Homework 4, Homological Algebra, 253, Spring 2008

1. **Computing group homology.** Compute the group homology of
   (a) the fundamental groups of all compact surfaces,
   (b) finite groups acting freely on S^3,
   (c) any other groups for which you can write down the entire group homology.

2. **Adjoint functors.** Let $L : \mathsf{C} \longrightarrow \mathsf{D}$ be a left adjoint to $R : \mathsf{D} \longrightarrow \mathsf{C}$.
   (a) Show that L preserves all colimits (e.g. coproducts, cokernels, etc.) and R preserves all limits (e.g. products, kernels, etc.).
   (b) If C and D are abelian and L, R are additive, conclude that L is right exact and R is left exact. Apply this to $\otimes$ and Hom.
   (c) If L is in addition exact, show that R preserves injectives. Conclude that for any ring S, $\mathrm{Hom}_{\mathsf{Ab}}(S, \mathbb{Q}/\mathbb{Z})$ is an injective S-module and that any S-module can be embedded into an injective S-module.
   (d) Give examples of adjoint pairs, where L is a 'free' functor and R is a 'forgetful' functor. One such example has $\mathsf{C} = \mathsf{Set}$ and $\mathsf{D} = \mathsf{Group}$.

3. **Sheaf cohomology.**
   (a) Let X be a topological space and $x \in X$. Let $\mathsf{C} = \mathsf{Sheaf}(X)$ be the category of sheaves of abelian groups on X and $\mathsf{D} = \mathsf{Ab}$. Show that the stalk functor $L(\mathcal{F}) = \mathcal{F}_x$ is left adjoint to the pushforward functor $R(A) = x_*A$. Here the *skyscraper sheaf* is defined for an abelian group A:
   $$x_*(A)(U) = \begin{cases} A & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$
   Conclude that $\mathsf{Sheaf}(X)$ has enough injectives, i.e. every sheaf can be embedded into an injective sheaf.
   (b) Use the method of adjoint functors to show that the *global sections* functor $\Gamma : \mathsf{Sheaf}(X) \to \mathsf{Ab}$, $\Gamma(\mathcal{F}) := \mathcal{F}(X)$, is left exact. Define $H^n(X; \mathcal{F})$ as the cohomology groups of the cochain complex $\Gamma(\mathcal{I}^*)$, for any injective resolution $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \dots$. Show independence of $\mathcal{I}^*$.
   (c) Use the method of double complexes to show that for a space X with an $\mathcal{F}$-acyclic cover $\mathcal{U}$, there is an isomorphism $H^n(X; \mathcal{F}) \cong \check{H}^n(\mathcal{U}; \mathcal{F})$.

PLEASE RETURN PROBLEM 1 IN THE DISCUSSION SESSION ON FRIDAY, FEB. 22.