# Homework 10, Homological Algebra, 253, Spring 2008

1. **A Serre Spectral Sequence.** Compute $H_2(D_{2n})$ for the dihedral group of order $2n$ and show that $D_{2n}$ has 4-periodic cohomology if and only if $n$ is odd.

2. **Filtered Abelian Groups.** Consider the category $\mathsf{AbF}$ of filtered abelian groups, i.e abelian groups $A$ together with an ascending filtration indexed by integers $n \in \mathbb{Z}$:
$$0 \subseteq \cdots \subseteq F_n A \subseteq F_{n+1} A \subseteq \cdots \subseteq A$$
The morphisms are homomorphisms that preserve the given filtrations.
   (a) Show that $\mathsf{AbF}$ is enriched over $(\mathsf{Ab}, \otimes)$ and has all coproducts.
   (b) Show that kernels and cokernels exist in $\mathsf{AbF}$.
   (c) Find a non-isomorphism in $\mathsf{AbF}$ with trivial kernel and cokernel. (This means that $\mathsf{AbF}$ is not an abelian category!)
   (d) Give a similar example in the category of topological abelian groups.

3. **Simplicial resolutions.** Let $F : \mathsf{D} \longrightarrow \mathsf{C}$ be a left adjoint to $U : \mathsf{C} \longrightarrow \mathsf{D}$ and denote by $\sigma$ (respectively $\delta$) the unit (respectively counit) of the adjunction. So $\sigma(d) \in \mathsf{D}(d, UF(d))$ corresponds to $id_{F(d)}$ and $\delta(c) \in \mathsf{C}(FU(c), c)$ to $id_{U(c)}$.
   (a) For any object $c \in \mathsf{C}$, show that the following formulas define a simplicial object $\perp_\bullet (c)$ in $\mathsf{C}$: Let $\perp_n (c) := (FU)^{n+1}(c)$ and
$$d_i := (FU)^i \left(\delta((FU)^{n-i}(c))\right) : (FU)^{n+1}(c) \longrightarrow (FU)^n(c),$$
$$s_i := (FU)^i F \left(\sigma(U(FU)^{n-i}(c))\right) : (FU)^{n+1}(c) \longrightarrow (FU)^{n+2}(c).$$
   (b) Show that $\delta(c)$ induces a simplicial map $\epsilon(c) : \perp_\bullet (c) \to c$, where the right hand side denotes the constant simplicial $\mathsf{C}$-object.
   (c) Assume that there is a functor $K : \mathsf{D} \to \mathsf{Set}$ such that $KU(\perp_\bullet (c))$ is Kan. Show that the augmentation $\epsilon(c)$ gives a weak equivalence $KU(\epsilon(c))$, i.e. it induces an isomorphism on all homotopy groups (that vanish above dimension zero for the constant functor).
   (d) Apply this to your favorite pair of adjoint functors and see what you get. For example, you could use $\mathsf{C} = R\text{-}\mathsf{Mod}$ or $\mathsf{Ring}$ and $\mathsf{D} = \mathsf{Ab}$ or $\mathsf{Set}$. If $\mathsf{C}$ happens to be an abelian category, one can apply the (normalized) chain complex to $\epsilon(c)$ and get all resolutions we have studied in class so far!

PLEASE RETURN PROBLEMS IN THE DISCUSSION SESSION ON FRIDAY, APRIL 11.