FREE ACTIONS OF GROUPS THROUGH COHOMOLOGY

Abstract. It is a classic result of Milnor that the symmetric group on three elements does not act freely on the 3-sphere. This will be shown, as well as an explicit construction of a CW-complex homotopy equivalent to S^3 that S_3 does act freely on. Through the techniques of periodic group cohomology, we will attempt to classify some of the finite groups which act freely on a complex homotopy equivalent to a sphere.

1. Introduction

While determining homotopy equivalence of topological spaces is important, for a great deal of work it is necessary to distinguish homotopy equivalence from homeomorphism. How can we make such a distinction, when homotopy and homology groups cannot detect a homeomorphism? By combining results of Milnor and Swan on the actions of groups on spaces, we are given some insight as to how making such distinctions might be possible for the sphere. We show the existence of a space that is homotopy equivalent to S^3, but not homeomorphic, by demonstrating that it admits a free action by S_3 and S^3 does not. Regarding S_3 as a cyclic group extension, this theory will be extrapolated, as we use group cohomology to determine which cyclic extensions act freely on homotopy spheres. I would like to thank Peter Teichner for proposing the problem that is tackled in this paper, as well as setting up the framework for how to approach and solve it.

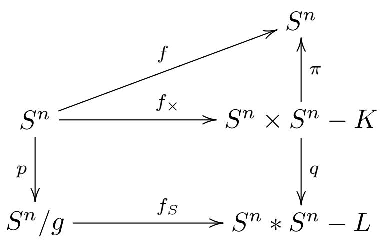
2. Actions of S_3 on S^3 and its Homotopy Equivalents

2.1. **S_3 does not act freely on S^3.** In order to see that there is no free action of the symmetric group on three elements, S_3, on the 3-sphere (and in fact no symmetric group S_n for $n \geq 3$) we observe that S_3 contains an element of order 2, namely the cycle (12), but the center is trivial. We now state and prove the following result of Milnor [4] in its classic form, with which we will see that these properties of S_3 imply that a free action is impossible on any sphere.

Milnor (1957) 2.1. *Suppose $g : S^n \to S^n$ is a map with $g^2 = Id_{S^n}$ such that g has no fixed points. Then for all $f : S^n \to S^n$ with odd degree, there is an x in S^n such that $g(f(x)) = f(g(x))$.* (The original proof is for any manifold equivalent to the n-sphere under mod 2 homology, but since we are showing S_3 does not act freely on the 3-sphere, we may weaken the statement.)

Proof. We will argue by contradiction. Assume that no such x existed. We define the following sets and maps to construct a commutative diagram. First, take K to be $\{(x, g(x)) : x \in S^n\} \subseteq S^n \times S^n$. Now let $S^n * S^n$ be the symmetric product of two copies of S^n, $(S^n \times S^n)/\sim$, where $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 = x_2, y_1 = y_2$ or $x_1 = y_2, y_1 = x_2)$, with L composed of the points $\{x, g(x)\} \in S^n * S^n$. We take the identification maps $S^n \to S^n/g$ and $S^n \times S^n - K \to S^n * S^n - L$ to be p and q respectively. By defining the maps $f_\times : S^n \to S^n \times S^n - K$ by $x \mapsto (f(x), f(g(x)))$,

$f_S : S^n/g \to S^n * S^n - L$ by $x \mapsto \{f(x), f(g(x))\}$, and $\pi : S^n \times S^n - K \to S^n$ by the projection map $(x_1, x_2) \mapsto x_1$, we obtain the commutative diagram below:



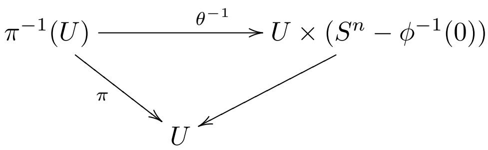
We now approach this proof in two steps. First we show that the induced homomorphism, $q_* : H_*(S^n \times S^n - K; \mathbb{Z}_2) \to H_*(S^n * S^n - L; \mathbb{Z}_2)$ is an isomorphism. (From now on we will assume coefficients in $\mathbb{Z}_2$.) This fact is then used to show that the induced homomorphism $f_* : H_*(S^n) \to H_*(S^n)$ is the zero homomorphism, contradicting f having odd degree. The following claim about the diagram will help to tackle the first step.

Claim: $\pi : S^n \times S^n - K \to S^n$ is a fiber bundle, with fiber $S^n - g(x)$ over each x in S^n.

