The rational bordism ring

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1 Introduction

The (oriented) cobordism ring Ω^* was defined and studied by Thom. The aim of this short paper is to present the main results (most of them with proofs) that led to the description of this ring. The main reference is Milnor's book on Characteristic Classes [2].

The most effective tool for this problem is provided by the Pontrjagin classes, described in the first part. For any oriented real bundle, they live in the cohomology ring of the base space; one can define them in terms of the Chern classes of the complexified bundle. Pontrjagin classes are somewhat better than Stiefel-Whitney classes (they are well defined for cohomology with $\mathbf{Z}$ coefficients), but on the other hand they exist only in degrees of the form $4i$. One can then introduce the Pontrjagin numbers of a manifold, which are integers obtained by evaluating the fundamental class of the manifold on certain products of Pontrjagin classes of the tangent bundle.

We will prove that $\Omega^* \otimes \mathbf{Q}$ has the structure of a polynomial ring. One part of this, which amounts to saying that any manifold which is a boundary has Pontrjagin numbers equal to 0, is fairly easy; the other part involves finding a lower bound for the rank of Ω^n. We use transversality arguments to reduce this to a homotopy theory problem.

The end of the paper contains a brief outline of results continuing Thom's work and some applications, among which an outline of the ideas that led to Hirzebruch's signature theorem.

2 Pontrjagin classes and numbers

Throughout this paper all manifolds are assumed to be compact, oriented and differentiable unless otherwise stated.

To any real n-dimensional vector bundle $\pi : E \to B$ we can associate a complexified vector bundle (which we'll denote by $E_{\mathbf{C}}$) defined in the following way: the base space B is the same and the fibers over each point $b \in B$ are $\pi^{-1}(b) \oplus \pi^{-1}(b)$ with multiplication over $\mathbf{C}$ given (in each fiber) by $i(x, y) = (-y, x)$.

Lemma 2.1. If i is odd then $2c_i(E_{\mathbf{C}}) = 0$ where $c_i(E_{\mathbf{C}})$ is the i-th Chern class.

Proof: The bundle map $f : E_{\mathbf{C}} \to E_{\mathbf{C}}$ given by $f(x, y) = (x, -y)$ is an isomorphism $E_{\mathbf{C}} \cong \overline{E_{\mathbf{C}}}$ of complex bundles. To see this note that:

$$f[i(x, y)] = f(-y, x) = (-y, -x) = -i(x, -y)$$

Hence we get:

$$1 + c_1(E_{\mathbf{C}}) + c_2(E_{\mathbf{C}}) + \cdots = c(E_{\mathbf{C}}) = c(\overline{E_{\mathbf{C}}}) = 1 - c_1(E_{\mathbf{C}}) + c_2(E_{\mathbf{C}}) + \cdots$$

and the result follows.

Having convinced ourselves that the odd dimensional classes are not very interesting we can restrict our attention to the even dimensional ones.

Definition 2.2. *For any real n-bundle E we define the i-th Pontrjagin class $p_i(E)$ to be $(-1)^i c_{2i}(E_{\mathbf{C}}) \in H^{4i}(B)$. The total Pontrjagin class is defined to be*

$$p(E) = 1 + p_1(E) + \cdots + p_{[n/2]}(E) \in H^*(B).$$

The basic example we will need is the tangent bundle of the projective space $\mathbf{P}^n_{\mathbf{C}}$. This is already a complex bundle; to get its Pontrjagin class we have to regard it as a real bundle and then complexify it. This procedure will of course double its dimension. To do this in full generality denote, for any complex bundle E, by $E_{\mathbf{R}}$ the real $2n$-dimensional bundle obtained by neglecting the complex structure. We have the following:

Lemma 2.3. *Let E be a complex bundle. Then $E_{\mathbf{RC}} \cong E \oplus \overline{E}$.*

Proof: Each fiber of $E_{\mathbf{RC}}$ consists of pairs (x, y) where x, y are complex numbers. We will denote by ix multiplication by i defined in E and by $i \cdot (x, y)$ the multiplication in $E_{\mathbf{RC}}$. Let E_1, E_2 be the subspaces of E consisting of all pairs $(x, -ix)$ and (x, ix) respectively. The following easy to check facts prove the lemma:

