

Homework 1, Homological Algebra, 253, Spring 2008

1. **G -Modules and $\mathbb{Z}G$ -Modules.** Given a ring R , let R^* denote the group of units.

(a) Prove the following *adjunction formula*,

$$\text{Hom}_{\text{Ring}}(\mathbb{Z}G, R) \cong \text{Hom}_{\text{Group}}(G, R^*)$$

(b) Explain in what sense this isomorphism is *natural*.

(c) Use this to produce an equivalence of categories between G -modules and $\mathbb{Z}G$ -modules.

2. **The Augmentation Ideal.** For any group G we define the *augmentation map* to be the ring homomorphism $\varepsilon : \mathbb{Z}G \rightarrow \mathbb{Z}$ such that $\varepsilon(g) = 1$ for all $g \in G$. The kernel of ε is called the *augmentation ideal* of $\mathbb{Z}G$ and is denoted I or IG .

(a) Show that the elements $g - 1$ ($g \in G$, $g \neq 1$) form a basis for I as a \mathbb{Z} -module.

(b) If S is a set of generators for G , show that the elements $s - 1$ ($s \in S$) generate I as a left ideal.

(c) Conversely, if S is a set of elements of G such that the elements $s - 1$ ($s \in S$) generate I as a left ideal, show that S generates G .

(d) Show that G is a finitely generated group if and only if I is finitely generated as a left ideal.

3. Let H be a subgroup of G and let E be a set of representatives for the *right* cosets Hg . Show that $\mathbb{Z}G$, regarded as a left-module over its subring $\mathbb{Z}H$, is free with basis E .

PLEASE RETURN PROBLEM 1 IN THE DISCUSSION SESSION ON FRIDAY, FEB. 1.