d$  bimodules, the bimodule  $\mathcal{D}_\Lambda^d$  corresponds to  $\mathcal{O}\mathcal{C}_\Lambda^d$ . Also,  $\mathcal{D}_\Lambda^d$  is flat.

Thus, by lemma 4.4.4 if  $M$  is a left  $\mathcal{D}_{open, \Lambda}^d$  module corresponding to a left  $\mathcal{O}_\Lambda^d$  module  $M'$ , then

$$\mathcal{O}\mathcal{C}_\Lambda^d(-, \beta) \otimes_{\mathcal{O}_\Lambda^d}^L M' \cong \mathcal{D}_\Lambda^d(-, \beta) \otimes_{\mathcal{D}_{open, \Lambda}^d} M$$

Denote by  $N(\beta)$  the left hand side of this equation. Then  $N(\beta)$  is h-split, if  $M$  is; the maps  $N(\beta) \otimes N(\beta') \rightarrow N(\beta \amalg \beta')$  are quasi-isomorphisms. This shows that  $N$  defines an

open-closed TCFT of dimension  $d$ , which is the homotopy universal open-closed TCFT associated to  $M'$ .

Finally, we have calculated the homology of the closed states of  $N$  to be the Hochschild homology of the associated  $A_\infty$  category.

## 8. APPENDIX

In this appendix a symmetric monoidal functor  $C_*$  from the category of topological spaces with local systems to chain complexes is constructed, which computes homology groups, and satisfies several nice properties. In particular, for a cell complex  $X$ , there is a map  $C_*^{cell}(X) \rightarrow C_*(X)$  which is natural for a strong notion of cellular map.

We recall the properties of homology with local coefficients. A  $\mathbb{K}$  local system on a space  $Y$  is a locally constant sheaf of  $\mathbb{K}$  vector spaces on  $Y$ . If  $E$  is a local system on  $Y$ , there are homology groups  $H_i(Y, E)$  with local coefficients. Spaces with local systems form a category; a map  $(Y, E) \rightarrow (Z, F)$  is a map  $f : Y \rightarrow Z$  and a map  $E \rightarrow f^*F$ . Homology with coefficients defines a functor from this category to the category of graded  $\mathbb{K}$  vector spaces.

This functor satisfies the following properties.

(1) If

$$0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$$

is a short exact sequence of local systems on  $Y$ , there is a corresponding long exact sequence of homology groups

$$\dots \rightarrow H_i(E_1) \rightarrow H_i(E_2) \rightarrow H_i(E_3) \rightarrow H_{i-1}(E_1) \rightarrow \dots$$

(2) If  $Y = U \cup V$  is written as a union of open subsets, there is a Mayer-Vietoris long exact sequence of homology groups

$$\dots \rightarrow H_i(U \cap V, E) \rightarrow H_i(U, E) \oplus H_i(V, E) \rightarrow H_i(Y, E) \rightarrow H_{i-1}(U \cap V, E) \rightarrow \dots$$

(3) Two maps  $f_0, f_1 : (Y, E) \rightarrow (Z, F)$  are homotopic if they extend to a map  $F : (Y \times I, \pi_1^*E) \rightarrow (Z, F)$ . Homotopic maps induce the same map on homology.

(4) If  $Y = *$  is a point, and  $E$  is a vector space, then  $H_i(*, E) = 0$  if  $i \neq 0$ , and  $H_0(*, E) = E$ .

On reasonable spaces, for example spaces with the homotopy type of finite cell complexes, the functor  $(Y, E) \rightarrow H_*(Y, E)$  is determined by these properties. We can define  $H_i(Y, E)$  using singular simplices  $f : \Delta_n \rightarrow Y$  with sections of  $f^*E \otimes \omega$ , where  $\omega$  is the orientation sheaf. There are also relative homology groups  $H_i(Y, Y', E)$  for a subspace  $Y' \subset Y$  and a local system  $E$  on  $Y$ , which fit into the obvious long exact sequence.

A finite regular cell complex is a space  $X$  obtained by attaching finitely many cells to a finite number of points, with the property that the boundary of one cell is a union of lower dimensional cells. Let  $X_i \subset X$  be the union of cells of dimension  $\leq i$ . A strong cellular map between finite regular cell complexes  $X, X'$  is a continuous map  $f : X \rightarrow X'$  such that  $f^{-1}(X'_i) = X_i$ . Thus we have a category  $\text{Cell}$  of finite regular cell complexes with these morphisms.

For a topological space  $Y$ , let  $\text{Cell}_Y$  be the category whose objects are finite regular cell complexes  $X$  with a map  $f : X \rightarrow Y$ , and whose morphisms are strong cellular maps  $X \rightarrow X'$  such that the obvious diagram commutes.