- both E_1 and E_2 are invariant under $i\cdot$; for example $i \cdot (x, ix) = (-ix, x)$;
- $E_{\mathbf{RC}} = E_1 \oplus E_2$: $(x, y) = (\frac{x+iy}{2}, \frac{y-ix}{2}) + (\frac{x-iy}{2}, \frac{y+ix}{2})$;
- $E_1 \cong E$ via the map f which sends $(x, -ix)$ to x: $if((x, -ix)) = ix = f((ix, -x)) = f(i \cdot (x, -ix))$;
- similarly one can prove $E_2 \cong \overline{E}$.

Applying the Whitney sum formula for the Chern classes we get:

$$c(E_{\mathbf{RC}}) = c(E)c(\overline{E}) = (1 + c_1(E) + c_2(E) + \cdots)(1 - c_1(E) + c_2(E) - \cdots).$$

Notice that the minus signs cancel all the odd dimensional classes; at the end of the day we get:

$$1 - p_1(E) + p_2(E) - \cdots = \sum_{k,j} (-1)^j c_k(E) c_j(E).$$

Example 2.4. We already know that $c(T\mathbf{P}^n_{\mathbf{C}}) = (1+\alpha)^{n+1}$ where $\alpha \in H^2(\mathbf{P}^n_{\mathbf{C}})$; hence the above formula gives $p(T\mathbf{P}^n_{\mathbf{C}}) = (1 + \alpha^2)^{n+1}$.

Example 2.5. It is known that the top Chern class of an Euler bundle is equal to its Euler class e. A nice relation between the top Pontrjagin class of a bundle and its Euler class occurs for bundles of even dimension $2n$ (the only ones for which we expect such a thing if we look at dimensions). For these bundles we have $p_n = e^2$.

As a more complicated example, let's use the Pontrjagin classes to compute the cohomology ring of the oriented Grassmanian $\tilde{G}_n$, i.e. the set of oriented planes in $\mathbf{R}^\infty$. This can be endowed with a manifold structure in the same way as the usual Grassmanian G_n. There is an obvious 2 to 1 covering map $\tilde{G}_n \to G_n$ given by ignoring the orientation. It is sometimes useful to regard $\tilde{G}_n$ as the direct limit of $\tilde{G}_{n,k}$ of the Grassmanian of oriented n planes in $\mathbf{R}^{n+k}$ as $k \to \infty$. The reason for this is that cohomology commutes with direct limit. We'll denote by V_n the universal bundle over $\tilde{G}_n$, whose fiber over each point given by an n plane is precisely that plane.

Proposition 2.6. *$H^*(\tilde{G}_n; \mathbf{Q})$ is a polynomial ring generated by the classes $p_1(V_n), p_2(V_n), \dots p_{[n/2]}(V_n)$ if n is odd and by $p_1(V_n), \dots p_{[n/2]-1}(V_n), e(V_n)$ if n is even, where by e we denote the Euler class.*

Proof: Recall that for every oriented n plane bundle $\pi : E \to B$ we have the Gysin exact sequence:

$$\dots \to H^i(B) \overset{\cup e}{\to} H^{i+n}(B) \overset{\pi_0^*}{\to} H^{i+n}(E_0) \to \dots$$

where E_0 is E minus the 0 section and $\pi_0 = \pi|_{E_0}$. In our case $V^0_{n,k}$ is the space consisting of pairs $(H, \vec{v})$ where $\vec{v}$ is a nonzero vector in H. We can define a metric on $\mathbf{R}^{n+k}$, and use it to define a map $\rho : V^0_{n,k} \to \tilde{G}_{n-1,k+1}$ which sends the pair $(H, \vec{v})$ to the orthogonal complement of $\vec{v}$ in $\mathbf{R}^n$. This is a fiber bundle with fiber $\mathbf{R}^{k+1} - \{0\}$. For $i \le k$ the Gysin sequence of this bundle gives isomorphisms $\rho^* : H^i(\tilde{G}_{n-1,k+1}) \to H^i(V^0_{n,k})$.

