# Homework 9, Homological Algebra, 253, Spring 2008

1. **Quaternion Group Extensions.** In this problem you will use the techniques explained in class to classify all extensions of the form

$$1 \to Q_8 \to E \to \mathbb{Z}/2 \to 1 \qquad (*)$$

where $Q_8$ is the quaternion group of order 8. Proceed by following these steps:

   (a) Show that the inner automorphisms of $Q_8$ are isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$, $\mathrm{Aut}(Q_8) \cong S_4$ and $\mathrm{Out}(Q_8) \cong S_3$. Hint: Any automorphism must fix the center of $Q_8$, so can be viewed as acting on $\pm i, \pm j, \pm k$. Then argue geometrically in $\mathbb{R}^3$.

   (b) Find all extensions (*) which induce the trivial map $\rho : \mathbb{Z}/2 \to \mathrm{Out}(Q_8)$. Are the resulting groups $E$ isomorphic?

   (c) Let $\rho : \mathbb{Z}/2 \to \mathrm{Out}(Q_8)$ be non-trivial. Show that $\rho^*$ induces an epimorphism $H^3(\mathrm{Out}(Q_8); \mathbb{Z}/2) \twoheadrightarrow H^3(\mathbb{Z}/2; \mathbb{Z}/2)$ (so there could be obstructions for an extension (*).) Hint: $\mathrm{Out}(Q_8) \cong S_3 \cong \mathbb{Z}/3 \rtimes \mathbb{Z}/2$.

   (d) Show that the universal obstruction class $[\alpha] \in H^3(\mathrm{Out}(Q_8); \mathbb{Z}/2)$ vanishes, where $\alpha$ is the crossed module

$$0 \to \mathbb{Z}/2 \to Q_8 \xrightarrow{\alpha} S_4 \to S_3 \to 1$$

   (e) In total, how many extensions (*) are there? Which of the corresponding groups $E$ are isomorphic?

2. **Homotopy for Kan Simplicial Sets.** We say that two simplical maps $f_0, f_1 : X_\bullet \to Y_\bullet$ are *homotopic*, $f_0 \simeq f_1$, if there exists a *homotopy* $H : X_\bullet \times \Delta^1_\bullet \to Y_\bullet$ extending the map $f_0 \amalg f_1$ on $X_\bullet \times \partial\Delta^1_\bullet$.

   (a) Show that $\simeq$ is an equivalence relation if $Y_\bullet$ is Kan.

   (b) Define an analogous concept of homotopy for pairs of simplicial sets, i.e. for maps $f_i : (X_\bullet, A_\bullet) \to (Y_\bullet, B_\bullet)$. Find a condition on $(Y_\bullet, B_\bullet)$ so that homotopy, relative to $A_\bullet$, is an equivalence relation.

   (c) Show that for a pointed simplicial Kan set $Y_\bullet$, we have isomorphisms

$$\pi_n(Y_\bullet, *) \cong [(\Delta^n_\bullet, \partial\Delta_\bullet), (Y_\bullet, *)]$$

   where the right hand side denotes the homotopy classes of maps of pairs.

3. **Skeletal Picard Groupoids.** Consider a monoidal category $(\mathsf{G}, \otimes, 1, a, \ell, r)$ that is a groupoid. (We're avoiding the term 'monoidal groupoid'!)

   (a) Show that $A := \pi_1(\mathsf{G}, 1)$ is an abelian group.

   (b) Show that $Q := \pi_0(\mathsf{G})$ is a group if $\mathsf{G}$ is *Picard* (i.e. for each $x \in \mathsf{G}$ there exists an $\overline{x} \in \mathsf{G}$ such that $x \otimes \overline{x} \cong 1 \cong \overline{x} \otimes x$). Prove that in this case, $\pi_1(\mathsf{G}, x) \cong A$ for any other object $x$.

   (c) Finally, assume that $\mathsf{G}$ is *skeletal* (i.e. for all $x, x' \in \mathsf{G}$ we have $x \cong x'$ if and only if $x = x'$) and show that $Q$ acts on $A$ and that the collection of all morphisms in $\mathsf{G}$ can be identified with $A \rtimes Q$, with $\otimes$ matching the group structure on this semi-direct product. (Note that there can still be non-trivial associators and identity transformations).

PLEASE RETURN PROBLEMS IN THE DISCUSSION SESSION ON FRIDAY, APRIL 4.

**Enjoy your spring break!**