Homework 8, Homological Algebra, 253, Spring 2008

We received feedback that the class and homework are turning from 'too easy' to 'too hard'. Even though the simplicial language is purely combinatorial, it does require time to getting used to, please remain patient. The rewards will be important insights into many constructions all over mathematics that otherwise seem totally ad hoc. On the current homework, we pushed the hardest problems into notes and added hints to all remaining problems that might be hard. Please continue to give us feedback!

1. **Homotopy Groups of Kan Simplicial Sets.** Let $X_\bullet$ be a simplicial set with basepoint $* \in X_0$ (leading to degenerate basepoints $(s_0)^n(*) \in X_n$, again denoted by $*$). Consider the set

$$Z_n(X_\bullet, *) := \{x \in X_n \mid d_i(x) = *, i = 0, \dots, n\}$$

For $x, x' \in Z_n$ write $x \sim x'$ if there is a $y \in X_{n+1}$ such that $d_i(y) = *$ for $i = 0, \dots, n-1$ and $d_n(y) = x, d_{n+1}(y) = x'$. Ignoring the basepoint and all conditions involving it, this definition makes good sense also for $n = 0$.

   (a) Show that the relation $\sim$ is reflexive. If $X_\bullet$ is Kan, show that it is an equivalence relation. *In this case*, define $\pi_n(X_\bullet, *) := Z_n(X_\bullet, *)/\sim$.

   (b) Show that for topological spaces X, the corresponding simplicial sets $\Delta_\bullet(X)$ are Kan and there are natural bijections

$$\pi_n(X, *) \cong \pi_n(\Delta_\bullet(X), *)$$

   (c) Recall from HW7 that a simplicial abelian group $A_\bullet$ is Kan. Show that

$$\pi_n(A_\bullet, 0) \cong H_n(N_*(A_\bullet))$$

Note: The sets $\pi_n(X_\bullet, *)$ are in fact groups for $n \geq 1$, abelian for $n \geq 2$. One can use cellular approximation theorems to show that

$$\pi_n(|K_\bullet|, *) \cong \pi_n(K_\bullet, *)$$

for any Kan set $K_\bullet$. Together with (b) this implies that $|\Delta_\bullet(X)| \to X$ is a weak equivalence and hence gives a functorial CW-approximation (or *cofibrant replacement*) of any space X. The other functorial map $X_\bullet \to \Delta_\bullet(|X_\bullet|)$ is a Kan-approximation (or *fibrant replacement*) of any simplicial set $X_\bullet$.

2. **Natural Transformations are Homotopies.** Let $\eta : F_0 \Rightarrow F_1$ be a natural transformation of functors $F_i : \mathsf{C} \to \mathsf{D}$ between small categories.

(a) Show that the Nerve construction leads to a canonical homotopy

$$N_\bullet(\eta) : N_\bullet(\mathsf{C}) \times \Delta^1_\bullet \to N_\bullet(\mathsf{D})$$

between the simplicial maps $N_\bullet(F_i)$. Hint: Write $N_\bullet(\mathsf{C}) \times \Delta^1_\bullet$ as the nerve of a category.

(b) Conclude that an equivalence of categories, $\mathsf{C} \simeq \mathsf{D}$, leads to a homotopy equivalence of classifying spaces $|N_\bullet(\mathsf{C})| \simeq |N_\bullet(\mathsf{D})|$. Hint: Use that geometric realization preserves products.

(c) Recall from HW7 that the nerve $N_\bullet(G)$ of a small groupoid G is a Kan set. Show that $\pi_n(G, *) := \pi_n(N_\bullet(G), *) = 0$ for all $n \geq 2$. Hint: Use that there are *unique* fillers $f : \Delta^n_\bullet \to N_\bullet(G)$ for all horns if $n \geq 2$.

(d) Show that a functor $\Phi : G \to H$ between small groupoids is an equivalence of categories if and only if it induces isomorphisms $\pi_0(G) \cong \pi_0(H)$ and $\pi_1(G, *) \cong \pi_1(H, \Phi(*))$ for all basepoints $* \in G$.

3. **Realization of Simplicial Sets.**

(a) Let $X_\bullet$ be a simplicial set and consider the geometric realization

$$|X_\bullet| := \coprod_n (X_n \times \Delta^n)/ \sim,$$

where $\sim$ is the equivalence relation given by $(x, \alpha_*(t)) \sim (\alpha^*(x), t)$, for all $x \in X_n, t \in \Delta^m$ and $\alpha : [m] \to [n]$. Show that $|X_\bullet|$ is a CW-complex with exactly one n-cell for each *non-degenerate* n-simplex in X_n. Hint: Show that $|X_\bullet|$ is a quotient of the realization of the underlying semi-simplicial set, studied in HW 5, problem 1 (d): Exactly simplices of the form $s_j(x)$ are identified, where $s_j : [n+1] \to [n]$ is one of the degeneracy maps.

(b) Conclude that $|\Delta^1_\bullet \times \Delta^1_\bullet|$ has exactly 4 vertices, 5 edges and 2 faces. Hint: Show combinatorially that these are the only non-degenerate simplices.

Note: Let $L : \mathsf{ssSet} \to \mathsf{sSet}$ be left-adjoint to the forgetful functor. Then for every semi-simplicial set $Y_\bullet$, there is a CW-homeomorphism $\mathrm{Re}(Y_\bullet) \cong |L(Y_\bullet)|$, where Re is the realization of a *semi*-simplicial set as in HW 5. This is the precise analogue of problem 3(b) from HW7 in the non-linear world. The analogue of 3(c) is the fact that the canonical quotient map (identifying degenerate simplices) $\mathrm{Re}(X_\bullet) \twoheadrightarrow |X_\bullet|$ is a homotopy equivalence for simplicial sets $X_\bullet$.

PLEASE RETURN PROBLEMS IN THE DISCUSSION SESSION ON FRIDAY, MARCH 21.