We can now let $k \to \infty$ and write the Gysin sequence of the bundle $\pi : V_n \to \tilde{G}_n$ in the following way:

$$\dots \to H^i(\tilde{G}_n) \overset{\cup e}{\to} H^{i+n}(\tilde{G}_n) \overset{\rho^{*-1}\pi_0^*}{\to} H^{i+n}(\tilde{G}_{n-1}) \to \dots$$

We'll use this exact sequence to prove our claim by induction. Since $\tilde{G}_0$ is a point the starting step is trivial. The key observation here is that $\rho^{*-1}\pi_0^*$ carries the Pontrjagin classes of $\tilde{G}_n$ into those of $\tilde{G}_{n-1}$. To see this, one needs to use an alternative definition of the Chern classes given in [2]: the top Chern class of a bundle $\pi : E \to B$ is defined to be the Euler class, and the lower dimensional ones are defined to be $\pi_0^{*-1}(c_i)(E')$ where E' is the $n-1$ dimensional bundle over E_0 whose fiber at each point $(H, \vec{v})$ is the orthogonal complement of $\vec{v}$. It is not difficult to prove that these classes satisfy the properties that define Chern classes uniquely : naturality, Whitney sum formula and normalization (i.e. there is a distinguished line bundle over $\mathbf{P}^1_{\mathbf{C}}$ with nontrivial Chern class). This construction was done, of course, for complex bundles but since Pontrjagin classes are defined in terms of Chern classes, we can imitate it for real bundles. Once we believe this it's enough to look at the diagram below

$$\begin{array}{ccc} \pi_0^*(V_n') & \longrightarrow & V_{n-1} \\ \downarrow & & \downarrow \\ V_n^0 & \overset{\rho}{\longrightarrow} & \tilde{G}_{n-1} \\ {\scriptstyle \pi_0}\downarrow & & \\ \tilde{G}_n & & \end{array}$$

where the second horizontal map is ρ and the first column second map is π_0, to convince ourselves that $\rho^{*-1}\pi_0^*$ carries the Pontrjagin classes of $\tilde{G}_n$ into those of $\tilde{G}_{n-1}$. Going back to our induction, we have 2 cases to consider:

1. assume the result is true for $n = 2m - 1$; then $\rho^{*-1}\pi_0^*$ is surjective by what we've said above (and the induction hypothesis, which says that $H^*(\tilde{G}^{2m-1})$ is generated by $p_1, \dots p_{m-1}$). The Gysin sequence looks like this:

$$\dots \xrightarrow{0} H^i(\tilde{G}_{2m}) \xrightarrow{\cup e} H^{i+2m}(\tilde{G}_{2m}) \xrightarrow{\rho^{*-1}\pi_0^*} H^{i+2m}(\tilde{G}_{2m-1}) \xrightarrow{0} \dots$$

   It is now fairly easy to see that $H^*(\tilde{G}_{2m})$ is generated by $p_1, \dots p_{m-1}$: for a given cocycle $a \in H^*(\tilde{G}_{2m})$ we have $\rho^{*-1}\pi_0^*(a) = Q(p_1, \dots p_{m-1}) \in H^*(\tilde{G}_{2m-1})$; look now at $a - Q(p_1, \dots p_{m-1}) = a' \smile e \in H^*(\tilde{G}_{2m})$ by the exact sequence, where a' has smaller degree, and an induction on the degree solves the problem.

2. assume the claim to be true for $n = 2m$. Euler classes of odd dimensional bundles have order 2; since we are looking at the cohomology with rational coefficients, we have $e(V_{2m+1}) = 0$. Then the Gysin sequence looks like this:

$$\dots \to H^i(\tilde{G}_{2m+1}) \xrightarrow{0} H^{i+2m+1}(\tilde{G}_{2m+1}) \xrightarrow{\rho^{*-1}\pi_0^*} H^{i+2m+1}(\tilde{G}_{2m}) \to \dots$$

   We can thus regard $H^*(\tilde{G}_{2m+1})$ as a subring of $H^*(\tilde{G}_{2m})$, which certainly contains $p_1, \dots p_{m-1}$ and $p_m = e^2$. To prove their equality, one can use the information from the Gysin sequence and the analogous result for $H^*(\tilde{G}_{2m})$ and show they have equal ranks.