There is a functor  $C_*^{cell} : \text{Cell}_Y \rightarrow \text{Comp}_{\mathbb{K}}$ , which takes  $X$  to the  $\mathbb{K}$  cellular chain complex  $C_*^{cell}(X, \mathbb{K})$ . (Of course we could use any coefficient ring). If  $E$  is a local system on  $Y$ , then pulling back  $E$  gives a local system on each object  $X \in \text{Cell}_Y$ , and there is a functor of cellular chains with coefficients from  $\text{Cell}_Y \rightarrow \text{Comp}_{\mathbb{K}}$ . This functor applied to  $X \in \text{Cell}_Y$  is denoted  $C_*^{cell}(X, E)$ . By definition,  $C_n^{cell}(X, E) = H_n(X_n, X_{n-1}, E)$  is the relative sheaf homology.  $C_n^{cell}(X, E)$  is naturally isomorphic to the space of sections over  $X_n \setminus X_{n-1}$  of the sheaf  $E \otimes \omega_{X_n \setminus X_{n-1}}$ , where  $\omega_{X_n \setminus X_{n-1}}$  is the orientation sheaf.

Define  $C_*(Y, E)$  by

$$C_*(Y, E) = \lim_{X \in \text{Cell}_Y} C_*^{cell}(X, E)$$

to be the direct limit over the cellular chain groups of objects of  $\text{Cell}_Y$ .

It is clear that  $C_*$  is functorial. Denote by  $H'(Y, E)$  the homology of the chain complex  $C_*(Y, E)$ .

**Proposition 8.0.4.** *The functor  $H'(Y, E)$  satisfies the axioms (1)-(4) above, and so coincides with usual homology with local coefficients on reasonable spaces.*

*Proof.* Axiom (1) is straightforward; the sequence of complexes

$$0 \rightarrow C_*(Y, E_1) \rightarrow C_*(Y, E_2) \rightarrow C_*(Y, E_3) \rightarrow 0$$

is exact. Axiom (3) follows as if  $\iota_0, \iota_1 : Y \hookrightarrow Y \times I$  are the inclusions, there is a canonical chain homotopy between the induced maps  $C_*(Y, E) \rightarrow C_*(Y \times I, \pi_1^* E)$ . Axiom (4) is also quite straightforward; for any  $n$  cell complex  $X$  over a point, with  $n > 1$ , there is a cellular isomorphism  $X \rightarrow X$  changing the orientation on the  $n$  cells.

It remains to prove the Mayer-Vietoris axiom. The sequence of complexes

$$0 \rightarrow C_*(U \cap V, E) \rightarrow C_*(U, E) \oplus C_*(V, E) \rightarrow C_*(Y, E) \rightarrow 0$$

is actually exact. Exactness on the left and in the middle is straightforward. Exactness on the right is more difficult; this can be proved by showing, inductively on the dimension of the cells, that for any  $n$  cell complex  $X \in \text{Cell}_Y$ , we can find a refinement  $X'$  of the cell structure on  $X$  such that any closed cell of  $X'$  lands in either  $U$  or  $V$ .  $\square$

If  $X_1, X_2$  are cell complexes, and  $E_i$  are finite dimensional  $\mathbb{K}$  local systems on  $X_i$ , then there is an isomorphism

$$C_*^{cell}(X_1, E_1) \otimes C_*^{cell}(X_2, E_2) \xrightarrow{\cong} C_*^{cell}(X_1 \times X_2, E_1 \boxtimes E_2)$$

This induces maps

$$C_*(Y_1, E_1) \otimes C_*(Y_2, E_2) \rightarrow C_*(Y_1 \times Y_2, E_1 \boxtimes E_2)$$

making  $C_*$  into a symmetric monoidal functor from spaces with finite dimensional  $\mathbb{K}$  local systems to chain complexes.

