Homework 3, Homological Algebra, 253, Spring 2008

1. **Ext for Abelian Groups.** In this problem A and B will denote abelian groups and we'll calculate $\mathrm{Ext}^n_{\mathbb{Z}} =: \mathrm{Ext}^n$ for the ring $\mathbb{Z}$ as best we can. Prove:

   (a) $\mathrm{Ext}^n(A, B) = 0$ for $n \geq 2$,

   (b) $\mathrm{Ext}^1(\mathbb{Z}/p, B) \cong B/pB$,

   (c) $\mathrm{Ext}^0(\mathbb{Z}/p, B) \cong \{b \in B \mid pb = 0\}$, the p-torsion of B,

   (d) $\mathrm{Ext}^1(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}) \cong \mathbb{Z}_p/\mathbb{Z}$,where $\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^n$ are the p-adic integers.

   (e) $\mathrm{Ext}^1(A, \mathbb{Z}) \cong \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ for all torsion groups A.

   Conclude that Ext^1 does not vanish for all flat groups and is in general not a torsion group.

2. **More groups with periodic cohomology.** Show that every abelian subgroup of a finite group G is cyclic if and only if G satisfies the p^2-conditions for all primes p: every subgroup of order p^2 is cyclic.

   Now let m and n be two coprime integers and let $G := \mathbb{Z}/m \rtimes \mathbb{Z}/n$ be the semi-direct product, where $\mathbb{Z}/n$ acts on $\mathbb{Z}/m$ via multiplication by some integer r.

   (a) Show that every abelian subgroup of G is cyclic.

   (b) Show that there is a free linear G-action on S^{2k-1} under the following assumption: Let k be the order of r in $(\mathbb{Z}/m)^\times$ (in particular, k divides n). Assume that every prime that divides n also divides $\frac{n}{k}$.

3. **Our signs are consistent.** Let C, C' and C'' be chain complexes and recall from class the definitions of tensor product and (inner) $\underline{\mathrm{Hom}}$ in the category CHAIN. Show that composition of maps induces a chain map

   $$\underline{\mathrm{Hom}}(C', C'') \otimes \underline{\mathrm{Hom}}(C, C') \longrightarrow \underline{\mathrm{Hom}}(C, C'')$$

   Similarly, the evaluation map induces a chain map

   $$\underline{\mathrm{Hom}}(C, C') \otimes C \longrightarrow C'$$

   Note that our sign conventions agree with Brown but not Weibel.

PLEASE RETURN PROBLEM 1 IN THE DISCUSSION SESSION ON FRIDAY, FEB. 15.