Closely related to Pontrjagin classes one can define the Pontrjagin numbers. Let M^n be a compact, oriented differentiable manifold with tangent bundle TM. Denote by $[M]$ its fundamental homology class. For a given integer k we will denote by $\Gamma(k)$ the set of partitions of k.

Definition 2.7. *Choose an integer k and a partition $i_1, \dots i_r$ of k. Then the integer $< p_{i_1}(TM) \cdots p_{i_r}(TM), [M] >$ is called the $(i_1 \dots i_r)$-th Pontrjagin number of M, where $<,>$ denotes the usual pairing between homology and cohomology classes. We denote this number by $p_{i_1} \cdots p_{i_r}(M)$.*

Notice that dimensional reasons force the Pontrjagin numbers to be 0 if $n \neq 4k$. An important class of nontrivial examples are again the projective spaces.

Example 2.8. Keeping the notations from example 2.4 we know that:

$$p_i(\mathbf{P}^{2n}_{\mathbf{C}}) = \binom{2n+1}{i}\alpha^{2i}$$

Hence:

$$p_{i_1} \cdots p_{i_r}(\mathbf{P}^{2n}_{\mathbf{C}}) = \binom{2n+1}{i_1} \cdots \binom{2n+1}{i_r}$$

if $i_1 + \cdots i_r = n$ and 0 otherwise.

3 Cobordism

We can endow the set of isomorphism classes of manifolds with a graded ring structure as follows:

- addition $M_1^n + M_2^n$ is the disjoint union $M_1^n \amalg M_2^n$;
- multiplication is the product $M_1^n \times M_2^m$;
- the opposite $-M^n$ is the same manifold with opposite orientation;
- the neutral element is the vacuous manifold.

This will become a ring after we mod out by the following equivalence relation:

Definition 3.1. *$M_1^n \sim M_2^n$ if $M_1^n - M_2^n$ is a boundary, i.e. if there exists a manifold B^{n+1} such that its boundary is $M_1^n - M_2^n$ and moreover the differential structure induced from B^{n+1} coincides with the one originally given.* We say that *M_1^n and M_2^n belong to the same cobordism class.*

We denote by Ω^n the ring of cobordism classes in dimension n and by Ω^* the total ring of cobordism classes. The Pontrjagin classes provide a very effective tool for studying this ring. The first argument for the above claim is the folowing:

Theorem 3.2. *If M^n is a boundary then the Pontrjagin numbers $p_{i_1} \cdots p_{i_r}(M)$ are 0.*

Proof: Assume M is the boundary of an $n+1$ dimensional manifold B. Note that for any $\alpha \in H^*(B)$ and any $\mu \in H_*(B)$ we have $< \alpha, \partial\mu >=< \delta(\alpha), \mu >$, where ∂ and δ are the chain and cochain differentials. We want to use this in the case where $\alpha = p_{i_1}(TM) \cdots p_{i_r}(TM)$ and $\mu = [B]$ (which implies $\partial\mu = [M]$). This way the left hand side of the above equality becomes a Pontrjagin number. We want to prove that the right hand side is 0.

For this look at the tangent bundle $TB_{|M}$ of B restricted to M. Choosing a Riemannian metric on TB there is a unique inward unit vector normal to M. This generates a trivial one dimensional bundle θ. Hence we see that $TB_{|M} = TM \oplus \theta$. Otherwise stated, the inclusion $i : M \to B$ is covered by a map of bundles $f : TM \oplus \theta \to TB$. Since θ is trivial and one dimensional we see that $i^*(p_k(TB)) = p_k(TM)$. But we also have the exact sequence:

$$H^n(B) \xrightarrow{i^*} H^n(M) \xrightarrow{\delta} H^{n+1}(B, M).$$

Hence we can write:

$$< \delta(p_{i_1}(TM) \cdots p_{i_r}(TM)), \mu >=< \delta(i^*(p_{i_1}(TB) \cdots p_{i_r}(TB))), \mu >= 0.$$

This ends the proof.