Proof: We must show the local triviality condition. We choose a cell structure on S^n consisting of a 0-cell and an n-cell. Let e^n be the n-cell, and let x be in this cell. We choose an open neighborhood U containing x, inside of e^n, such that $g(U)$ is homeomorphic to the unit ball in $\mathbb{R}^n$, P. We denote the interior of P by P°. Pick a homeomorphism of $g(U)$ onto P, ϕ. We now define another map, ψ, from $P^\circ \times P$ to P, by taking a pair (s,t) to $t + (1 - \|t\|)s$. If we fix an s, it is easy to see that the map $\psi(s, \cdot)$ is a homeomorphism from P to itself, mapping 0 to s and is the identity map on the boundary. Finally, define a third map, θ from $U \times (S^n - \phi^{-1}(0))$ to $\pi^{-1}(U)$, by

$$\theta(s,t) = (s,t) \text{ for } y \notin g(P^\circ) \quad \theta(s,t) = (s, \phi^{-1}\psi(\phi(g(x)), \phi(y))) \text{ for } y \in g(P).$$

That θ is a homeomorphism follows from the fact that ϕ, ψ are homeomorphisms. By taking θ^{-1} with the neighborhood U, we have the diagram



which we see to be commutative. Therefore, the local triviality condition is satisfied.

Now since $S^n - \{y\} \simeq \mathbb{R}^n$, each fiber in the bundle is clearly contractible. Observe that this is not necessarily true for a space homotopy equivalent to S^n, so this method does not necessarily inhibit S_3 from acting on a homotopy sphere. Therefore, π_* induces the isomorphism $H_*(S^n \times S^n - K) \cong H_*(S^n)$ since it is just collapsing each copy of $\mathbb{R}^n$ to the point it is based at, which is clearly a homotopy equivalence from $S^n \times S^n - K$ to S^n. Because g has no fixed points, we see that the diagonal map from S^n into $S^n \times S^n - K$, with image D, is a section. The fibers over S^n from π are contractible, so we have that D is a deformation retract of $S^n \times S^n - K$. Therefore, $\tilde{H}_*(S^n \times S^n - K, D)$ is trivial. This implies that the inclusion map of D into $S^n \times S^n - K$ must be an isomorphism on the homology groups.

We now look at the map $q : S^n \times S^n - K \to S^n * S^n - L$ paired with the restriction from D to $q(D)$. Observe that any point $\{x, y\} \in (S^n * S^n - L) - q(D)$ is the image of both points (x, y) and (y, x) under q from $S^n \times S^n - K$. This tells us that off of D, $S^n \times S^n - K$ is a two-sheeted cover of $S^n * S^n - L$. It can be seen that if D was a two-sheeted cover over $q(D)$, then q would be a two-sheeted cover over all of $S^n * S^n - L$, and we would be able to apply the Gysin sequence to this covering. This is because a two sheeted cover of a total space, E, over a base space, B, corresponds to a fibration $S^0 \to E \to B$, so the sequence may be applied [3]. We solve this problem by taking fibrations that have a Gysin sequence, whose direct limit is the relative fibration above, which will allow us to use the sequence. We let N_D be a neighborhood of D such that $(x, y) \in N_D$ iff $(y, x) \in N_D$. This will be called a symmetric neighborhood of D. Now we use the following diagram

$$\begin{array}{ccccc}
(S^n \times S^n - K - D, N_D - D) & \longrightarrow & (S^n \times S^n - K, N_D) & \longleftarrow & (S^n \times S^n - K, D) \\
\downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle q} \\
(S^n * S^n - L - q(D), q(N_D) - q(D)) & \longrightarrow & (S^n * S^n - L, q(N_D)) & \longleftarrow & (S^n * S^n - L, q(D))
\end{array}$$

Since we remove the set D where the map is no longer a two-sheeted covering, we see that q_1 is a two-sheeted cover so we may apply the Gysin sequence to this fibration. By excision, we may apply the Gysin sequence to the map q_2 as well. If we examine all symmetric neighborhoods of D, N_D, then $H^*((S^n \times S^n - K)/D)$ is the direct limit over all such N_D of $H^*((S^n \times S^n - K)/N_D)$. By the same argument, we see that $H^*((S^n * S^n - L)/q(D))$ is the direct limit of the $H^*((S^n * S^n - L)/q(N_D))$. If we now take the direct limit of the exact sequences of the Gysin sequence, formed by the q_2 for each symmetric neighborhood N_D, we obtain an exact sequence. This gives us the Gysin sequence we initially tried to obtain for q.