## REFERENCES

- [Bar99] S. Barannikov, *Generalized periods and mirror symmetry in dimensions  $n > 3$* , (1999), math.AG/9903124.
- [Bar00] S. Barannikov, *Quantum periods I. Semi-infinite variations of Hodge structures*, (2000), math.AG/0006193.
- [BCOV94] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, Comm. Math. Phys. **165**(2), 311–427 (1994).
- [BK97] S. Barannikov and M. Kontsevich, *Frobenius manifolds and formality of Lie algebras of polyvector fields*, (1997), math.QA/9710032.
- [Coh04] R. Cohen, *Morse theory, graphs and string topology*, (2004), math.GT/0411272.
- [Cos04] K. Costello, *The  $A_\infty$  operad and the moduli space of curves*, (2004), math.AG/0402015.
- [Cos06] K. Costello, *A dual point of view on the ribbon graph decomposition of moduli space*, (2006), math.GT/0601130.
- [CS99] M. Chas and D. Sullivan, *String topology*, (1999), math.GT/9911159.
- [CS04] M. Chas and D. Sullivan, *Closed string operators in topology leading to Lie bialgebras and higher string algebra*, in *The legacy of Niels Henrik Abel*, pages 771–784, Springer, Berlin, 2004.
- [FOOO00] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian intersection Floer theory : anomaly and obstruction*, 2000.
- [Get94] E. Getzler, *Batalin-Vilkovisky algebras and two-dimensional topological field theories*, Comm. Math. Phys. **159**(2), 265–285 (1994).
- [HL04] A. Hamilton and A. Lazarev, *Homotopy algebras and noncommutative geometry*, (2004), math.QA/0410621.
- [Kau04] R. M. Kaufmann, *A proof of a cyclic version of Deligne’s conjecture via cacti*, (2004), math.QA/0403340.
- [Kel94] B. Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27**(1), 63–102 (1994).
- [Kon95] M. Kontsevich, *Homological algebra of mirror symmetry*, in *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 120–139, Basel, 1995, Birkhäuser.
- [Kon99] M. Kontsevich, *Operads and motives in deformation quantization*, Lett. Math. Phys. **48**(1), 35–72 (1999), Moshé Flato (1937–1998).
- [Kon03] M. Kontsevich, *Talk at the Hodge Centennial conference, Edinburgh*, (2003).
- [KR04] A. Kapustin and L. Rozansky, *On the relation between open and closed topological strings*, Comm. Math. Phys. **252**(1-3), 393–414 (2004), hep-th/0405232.
- [KS00] M. Kontsevich and Y. Soibelman, *Deformations of algebras over operads and the Deligne conjecture*, in *Conférence Moshé Flato 1999, Vol. I (Dijon)*, volume 21 of *Math. Phys. Stud.*, pages 255–307, Kluwer Acad. Publ., Dordrecht, 2000.
- [KS01] M. Kontsevich and Y. Soibelman, *Homological mirror symmetry and torus fibrations*, (2001), math.SG/0011041.
- [KSV95] T. Kimura, J. Stasheff and A. A. Voronov, *On operad structures of moduli spaces and string theory*, Comm. Math. Phys. **171**(1), 1–25 (1995), hep-th/9307114.
- [Laz01] C. I. Lazaroiu, *On the structure of open-closed topological field theory in two dimensions*, Nuclear Phys. B **603**(3), 497–530 (2001), hep-th/0010269.
- [Liu02] C.-C. M. Liu, *Moduli of  $J$ -holomorphic curves with Lagrangian boundary conditions and open Gromov-Witten invariants for an  $S^1$ -equivariant pair*, (2002), math.SG/0210257.
- [Mar04] M. Markl, *Transferring  $A_\infty$  (strongly associative) structures*, (2004), math.AT/0401007.
- [Mer99] S. A. Merkulov, *Strong homotopy algebras of a Kähler manifold*, Internat. Math. Res. Notices (3), 153–164 (1999), math.AG/9809172.
- [ML98] S. Mac Lane, *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, second edition, 1998.

- [Moo01] G. Moore, Some comments on branes,  $G$ -flux, and  $K$ -theory, in *Strings 2000. Proceedings of the International Superstrings Conference (Ann Arbor, MI)*, volume 16, pages 936–944, 2001.
- [MS02] J. E. McClure and J. H. Smith, A solution of Deligne’s Hochschild cohomology conjecture, in *Recent progress in homotopy theory (Baltimore, MD, 2000)*, volume 293 of *Contemp. Math.*, pages 153–193, Amer. Math. Soc., Providence, RI, 2002.
- [Seg99] G. Segal, *Topological field theory*, (1999), <http://www.cgtp.duke.edu/ITP99/segal/>, Notes of lectures at Stanford university.
- [Seg04] G. Segal, The definition of conformal field theory, in *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 421–577, Cambridge Univ. Press, Cambridge, 2004, First circulated in 1988.
- [Sei01a] P. Seidel, More about vanishing cycles and mutation, in *Symplectic geometry and mirror symmetry (Seoul, 2000)*, pages 429–465, World Sci. Publishing, River Edge, NJ, 2001.
- [Sei01b] P. Seidel, Vanishing cycles and mutation, in *European Congress of Mathematics, Vol. II (Barcelona, 2000)*, volume 202 of *Progr. Math.*, pages 65–85, Birkhäuser, Basel, 2001.
- [Sei02] P. Seidel, Fukaya categories and deformations, in *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pages 351–360, Beijing, 2002, Higher Ed. Press.
- [Sta63] J. D. Stasheff, *Homotopy associativity of  $H$ -spaces. I, II*, *Trans. Amer. Math. Soc.* 108 (1963), 275–292; *ibid.* 108, 293–312 (1963).
- [Tam98] D. E. Tamarkin, *Another proof of M. Kontsevich formality theorem*, (1998), [math.QA/9803025](http://math.QA/9803025).
- [TZ04] T. Tradler and M. Zeinalian, *On the cyclic Deligne conjecture*, (2004), [math.QA/0404218](http://math.QA/0404218).
- [Wit91] E. Witten, *Mirror manifolds and topological field theory*, (1991), [hep-th/9112056](http://hep-th/9112056).

DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE LONDON  
*E-mail address:* [k.costello@imperial.ac.uk](mailto:k.costello@imperial.ac.uk)