Hence for each element $i_1 \dots i_r \in \Gamma(k)$ we have a well defined morphism of groups

$$\Omega^{4k} \to \mathbf{Z} \qquad M^{4k} \mapsto p_{i_1} \cdots p_{i_r}(M).$$

A less obvious consequence is the following:

Corollary 3.3. For each partition $i_1 \dots i_r$ of k, the manifolds $\mathbf{P}_{\mathbf{C}}^{2i_1} \times \cdots \times \mathbf{P}_{\mathbf{C}}^{2i_r}$ represent linearly independent elements of the cobordism group Ω^{4k}.

Proof: We will just sketch the proof, refering the reader to [2] for details. According to the previous theorem, it is enough to prove that the square matrix with entries:

$$p_{j_1} \dots p_{j_s}[\mathbf{P}_{\mathbf{C}}^{2i_1} \times \cdots \times \mathbf{P}_{\mathbf{C}}^{2i_r}]$$

has maximal rank $|\Gamma(k)|$, where $j_1 \dots j_s$ and $i_1 \dots i_r$ range over all the elements in $\Gamma(k)$. Moreover one can replace the Pontrjagin numbers by linear combinations of them ; it turns out that one can find linear combinations $s_{j_1 \dots j_s}[M]$ behaving nicely with respect to products, namely satisfying:

$$s_{j_1 \dots j_s}[M_1 \times M_2 \cdots \times M_s] = \prod_{i=1}^{s} s_{j_i}[M_i];$$

Then, since we know the Pontrjagin numbers of the projective spaces, we can determine the numbers $s_i[\mathbf{P}_{\mathbf{C}}^{2i}]$ and prove that the matrix obtained this way is nonsingular (in fact it is triangular).

4 The main theorem

Given a vector bundle $E \to B$, we define its Thom space to be the one point compactification of E; we denote it by $M(E)$ and the point at infinity by t_0. We will often identify B with the subspace of E given by the 0 section of the bundle. Notice that if B is a finite complex then $M(E)$ is $n-1$ connected (n is the dimension of the fiber) because preimages of cells of B together with t_0 cover $M(E)$.

Lemma 4.1. If $E \to B$ is an oriented n dimensional vector bundle then $H^{n+k}(M(E), t_0) \cong H^k(B)$.

Proof: Both rings are isomorphic to $H^{n+k}(E, E-B)$: the first one via excision and the second one via the Thom isomorphism.

At this point we want to quote some transversality results which are useful for our purposes; first recall that:

Definition 4.2. *A differentiable map $f : M \to N$ is transverse regular on a manifold $N' \subset N$ if for every $y \in N'$ and every $x \in f^{-1}(y)$ the induced map Tf from the tangent space at x to the normal space of y: $T_xM \to T_yN \to T_yN/T_yN'$is surjective.*

If f is regular transverse on N' then the inverse image $f^{-1}(N')$ is a submanifold of M of the "right" dimension dim(M)- dim(N)+ dim(N'). The main tool we need is the following

Theorem 4.3. *Given a manifold N, a bundle $\pi : E \to B$ and a continuous map $f : N \to M(E)$, there exists a map $h : N \to M(E)$ which is homotopic to f, transverse regular on B and differentiable on $h^{-1}(E)$ (i.e. wherever differentiability makes sense).*

Proof: First replace f by a map g which is differentiable on $g^{-1}(E)$ and coincides with f on $f^{-1}(t_0)$ (any map can be perturbed to a differentiable map). Now we just need to deform g to a map transverse regular on B. Although it is intuitively clear that this should be possible, let's go into the technical details for once: first let's choose a open covering $\{B_j\}$ of B; denote by p_j the map $p_j : \pi^{-1}(B_j) \to \mathbf{R}^k$ given by trivializing and then projecting on the second coordinate. Also we can choose a finite open cover $U_1, \ldots, U_m$ of (the compact set) $g^{-1}(B)$ such that each U_i is diffeomorphic to an open set of $\mathbf{R}^n$ and such that each $g(U_i)$ is contained in some $\pi^{-1}(B_j)$. Refine this cover by choosing smaller open sets W_i, V_i such that the following conditions hold:

- $\overline{W_i} \subset V_i \subset \overline{V_i} \subset U_i$;
- $g^{-1}(B) \subset W := \cup W_i$.