This sequence will be:

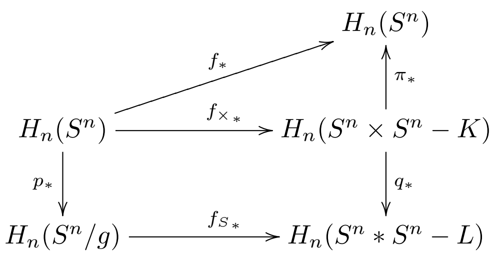
$$\cdots \to H^k(S^n \times S^n - K, D) \to H^k(S^n * S^n - L, q(D))$$
$$\to H^{k+1}(S^n * S^n - L, q(D)) \to H^{k+1}(S^n \times S^n - K, D) \to \dots$$

From our previous calculations, we were able to show that $H^k(S^n \times S^n - K, D) = 0$ for $k > 0$, since all of $\tilde{H}_*$ was trivial. Therefore, since $H^0(S^n * S^n - L, q(D))$ is zero, we easily see that $H^k(S^n * S^n - L, q(D))$ is 0 for all $k > 0$. Since the entire cohomology ring is trivial, we have that $H_*((S^n * S^n - L)/q(D)) = 0$. This tells us that the inclusion map from $q(D)$ into $S^n * S^n - L$ induces an isomorphism between $H_*(q(D))$ and $H_*(S^n * S^n - L)$. Using the analogous isomorphism from above, we have the commutative diagram

$$\begin{array}{ccc}
H_*(S^n \times S^n - K) & \xrightarrow{q} & H_*(S^n * S^n - L) \\
\cong\downarrow & & \downarrow\cong \\
H_*(D) & \xrightarrow[\cong]{q|_D} & H_*(q(D))
\end{array}$$

We now observe that while q is a two-sheeted cover of $(S^n \times S^n - K) - D$ over $(S^n * S^n - L) - q(D)$, D is actually a one-sheeted cover of $q(D)$. Moreover, $q|_D$ is a homeomorphism from D to $q(D)$. This is because q maps D injectively, since the projection map of $S^n \times S^n - K$ onto $S^n * S^n - L$ takes any point (x, y) to the element $\{x, y\}$. But since D is the diagonal, $t \in D$ implies $t = (x, x)$, so $q(t) = x$, so only the point t could have image x. Since D compact, $q(D)$ Hausdorff, we have the result. We have seen each map to be an isomorphism, except for $q_* : H_*(S^n \times S^n - K) \to H_*(S^n * S^n - L)$. Since this diagram commutes, we must have that q_* is an isomorphism as well, which proves our claim.

With this result, the proof will be easily completed. By naturality of homology, the first commutative diagram of spaces has a similar diagram of induced maps on H_*.



We first show the map $p_* : H_n(S^n) \to H_n(S^n/g)$ is the trivial map. This follows from the fact that g has degree 2. We can thus factor the map by $H_n(S^n) \xrightarrow{g_*} H_n(S^n) \xrightarrow{p_*} H_n(S^n/g)$, since $p(g(x)) = p(x)$. But since g_* is multiplication by 2, this must be the 0 map, since we recall that we are working with $\mathbb{Z}_2$ coefficients. Therefore, p_* is the 0 map. By looking at the bottom square, since q_* is an isomorphism, $f_{\times *}$ must be the 0 map, by commutativity. This finally gives us that f_* must be 0, contradicting f having odd degree. □

We finally are able to prove Milnor's theorem that gives us a powerful necessary condition for a finite group to act freely on S^n.

Theorem 1. *Let G act freely on S^n. If x is an element of G with order 2, then x is in the center of G.* (Again we observe this holds for any manifold equivalent to S^n by mod 2 homology.)

Proof. Choose y in G. The degree of y must be 1 or -1, since by Hatcher [3], these are the only possible degrees that do not have fixed points. If x has order two, then by our theorem, we have that there exists $s \in S^n$ such that $xy(s) = yx(s)$. However, we know that the action is free, so $xy = yx$. Therefore, x must be in the center of G. □

This immediately gives us our result for S_3 due to the properties discussed earlier. While it cannot act freely on S^3, or any other sphere, in the next section we will be able to show that S_3 does act freely on a CW-complex homotopy equivalent to S^3. While the methods used to show Milnor's theorem were purely topological, Swan's approach to group actions on spaces was almost completely based on group cohomology, through the use of cohomological period.

