Homework 5, Homological Algebra, 253, Spring 2008

We first straighten out some terminology from class. For a functor $\phi : \mathsf{C} \to \mathsf{D}$ and an object $d \in \mathsf{D}$, there are various versions of a *fibre category*:

- The *strict fibre* $\phi^{-1}(d)$ is the full subcategory of C consisting of objects mapping to d and morphisms mapping to Id_d.
- The objects of the *homotopy fibre* are pairs (c, g) where $c \in \mathsf{C}$ and $g : d \to \phi(c)$ is an *isomorphism* in D. The morphisms from (c, g) to (c', g') are morphisms $c \to c'$ in C making the obvious triangle in D commute.
- The *under category* $(d \downarrow \phi)$ is as above except that g is any morphism.
- The *over category* $(\phi \downarrow d)$ is as above except that $g : \phi(c) \to d$ is any morphism.

1. Semi-Simplicial Sets. Let $r : \mathsf{C} \to \mathsf{Pre}(\mathsf{C}) = \mathsf{Fun}(\mathsf{C}^{op}, \mathsf{Set})$ be the Yoneda embedding of a small category C into the category of presheaves on C.

(a) Using colimits over index categories of the form $(r \downarrow \mathcal{F})$, show that any functor $\gamma : \mathsf{C} \to \mathsf{D}$ into a cocomplete category D may be uniquely factored through a colimit preserving map $\mathrm{Re} : \mathsf{Pre}(\mathsf{C}) \to \mathsf{D}$:

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "r"] \arrow[dr, "\gamma"'] & \mathsf{Pre}(\mathsf{C}) \arrow[d, dashed, "\mathrm{Re}"] \\
 & \mathsf{D}
\end{tikzcd}$$

(b) Show that the map Re admits a canonical right adjoint $\mathrm{Sing} : \mathsf{D} \to \mathsf{Pre}(\mathsf{C})$, given on objects $c \in \mathsf{C}, d \in \mathsf{D}$ by $\mathrm{Sing}(d)(c) = \mathsf{D}(\gamma(c), d)$.

(c) Consider the category C_Δ with objects $\mathbb{N}_{\geq 0} = \{[n]\}$, where $[n]$ is the totally ordered set $\{0 < 1 < 2 < \cdots < n\}$. The morphisms of C_Δ are the *strictly* order preserving maps and a presheaf on C_Δ is called a *semi-simplicial set.*

Show that a semi-simplicial set $X_\bullet$ consists of the following data: a sequence of sets X_n of *n-simplices*, $n \geq 0$, together with *face maps* $d_i : X_n \to X_{n-1}$, $0 \leq i \leq n$, such that

$$d_i d_j = d_{j-1} d_i \quad \text{for} \quad i < j.$$

(d) Show that there is a functor $\gamma : \mathsf{C}_\Delta \to \mathsf{Top}$ that sends $[n]$ to the standard n-simplex $\Delta^n \subset \mathbb{R}^{n+1}$ and a morphism in C_Δ to an affine linear embedding. Parts (a) and (b) above yield an adjunction

$$\mathrm{Re} : \mathsf{Pre}(\mathsf{C}_\Delta) \rightleftarrows \mathsf{Top} : \mathrm{Sing}_\bullet$$

Show that the *geometric realization* Re of a semi-simplicial set $X_\bullet$ satisfies

$$\mathrm{Re}(X_\bullet) \approx \coprod_n (X_n \times \Delta^n)/\sim$$

where $\sim$ is the equivalence relation generated by $(x, \partial_i(t)) \sim (d_i(x), t)$, for $x \in X_n, t \in \Delta^{n-1}$ and $\partial_i : \Delta^{n-1} \to \Delta^n$ the i-th face map. Conclude that $\mathrm{Re}(X_\bullet)$ is a CW-complex with exactly one n-cell for every n-simplex in X_n.

(e) For a topological space S, describe the semi-simplicial set $\mathrm{Sing}_\bullet(S)$ explicitly.

2. Small Group Extensions. Show that, up to equivalence, there are exactly eight group extensions of the form

$$1 \to \mathbb{Z}/2 \to G \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1$$

How many isomorphism classes of groups G are there? Write down those groups and explain your answer in terms of extension theory.

3. Limits and Colimits. Let $D \in \mathsf{Fun}(\mathsf{J}, \mathsf{C})$ be a diagram and consider the 'restriction to J' functor $\phi^\triangleleft : \mathsf{Fun}(\mathsf{J}^\triangleleft, \mathsf{C}) \to \mathsf{Fun}(\mathsf{J}, \mathsf{C})$ as well as the diagonal functor $\Delta : \mathsf{C} \to \mathsf{Fun}(\mathsf{J}, \mathsf{C})$ that maps $c \in \mathsf{C}$ to the constant diagram D_c.

(a) Show that if $\lim D$ exists, then it gives a final object in the strict fibre $(\phi^\triangleleft)^{-1}(D)$ as well as in the over category $(\Delta \downarrow D)$. Conversely, a final object in $(\phi^\triangleleft)^{-1}(D)$ (respectively in $(\Delta \downarrow D)$) gives a limit of D.

Does this also work for the other versions of the fibre categories for $\phi^\triangleleft$?

Show that $\ell \in \mathsf{C}$ is a limit of D if and only if there is a natural transformation $D_\ell \to D$ that induces bijections for all $c \in \mathsf{C}$ as follows:

$$\mathsf{Fun}(\mathsf{J}, \mathsf{C})(D_c, D) \cong \mathsf{C}(c, \ell)$$

(b) Formulate and prove the anologous properties of colimits.

PLEASE RETURN PROBLEM 1 IN THE DISCUSSION SESSION ON FRIDAY, FEB. 29.