Homework 12, Homological Algebra, 253, Spring 2008

1. **Künneth Spectral Sequence.** Let P, Q be chain-complexes of R-modules, concentrated in non-negative degree. Assume that all P_k are flat and show that there is a first quadrant spectral sequence

$$E^2_{p,q} \cong \bigoplus_{q'+q''=q} \mathrm{Tor}^R_p(H_{q'}(P), H_{q''}(Q)) \Longrightarrow H_{p+q}(P \otimes_R Q).$$

   Derive the usual Künneth theorem for topological spaces.

   Hint: The relevant double complex arises from a *Cartan-Eilenberg resolution* of Q that you need to construct. This is a resolution in $\mathsf{Chain}(R\text{-}\mathsf{Mod})$:

$$\ldots \longrightarrow Q^n \longrightarrow \ldots \longrightarrow Q^1 \longrightarrow Q^0 \xrightarrow{\epsilon} Q \to 0$$

   such that the following are *projective resolutions*:

   (a) $\ldots \longrightarrow Q^n_k \longrightarrow \ldots \longrightarrow Q^1_k \longrightarrow Q^0_k \longrightarrow Q_k \longrightarrow 0$ for all k,

   (b) $\cdots \to H_k(Q^n) \to \cdots \to H_k(Q^1) \to H_k(Q^0) \to H_k(Q) \to 0$ for all k,

   (c) $\cdots \to Z_k(Q^n) \to \cdots \to Z_k(Q^1) \to Z_k(Q^0) \to Z_k(Q) \to 0$ for all k,

2. **Hurewicz Theorem mod $\mathcal{C}$.** Let $\mathcal{C}$ denote one of the following classes of *abelian* groups: finitely generated groups, finite groups, p-groups, trivial groups.

   Let X be a simply-connected space such that $\pi_i(X)$ is in $\mathcal{C}$ for all $1 < i < n$. Conclude that $H_i(X)$ is in $\mathcal{C}$ for all $1 < i < n$ and that the Hurewicz map $\pi_n(X) \to H_n(X)$ is a mod $\mathcal{C}$ isomorphism, i.e. its kernel and cokernel lie in $\mathcal{C}$.

   Hint: The induction step is achieved by looking at fibrations

$$F \to X \to K(\pi_2(X), 2) \quad \text{and} \quad \Omega F \to PF \to F$$

   The main lemma is to show that for $A, B \in \mathcal{C}$, all groups $H_i(K(A, 2); B)$ are in $\mathcal{C}$. From this proof, you can derive axiomatic properties for a class $\mathcal{C}$ to have a Hurewicz theorem.

3. **Finiteness of homotopy groups of spheres.** Show that $\pi_k(S^n)$ is finite, except for $k = n$ and $(k, n) = (4r - 1, 2r)$. Show also that in these exceptional cases, $\dim_{\mathbb{Q}} \pi_k(S^n) \otimes \mathbb{Q} = 1$.

   Hint: Apply the Hurewicz theorem mod finite groups and the Leray-Serre spectral sequence with $\mathbb{Q}$-coefficients to a fibration $S^n \to K(\mathbb{Z}, n)$.

Please return problems in the discussion session on Friday, April 25.