Homework 7, Homological Algebra, 253, Spring 2008

1. **Kan Condition.** A semi-simplicial set $X_\bullet$ is called *Kan* if it satisfies the *Kan condition* : Every *horn* h has a *filler* f, making the following diagram commute:

$$\begin{array}{ccc} \Lambda^k[n]_\bullet & \xrightarrow{h} & X_\bullet \\ \big\downarrow & \overset{f}{\nearrow} & \\ \Delta[n]_\bullet & & \end{array}$$

Here $\Lambda^k[n]_\bullet \subset \Delta[n]_\bullet$ is the semi-simplicial set obtained from $\Delta[n]_\bullet$ by deleting the n-simplex and the $(n-1)$-simplex opposite the k-th vertex. A simplicial set is called Kan (or fibrant) if its underlying semi-simplicial set is Kan.

   (a) Show that a Kan simplicial set $X_\bullet$ is either the constant functor or X_n constains a non-degenerate n-simplex for every $n \geq N$, for some N. Decide which simplicial n-simplices (i.e. representable simplicial sets) are Kan.

   (b) Show that the underlying semi-simplicial set of a simplicial group is Kan.

   (c) Show that the nerve of a category is Kan if and only if the category is a groupoid.

2. **Estimating Group Homology.**

   (a) Let X be a connected CW-complex with fundamental group G such that $\pi_i(X) = 0$ for $i = 2, \ldots, n$. Show that there are isomorphisms $H_i(X) \cong H_i(G)$ for $i = 1, \ldots, n$ and an epimorphism $H_{n+1}(X) \twoheadrightarrow H_{n+1}(G)$.

   (b) Let G be group with g generators and r relations. Construct epimorphisms

$$\mathbb{Z}^g \twoheadrightarrow H_1(G) \quad \text{and} \quad \mathbb{Z}^{r-g+k} \twoheadrightarrow H_2(G)$$

   if $\mathbb{Z}^k \cong H_1(G)/\,\text{torsion}$. Can you continue the pattern for $H_3(G)$?

   (c) Let G be a *balanced* group, i.e. $g = r$ above. Show that if G is abelian then it is isomorphic to $\mathbb{Z}/n, \mathbb{Z} \times \mathbb{Z}/n$ or $\mathbb{Z}^3$, for some $n \in \mathbb{N}_0$.

Addendum: For a closed 3-manifold, a handlebody decomposition gives a balanced presentation of the fundamental group. Do all abelian groups listed above arise that way?

3. **Chain Complexes for Simplicial Abelian Groups.** Let $A_\bullet$ be a semi-simplicial abelian group. The standard chain complex $\mathrm{Alt}_*(A_\bullet)$ has groups

$$\mathrm{Alt}_n(A_\bullet) := A_n$$

with the differential given by the alternating sum of the d_i. The *normalized (or Moore) complex* $N_*(A_\bullet)$ of $A_\bullet$ is defined as

$$N_n(A_\bullet) := \bigcap_{i>0} \ker d_i$$

with the differential given by d_0. This is clearly a sub-complex of $\mathrm{Alt}_*(A_\bullet)$.

(a) Give an example where these two chain complexes have non-isomorphic homology.

(b) Let $L : \mathsf{ssAb} \to \mathsf{sAb}$ be left-adjoint to the forgetful functor. Show that for every semi-simplicial abelian group $A_\bullet$, there is a chain isomorphism

$$N_*(A_\bullet) \cong N_*(L(A_\bullet))$$

(c) Let $A_\bullet$ be a *simplicial* abelian group and let $D_*(A_\bullet)$ be the subcomplex of $\mathrm{Alt}_*(A_\bullet)$ consisting of degenerate simplices. Show that $D_*(A_\bullet)$ is contractible and that

$$\mathrm{Alt}_*(A_\bullet) = N_*(A_\bullet) \oplus D_*(A_\bullet)$$

Conclude that the inclusion $N_*(A_\bullet) \subset \mathrm{Alt}_*(A_\bullet)$ is a chain homotopy equivalence.

Note: In the non-linear world, we replace Ab by Set and Chain by Top. Then the geometric realization of a semi-simplicial set plays the role of Alt_*, and the realization of a simplicial set (using the degeneracies for further identifications) plays the role of Alt_* / D_*. The analogues of (b) and (c) are discussed in class. The isomorphism

$$\mathrm{Alt}_*(A_\bullet)/D_*(A_\bullet) \cong N_*(A_\bullet)$$

shows that the left hand side can actually be already defined on semi-simplicial abelian groups. This is a coincidence and does not hold in the non-linear case.

Please return problems in the discussion session on Friday, March 14.