2.2. **A CW-Complex on which S_3 Acts Freely.** Although we found above that S^3 does not admit a free action from S_3, we do not give up hope. We now prove the existence of a CW-complex homotopy equivalent to the 3-sphere that S_3 acts freely on.

Theorem 2. *The symmetric group on three elements, S_3, acts freely on a CW-complex that is homotopy equivalent to the 3-sphere.*

In order to prove this theorem, we must use the methods of group cohomology. Therefore, a brief introduction to the subject will be given; proofs will be omitted since this is not the focus of the paper. Given a group G, let $K(G,1)$ be the Eilenberg-Maclane space for G,1; this is a CW complex with $\pi_1 \cong G$ and all higher homotopy groups trivial. Recall that $K(G,1)$ is unique up to homotopy equivalence. Thus, the cohomology ring of a group, G, with $\mathbb{Z}$ coefficients, can be defined as

$$H^*(G;\mathbb{Z}) = H^*(K(G,1);\mathbb{Z}).$$

It is also important to know the homology of a group as well, as this will be used later in calculating cohomological period later on. This is exactly the same definition as for cohomology:

$$H_*(G;\mathbb{Z}) = H_*(K(G,1);\mathbb{Z}).$$

In order to discuss the main property that is directly linked to free actions on homotopy spheres, we must present another cohomology theory, Tate cohomology. This ring, $\hat{H}^*$, contains the information of both homology and cohomology of the group. We define it for all $n \in \mathbb{Z}$ such that $n \neq -1, 0$ as follows:

$$\hat{H}^n(G;\mathbb{Z}) = \begin{cases} H^n(G;\mathbb{Z}) & \text{for } n \geq 1 \\ H_{-n-1}(G;\mathbb{Z}) & \text{for } n \leq -2. \end{cases}$$

We omit the Tate cohomology groups for $n = -1, 0$, since these have complicated definitions and will not make use of them in the proof. (The reader is referred to [2] for more details on Tate cohomology). This now allows us to present the following definition:

Definition 1. *Suppose there exists some $u \in \hat{H}^d(G;\mathbb{Z})$ such that $d \neq 0$ and u is invertible in $\hat{H}^*$. We then say that G has periodic cohomology, where the least such d is called the period of G.*

It is a well know fact of group cohomology that if a group acts freely on a homotopy sphere of dimension $2k-1$, the group has periodic cohomology $2k$. However, a theorem of Swan [5] shows the converse to hold as well. He does this by first showing that a group with periodic cohomology will admit a free resolution that is periodic. This resolution is then used to construct a CW-complex for the group, say G, to act freely on. Swan chooses a complex with $\pi_1 \cong G$, and looks at its universal cover. By attaching the correct cells, this complex is seen to admit a free action from the group and have the same homology groups as a sphere. We will show that S_3 has group cohomology of period 4, so the result will follow. The reader is referred to [5] for both an explicit construction of the CW-complex that S_3 acts freely on, as well as the general proof of existence for such complexes. We observe that by Milnor's theorem, this CW-complex cannot be a manifold.

We observe that the map $x \mapsto u \smile x$ induces an isomorphism from $\hat{H}^k(G;\mathbb{Z})$ onto $\hat{H}^{k+d}(G;\mathbb{Z})$ for all $k, k+d \neq 0$. The following theorem of Brown [2] immediately shows that S_3 has periodic cohomology, and will indicate how to go about finding the period.

Theorem 3. *Given any group G, the following are equivalent:*
(1) G has periodic cohomology
(2) All abelian subgroups of G are cyclic
(3) There exists $d \neq 0$ such that $\hat{H}^d(G;\mathbb{Z}) \cong \mathbb{Z}/|G|$, where $|G|$ is the order of G.

This theorem also tells us that the smallest d satisfying (3) is the period of $\hat{H}^*$. Since S_3 has order 6, the only possible subgroups it can have must be of either order 2 or 3. Since the groups of orders 2 and 3 are uniquely $\mathbb{Z}/2$ and $\mathbb{Z}/3$ respectively, the criterion for S_3 to have periodic cohomology holds. We therefore show that the smallest $n > 0$ s.t. $H^n(S_3;\mathbb{Z}) \cong \mathbb{Z}/6$ is 4, since $\hat{H}^n = H^n$ for $n > 0$. In order to see this, we must show the first three positive cohomology groups to be different from $\mathbb{Z}/6$ and that $H^4 \cong \mathbb{Z}/6$.

