

Homework 11, Homological Algebra, 253, Spring 2008

1. **Tic-Tac-Toe.** For a double complex $(D_{p,q}, d^v, d^h)$, $p, q \in \mathbb{Z}$, define

$$E_{p,q}^0 := D_{p,q}, \quad E_{p,q}^1 := H(D_{p,q}, d^v) \quad \text{and} \quad E_{p,q}^2 := H(E_{p,q}^1, [d^h])$$

as the iterated homology groups of $d_0 := d^v$ and $d_1 := [d^h]$.

(a) Consider the total complex $T_n = \bigoplus_p D_{p,n-p}$, $n \in \mathbb{Z}$, filtered by columns:

$$F^k T_n := \bigoplus_{p \leq k} D_{p,n-p}, \quad k \in \mathbb{Z}.$$

Show that the resulting spectral sequence coincides at its E^1 - and E^2 -terms with the data above.

(b) Show that the differentials $d_r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$ are derived from d^v, d^h by ‘tic-tac-toe’: Write $E_{p,q}^r = Z_{p,q}^r / B_{p,q}^r$ with $B_{p,q}^r \subseteq Z_{p,q}^r \subseteq D_{p,q}$ and find a representative $x \in Z_{p,q}^r$ of $[x] \in E_{p,q}^r$ such that the following process leads to $d_r[x] = [d^h(x_r)]$ for $r \geq 2$: Let $x_1 = x$ and find $x_i \in D_{*,*}$ such that

$$d^h(x_1) = d^v(x_2), d^h(x_2) = d^v(x_3), \dots, d^h(x_{r-1}) = d^v(x_r).$$

(c) Discuss the convergence of this spectral sequence.

2. **Rational Cohomology of Eilenberg-Mac Lane Spaces.** For an abelian group A and $n \geq 1$, let $K(A, n)$ be a CW-complex with a single homotopy group $\pi_n = A$. Consider cohomology with \mathbb{Q} -coefficients in the following:

(a) If A is finite, show that $\tilde{H}^*(K(A, n)) = 0$.

(b) If A is free abelian, show that $H^*(K(A, n))$ is the free $\mathbb{Z}/2$ -graded commutative \mathbb{Q} -algebra on A^* in degree n . Hint: The rank 1 case is an induction on n , where the Leray-Serre spectral sequence is used for the induction step. Then argue by the Künneth theorem for the general case.

(c) What happens if A is a \mathbb{Q} -vector space?

3. **Presentations of Quaternion Groups.** Consider the group presentations

$$G_n = (x, y \mid x^2 = y^n = (xy)^2) \quad \text{for } n \geq 1.$$

Show that G_n is isomorphic to the quaternion group Q_{4n} of order $4n$.

Hint: Divide out by the relation $x^2 = 1$ and show that this gives the dihedral group D_{2n} . Then compute the kernel of $G_n \rightarrow D_{2n}$, using the 5-term exact sequence in homology and homework problems 7-2(b) and 10-1.

PLEASE RETURN PROBLEMS IN THE DISCUSSION SESSION ON FRIDAY, APRIL 18.