Homework 2, Homological Algebra, 253, Spring 2008

1. **Tor for Abelian Groups.** In this problem A and B will denote abelian groups and we'll calculate $\mathrm{Tor}_n^{\mathbb{Z}} =: \mathrm{Tor}_n$ for the ring $\mathbb{Z}$. Show that

   (a) $\mathrm{Tor}_0(\mathbb{Z}/p\mathbb{Z}, B) \cong B/pB$,

   (b) $\mathrm{Tor}_1(\mathbb{Z}/p\mathbb{Z}, B) \cong \{b \in B \mid pb = 0\}$, the p-torsion of B,

   (c) $\mathrm{Tor}_n(\mathbb{Z}/p\mathbb{Z}, B) = 0$ for $n \geq 2$,

   (d) $\mathrm{Tor}_1(A, B)$ is a torsion group and $\mathrm{Tor}_n(A, B) = 0$ for $n \geq 2$,

   (e) $\mathrm{Tor}_1(A, B) = 0$ for all $B \Longleftrightarrow A$ is torsion free.

2. **The Acyclic Assembly Lemma.** A *double complex* is a family $\{C_{p,q}\}_{p,q\in\mathbb{Z}}$ of abelian groups together with *horizontal* and *vertical* differentials

$$d^h : C_{p,q} \to C_{p-1,q} \quad \text{and} \quad d^v : C_{p,q} \to C_{p,q-1}$$

   satisfying $d^h \circ d^h = d^v \circ d^v = d^v d^h + d^h d^v = 0$. This means that the following diagram is commutative only up to sign:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \\
\cdots & \xleftarrow{d^h} & C_{p-1,q+1} & \leftarrow & C_{p,q+1} & \leftarrow & C_{p+1,q+1} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xleftarrow{d^h} & C_{p-1,q} & \longleftarrow & C_{p,q} & \longleftarrow & C_{p+1,q} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xleftarrow{d^h} & C_{p-1,q-1} & \leftarrow & C_{p,q-1} & \leftarrow & C_{p+1,q-1} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

   Define (two versions of) the *total complex* of the double complex $C_{*,*}$ by:

$$\mathrm{Tot}^{\Pi}(C)_n = \prod_{p+q=n} C_{p,q} \quad \text{and} \quad \mathrm{Tot}^{\oplus}(C)_n = \bigoplus_{p+q=n} C_{p,q}$$

(a) Show that $d := d^h + d^v$ defines a *total differential* making $\mathrm{Tot}^\Pi(C)_*$ and $\mathrm{Tot}^\oplus(C)_*$ into chain complexes.

(b) Prove the following *acyclic assembly lemma*: For a double complex $C_{*,*}$,

- $\mathrm{Tot}^\Pi(C)$ is an acyclic chain complex, assuming either of the following:
  i. $C_{*,*}$ is an upper half-plane complex with exact columns.
  ii. $C_{*,*}$ is a right half-plane complex with exact rows.
- $\mathrm{Tot}^\oplus(C)$ is an acyclic chain complex, assuming either of the following:
  i. $C_{*,*}$ is an upper half-plane complex with exact rows.
  ii. $C_{*,*}$ is a right half-plane complex with exact columns.

(c) Let C be the periodic upper half-plane complex with $C_{p,q} = \mathbb{Z}/4\mathbb{Z}$ for all p and $q \geq 0$, and all nontrivial differentials being multiplication by 2. Show that $H_0(\mathrm{Tot}^\Pi(C)) \cong \mathbb{Z}/2\mathbb{Z}$, even though the rows of C are exact. Show that $\mathrm{Tot}^\oplus(C)$ is acyclic.

(d) Extend the previous double complex downward to form a double periodic double complex D where every $D_{p,q} \cong \mathbb{Z}/4\mathbb{Z}$ and every differential is multiplication by 2. Show that $H_0(\mathrm{Tot}^\Pi(D))$ maps surjectively onto $H_0(\mathrm{Tot}^\Pi(C)) \cong \mathbb{Z}/2\mathbb{Z}$. Hence $\mathrm{Tot}^\Pi(D)$ is not acyclic, even though every row and column of D is exact. Show that $\mathrm{Tot}^\oplus(D)$ is also not acyclic.

(e) Give an example of a 2^{nd} quadrant double chain complex E with exact columns for which $\mathrm{Tot}^\oplus(E)$ is not an acyclic chain complex. Similarly, give an example of a 4^{th} quadrant double complex F with exact columns, for which $\mathrm{Tot}^\Pi(F)$ is not acyclic.

PLEASE RETURN PROBLEM 1 IN THE DISCUSSION SESSION ON FRIDAY, FEB. 8.