The idea is now to change g by perturbing it on each V_i succesively such that after each step it becomes regular transverse on B over $\cup_{j=1}^{i-1}\overline{W_j}$; we can use induction: assume we have a map g_{i-1} regular transverse over $\cup_{j=1}^{i-1}\overline{W_j}$ and moreover satisfying $g^{-1}(B) \subset W$ (these conditions are obvious for $g_0 = g$).

Notice that transverse regularity of $g_{i-1}|U_i$ is the same as saying the composition $p_j g_{i-1} : U_i \to \mathbf{R}^k$ has 0 as regular value. So let's approximate $p_j g_{i-1}$ by a map g'_i such that the following hold:

1. $g'_i|\overline{W_i}$ has the origin as a regular value;
2. g'_i coincides with $p_j g_{i-1}$ outside V_i;
3. $g'_i|(\overline{W_1} \cup \cdots \cup \overline{W_{i-1}}) \cap U_i$ has the origin as regular value;
4. $0 \notin g'_i(U_i - W)$.

We can do this according to the 2 lemmata given below. Then we can define g_i by the conditions $\pi g_i(x) = \pi g(x)$ for all $x \in g^{-1}(E)$, $p_j g_i(x) = g'_i(x)$ for all $x \in U_i$ (we can do this because $p_j g_{i-1}(W_i)$ is determined by the projections π and p_j at every point) and $g_i(x) = g_{i-1}(x)$ for all $x \notin V_i$. To finish the proof, we can pick $h = g_m$ q.e.d.

Now let's state the facts needed to prove the existence of g'_i:

Lemma 4.4. For any differentiable map $f : U \to \mathbf{R}^k$, any compact $K \subset U$, any open V with $K \subset V \subset \overline{V} \subset U$ and for every $\epsilon > 0$ *there exists a differentiable map* $g : U \to \mathbf{R}^k$ *such that* $g|K$ *has the origin as absolute value,* $g|U - V = f|U - V$ *and* g *is arbitrarily close to* f *in the sense that it can be chosen such that the absolute value of the differences between all their partial derivatives at every point is bounded by* ϵ .

Sketch of proof: Consider a differentiable function $\lambda : U \to \mathbf{R}$ which takes the value 1 on K and 0 on $U - V$; for a given regular value $y \in \mathbf{R}^k$ of f the function $g = f - \lambda y$ satisfies the first 2 requirements of the lemma. But Sard's theorem assures us we can pick y sufficiently close to the origin, so that the third demand is also satisfied. Remark that "linearity in y" says that f and g are homotopic q.e.d.

Lemma 4.5. Let K be a compact subset of U and let $f : U \to \mathbf{R}^k$ be a differentiable map with 0 a regular value. Then there exists $\epsilon > 0$ such that for all $g : U \to \mathbf{R}^k$ satisfying

$$|f_i(x) - g_i(x)| < \epsilon \quad , \quad |\partial f_i(x)/\partial x_j - \partial g_i(x)/\partial x_j| < \epsilon$$

for all $x \in K$ for all possible i, j, then g also has 0 as regular value.

We have now developed enough technical tools to get closer to our final task:

Proposition 4.6. Let $f, g : N \to M(E)$ be homotopic, differentiable wherever possible, transverse regular on B. Then the oriented manifolds $f^{-1}(B)$ and $g^{-1}(B)$ belong to the same cobordism class.

Proof: Choose a homotopy H of f and g and extend this to a homotopy $H : N \times [0, 5] \to M(E)$ such that $H(x, t) = f(x)$ for $t \leq 2$, $H(x, t) = g(x)$ for $t \geq 3$ and H is differentiable on $H^{-1}(E)$. We can now imitate the proof of 4.3 to approximate the map H by a map H' which is transverse regular on B and which satisfies: $H'(x, t) = f(x)$ for $t \leq 1$ and $H'(x, t) = g(x)$ for $x \geq 4$. The reason we can do this is because we can choose (in the notations of 4.3) the open sets U_i in $N \times (1, 4)$ to cover the compact set $H^{-1}(B) \times [2, 3]$; perturbing the map H, we end up with a map H_n transverse regular over $N \times [2, 3]$. But now the assumption of transverse regularity on f and g assures us we can choose this perturbation "close enough" (see lemma 4.5) that the transverse regularity property is preserved on the rest of the domain. We can thus set $H' = H_n$; but then the manifold $H'^{-1}(B)$ has boundary diffeomorphic to $g^{-1}(B) - f^{-1}(B)$ q.e.d.

