Homework 6, Homological Algebra, 253, Spring 2008

To make the discussion session more lively, new rules were suggested for the homework. Let's try out the following in the coming weeks: Each group member writes up a *different* homework problem. The idea is that you discuss all the problems in your group but then write different problems (don't forget the proof-reading part).

1. **The Ring Ext.** Let k be a commutative ring and $R \to k$ an augmented k-algebra. Using the Yoneda product, we get another augmented k-algebra

$$\mathrm{Ext}_R^* := \bigoplus_n \mathrm{Ext}_R^n(k,k) \xrightarrow{p_0} \mathrm{Ext}_R^0(k,k) = k.$$

   Show that there are isomorphisms of augmented k-algebras

$$\mathrm{Ext}_S^* \cong \Lambda \quad \text{and} \quad \mathrm{Ext}_\Lambda^* \cong S,$$

   where $S := S^*(V)$ is the symmetric and $\Lambda := \Lambda^*(V)$ the exterior k-algebra on a free k-module V of finite rank (both are *quotients* of the free k-algebra and carry the canonical augmentation). Do the rank one cases first!

2. **Locally Constant Sheaves.** A sheaf $\mathcal{F}$ on X is *locally constant* if every point in X has an open neighborhood U such that the 'restriction to stalks' maps $\mathcal{F}(U) \to \mathcal{F}_x$ are isomorphisms for all $x \in U$. Let X be a path connected space that is locally contractible, and set $G := \pi_1(X, x_0)$.

   (a) Show that the category of locally constant sheaves of abelian groups on X is equivalent to the category of G-modules.

   (b) Conclude that for a locally constant sheaf $\mathcal{F}$, the cohomology $H^n(X; \mathcal{F})$ can be computed from the cochain complex $\mathrm{Hom}_G(C_*(\tilde{X}), M_{\mathcal{F}})$, where $M_{\mathcal{F}}$ is the G-module associated to $\mathcal{F}$ and $\tilde{X}$ is a universal cover of X.

3. **Circle bundles.** Let $G = \pi_1(\Sigma) \neq 1$ be the fundamental group of a closed orientable surface Σ. From homework 4 we know that $H^2(G) \cong \mathbb{Z}$ and by extension theory an element $n \in \mathbb{Z}$ hence gives a central group extension

$$1 \to \mathbb{Z} \to G_n \to G \to 1$$

   Calculate the homology of G_n using a certain 3-manifold (and no spectral sequences). What happens for non-orientable surfaces?

PLEASE RETURN PROBLEMS IN THE DISCUSSION SESSION ON FRIDAY, MARCH 7.