

Homework 13, Homological Algebra, 253, Spring 2008

1. **Mapping Cones.** Let $f : A \rightarrow B$ be a chain map. The *mapping cone* $C(f)$ of f is the chain complex defined by

$$C(f)_n = A_{n-1} \oplus B_n \quad \text{and} \quad d_f(a, b) = (-d(a), d(b) + f(a))$$

Let ΣA denote the shifted complex and show that there is an *exact triangle*

$$A \xrightarrow{f} B \rightarrow C(f) \rightarrow \Sigma A$$

of chain maps in the following sense: For any chain complex X , denote by $[A, X]$ the (abelian group of) homotopy classes of chain maps $A \rightarrow X$. Then there are long exact sequences:

$$\begin{aligned} \dots H_{n+1}(C(f)) &\longrightarrow H_n(A) \xrightarrow{H(f)} H_n(B) \longrightarrow H_n(C(f)) \rightarrow H_{n-1}(A) \xrightarrow{H(f)} H_{n-1}(B) \dots \\ \dots &\rightarrow [X, \Sigma^{-1}C(f)] \rightarrow [X, A] \xrightarrow{f_*} [X, B] \rightarrow [X, C(f)] \rightarrow [X, \Sigma A] \xrightarrow{f_*} [X, \Sigma B] \rightarrow \dots \\ \dots &\leftarrow [\Sigma^{-1}C(f), X] \leftarrow [A, X] \xleftarrow{f^*} [B, X] \leftarrow [C(f), X] \leftarrow [\Sigma A, X] \xleftarrow{f^*} [\Sigma B, X] \leftarrow \dots \end{aligned}$$

Conclude that f is a quasi-isomorphism if and only if $C(f)$ is acyclic and f is a homotopy equivalence if and only if $C(f)$ is contractible.

2. **Injective/Projective versus Local/Colocal Objects.** Let \mathbf{A} be an abelian category and let $\mathbf{E} \subset \mathbf{hC} := \mathbf{hChain}(\mathbf{A})$ be the subcategory of quasi-isomorphisms.

- Let $I_* \in \mathbf{hC}^-$ be injective: all I_n are injective and there is an $N \in \mathbb{Z}$ such that $I_n \cong 0$ for $n > N$. Show that I_* is \mathbf{E} -local in \mathbf{hC} and conclude that \mathbf{hC}^- has all localizations if \mathbf{A} has enough injectives.
- Let $P_* \in \mathbf{hC}^+$ be projective: all P_n are projective and there is an $N \in \mathbb{Z}$ such that $P_n \cong 0$ for $n < N$. Show that P_* is \mathbf{E} -colocal in \mathbf{hC} and that \mathbf{hC}^+ has all colocalizations if \mathbf{A} has enough projectives.
- Give an example of an abelian category \mathbf{A} and a complex $P_* \in \mathbf{hC}$ with all P_n projective, but where P_* is not \mathbf{E} -colocal in \mathbf{hC} .
- Find morphisms f and g in \mathbf{hC}^+ so that $f = 0$ in the derived category $\mathbf{Der}(\mathbf{A}) = \mathbf{E}^{-1}\mathbf{hC}^+$ but not null-homotopic, and g induces 0 in homology but is not 0 in $\mathbf{Der}(\mathbf{A})$.

Note: To deal with *unbounded* chain complexes, one can introduce the following notion of *cellular* chain complexes $C \in \mathbf{Chain}(\mathbf{A})$. They are defined by having a filtration

$$0 \subseteq C^{(0)} \subseteq C^{(1)} \subseteq \dots \subseteq C^{(n)} \dots \subseteq C$$

such that $C^{(n+1)}$ is obtained from $C^{(n)}$ by forming the mapping cone of an *attaching map*

$$\alpha^{(n)} : P^{(n)} \rightarrow C^{(n)}$$

where $P^{(n)}$ consists of projective objects $P_k^{(n)} \in \mathbf{A}$ and has zero differentials. Assuming \mathbf{A} has filtered colimits, one defines $C = \operatorname{colim} C^{(n)}$, with the colimit taken degree-wise. This is a generalization (to the stable category) of CW-complexes, where each $P^{(n)}$ is like a disjoint union of spheres but of varying dimension.

Assuming filtered colimits are exact in \mathbf{A} , one shows by induction (using problem 1 above and a lemma from class) that these cellular chain complexes are colocal in \mathbf{hC} . If \mathbf{A} has in addition enough projectives, one can conclude that \mathbf{hC} has all colocalizations. This applies for example to $\mathbf{A} = R\text{-Mod}$.

Another approach is given in Spaltenstein [Comp. Math. 65, 1988], who showed that all localizations exist in \mathbf{hC} if \mathbf{A} is the category of sheaves of modules over a ringed space (X, \mathcal{O}_X) . In both cases, the (a priori large) category $\mathbf{Der}(\mathbf{A})$ is equivalent to a subcategory of \mathbf{hC} .

3. **Pontrjagin Products.** Let $K = K(\mathbb{Z}, 2)$ and $a \in H^2(K)$ be a generator. Use generators $z_k \in H_{2k}(K)$ satisfying $\langle z_k, a^k \rangle = 1$ and prove that the Pontrjagin product gives a *divided polynomial algebra* structure on $H_*(K)$:

$$z_k \cdot z_\ell = \binom{k + \ell}{k} z_{k+\ell}$$

Conclude that there is a fibration $K \rightarrow F \rightarrow S^3$ with $H_{2n-1}(F) = 0$ and $H_{2n}(F) \cong \mathbb{Z}/n$ for all $n > 0$ and use this to compute $\pi_5(S^3)$.

PLEASE RETURN PROBLEMS IN THE DISCUSSION SESSION ON FRIDAY, MAY 2.