Now consider $V_{k,l}$ - the universal bundle of the space $\tilde{G}_{k,l}$ introduced in section 2. We have the following

Proposition 4.7. For any k, l there is a group morphism

$$\lambda : \pi_{n+k}(M(V_{k,l})) \to \Omega^n.$$

which is an isomorphism for k, l sufficiently large.

Proof: We will only prove surjectivity, because we won't need the injectivity here. According to theorem 4.3, any element $[f] \in \pi_{n+k}(M(V_{k,l}))$ can be represented by a map f transverse regular on $\tilde{G}_{k,l}$. We can then define $\lambda([f]) := f^{-1}(\tilde{G}_{k,l})$. This correspondence is well defined by proposition 4.6 and it is also clearly a morphism (recall that addition in Ω^n is disjoint union).

For the second part we will use the Whitney theorem which states that any compact and differentiable n-dimensional manifold can be embedded in $\mathbf{R}^{2n}$. Hence we can embed N in $\mathbf{R}^{n+k}$ for any $k \geq n$. Denote the normal bundle by ν and consider the map $g : \nu \to V_{k,l}$ which associates to a k dimensional normal plane the parralel plane through the origin (for this we assume $l \geq n$ and take a parallel plane in some privileged $\mathbf{R}^{n+k} \subset \mathbf{R}^{k+l}$. Also pick ϵ such that the map which sends each vector in the normal bundle of length less than ϵ to its endpoint is a diffeomorphism onto a neighborhood U of N. Denote by ν_ϵ the set of such vectors (of length less than ϵ). We can then define a map $f : \mathbf{R}^{n+k} \to M(\nu)$:

$$f(x) = t_o \qquad if \qquad x \notin U$$

$$f(endpoint \quad of \quad \vec{v}) = \frac{\vec{v}}{\epsilon - |\vec{v}|} \quad if \quad x \in U.$$

Note that this is transverse regular on N as there are vectors $\vec{v}$ pointing in any direction in the normal bundle. Extending g to the Thom space and looking at the composition

$$gf : \mathbf{R}^{n+k} \to M(V_{k,l})$$

we see that $f^{-1}g^{-1}(\tilde{G}_{k,l}) = N$ because $\tilde{G}_{k,l} = N$ corresponds to 0 vectors, whose endpoints are obviously the points of N. We can now replace $\mathbf{R}^{n+k}$ by its one point compactification S^{n+k} to conclude that $\lambda([fg]) = N$ q.e.d.

We can now state the main result of this paper:

Proposition 4.8. *The algebra $\Omega^* \otimes \mathbf{Q}$ has the structure of a polynomial ring generated by $\mathbf{P}^{2i}_{\mathbf{C}}$.*

Proof: First note that the (real) dimension of these projective spaces is always divisible by 4. Since tensoring by $\mathbf{Q}$ is the same as killing torsion what we actually want to prove is that the group Ω^n has rank $|\Gamma(n)|$ if $n = 4j$ and 0 otherwise. The rank is at least this one by corollary 3.3, which also assures us that the multiplicative structure is the right one (i.e. taking products of projective spaces we get linearly independent "monomials" in our ring).

For the opposite inequality we want to use proposition 4.7 to prove that $\pi_{n+k}(M(V_{k,l}))$ has rank $|\Gamma(n)|$ or 0 depending whether $n = 4j$ or not (again for k, l sufficiently large). But the Hurewicz map

$$\pi_{n+k}(M(V_{k,l})) \to H_{n+k}(M(V_{k,l}))$$

is an isomorphism modulo the category of finite abelian groups. This follows from a more general fact about finite $n-1$ connected complexes: the above map is an isomorphism modulo the category of finite abelian groups for all $k \leq n-2$. This means that the kernel and cokernel of this map are finite groups, which implies that the groups have the same rank (here we look at the homology with $\mathbf{Z}$ coefficients). Moreover the universal coefficients theorem implies that $H_{n+k}(M(V_{k,l}))$ and $H^{n+k}(M(V_{k,l}))$ have the same rank ($M(V_{k,l})$ is a finite complex). The latter group is isomorphic by lemma 4.1 to $H^n(\tilde{G}_{k,l})$, whose cohomology we computed in proposition 2.6. It is a polynomial ring in the classes p_i, living in dimensions $4i$, hence for large k, l its rank is precisely the announced one q.e.d.

5 Related results and applications

Most of the facts presented in this paper were proved by Thom. However he also stated 2 conjectures about the cobordism ring. They were proved by Wall in his 1960 paper [3] (statements 2, 3 on the list below). Now even the torsion part of Ω^* has a complete description. We summarize here the main results, proved or at least outlined in [3]:

1. Ω^* has no odd torsion.

2. Ω^* contains no elements of order 4; hence Ω^n is a (finite) direct sum of $\mathbf{Z}$'s and $\mathbf{Z}/2$'s for every n.

3. 2 manifolds belong to the same cobordism class if and only if they have the same Stiefel and Pontrjagin numbers (Stiefel numbers are defined the

same way as Pontrjagin numbers, using Stiefel classes instead of Pontrjagin classes).

4. as generators for the torsion part one can pick degree (1,1) hypersurfaces in a product of projective spaces; e.g. $\Omega^5 = \mathbf{Z}/2$ generated by the degree (1,1) hypersurface $Y \subset \mathbf{P}^4_{\mathbf{R}} \times \mathbf{P}^2_{\mathbf{R}}$.

5. the only n for which $\Omega^n = 0$ are 1,2 and 3.

6. one can consider the nonoriented cobordism group Ω_n; there is an obvious map $\Omega^n \xrightarrow{r} \Omega_n$ given by ignoring orientation. There is an exact sequence:

$$\Omega^n \xrightarrow{2} \Omega^n \xrightarrow{r} \Omega_n.$$

Wall used these facts to deduce 2 nice properties:

Proposition 5.1. The product of an orientable and nonorientable class in $\Omega = \oplus_n \Omega_n$ is a nonorientable class.

Proposition 5.2. The square of any manifold is cobordant with an oriented manifold.

A more intricate problem where the description of the cobordism ring can be used is related to the signature of a manifold. This was originally done by Hirzebruch, but it is also presented in [2]. For a manifold M of dimension $4n$ one can look at the symmetric pairing :

$$H^{2n}(M) \times H^{2n}(M) \to \mathbf{Z}$$

given by cuping and then evaluating on the fundamental class.

Definition 5.3. *The signature $\sigma(M)$ is defined to be the signature of the above pairing, i.e. the dimension of the positive eigenspace minus the dimension of the negative one.*

It is clear that $\sigma(M)$ is a homotopy invariant. One can prove that:

Proposition 5.4. σ defines a ring homomorphism from Ω^* to $\mathbf{Z}$.

One can extend this morphism to a morphism $\Omega^* \otimes \mathbf{Q} \to \mathbf{Q}$. Hirzebruch proved that every morphism like that is given by evaluating some polynomial in the Pontrjagin classes (of the tangent bundle) on the fundamental class. Since we know the signature of the generators of $\Omega^* \otimes \mathbf{Q}$ (complex projective spaces have signature 1) we can compute the polynomial for this particular case. This is usually denoted by L and called the L-genus of a manifold. It follows that L is also homotopy invariant (this is in itself a very interesting fact because the Pontrjagin classes are not homotopy invariant). In particular we can determine the signature of a manifold if we know its Pontrjagin numbers.

References

[1] Hatcher A., Algebraic Topology, Cambridge University Press 2002;

[2] Milnor J., Lectures on Characteristic Classes, Princeton University 1976;

[3] Wall C.T.C., Determination of the Cobordism Ring, Annals of Math. 72 (1960) 292-231.