This will be done by proving two results from group cohomology [2] that will immediately show that the first three non-zero cohomology groups do not contain a unit.

Theorem 4. *If G has periodic cohomology d, then d is even.*

Proof. First, suppose G has order 2. From [2], we have that $H^k(\mathbb{Z}/2;\mathbb{Z})$ is trivial for k odd. Now suppose that $|G| > 2$ and the proposition does not hold. By definition, there exists some invertible element $\alpha \in \hat{H}^d(G;\mathbb{Z})$ such that there is an isomorphism $\beta \mapsto \alpha \smile \beta$, for all $\beta \in \hat{H}^*(G;\mathbb{Z})$. Therefore, we have that $\alpha \smile \alpha$ is an element of $\hat{H}^{2d}(G;\mathbb{Z})$. By anticommutativity of the cup product, we see $\alpha \smile \alpha = (-1)^{d^2}(\alpha \smile \alpha)$. Thus, $\alpha \mapsto 0$ under this map. This contradicts cupping with α being an isomorphism. □

This tells us that H^1 and H^3 cannot be $\mathbb{Z}/6$. Therefore only H^2 remains, but as the following theorem shows, this cannot occur unless $G \cong \mathbb{Z}/6$.

Theorem 5. *Let G be a finite group. G has cohomology period 2 if and only if G is cyclic.*

Proof. Observe that if G has periodic cohomology 2, then $\hat{H}^2(G) \cong \hat{H}^{-2}(G) = H_1(G)$, so $H_1(G) \cong \mathbb{Z}/|G|$. However, $H_1(G)$ is isomorphic to G mod its commutator subgroup, since $H_1(K(G,1);\mathbb{Z}) \cong \pi_1(K(G,1))_{ab}$, so this implies G is abelian. However, we require that all abelian subgroups of G be cyclic, so G itself must be cyclic.

Now if G is cyclic, then we have $\hat{H}^{-2}(G) \cong H_1(G) \cong G_{ab} \cong G \cong \mathbb{Z}/|G|$. Therefore, by Theorem 3 G has period 2, since an invertible element in H^{-2} clearly has inverse in H^2. □

The final step remaining is to calculate $H^4(S_3,\mathbb{Z})$. This will be done by looking at all of the p-primary components of H^4 and summing over them. For any finite group G, we see that $H^n(G;\mathbb{Z}) = \bigoplus_p H^n(G;\mathbb{Z})_p$, where $H^n(G;\mathbb{Z})_p$ represents the p-primary component of H^n, for each prime p dividing $|G|$.

By theorems of [2] and Adem & Milgram [1], we see that if H is a normal subgroup of a finite group with periodic cohomology, G, then $H^n(G;\mathbb{Z})_p \cong H^n(H;\mathbb{Z})^{G/H}$, where this is the fixed point set of $H^n(H;\mathbb{Z})$ under the action of G/H. In addition, if K is the 2-Sylow subgroup of G and $K \cong \mathbb{Z}/2^k$, then $H^n(G;\mathbb{Z})_2 \cong H^n(K;\mathbb{Z})$. Therefore we have

$$H^4(S_3,\mathbb{Z}) \cong H^4(\mathbb{Z}/2,\mathbb{Z}) \oplus H^4(\mathbb{Z}/3,\mathbb{Z})^{\mathbb{Z}/2}.$$

Since $\mathbb{Z}/2$ is cyclic, $H^4(\mathbb{Z}/2,\mathbb{Z}) \cong \mathbb{Z}/2$ by our previous theorem.

It is easy to see the following claim will now complete the proof that S_3 has cohomology period 4, and thus acts freely on a homotopy sphere of dimension 3.

Claim 1. $H^4(\mathbb{Z}/3,\mathbb{Z})^{\mathbb{Z}/2} \cong \mathbb{Z}/3$

Proof. We recall that S_3 can be represented as the semi-direct product of $\mathbb{Z}/3$ with $\mathbb{Z}/2$, where the homomorphism from $\mathbb{Z}/2$ into $\mathrm{Aut}(\mathbb{Z}/3)$ is given by taking 0 to the identity map and 1 to the automorphism $x \mapsto 2x$. Since the action $\mathbb{Z}/2$ induces on $\mathbb{Z}/3$ is the action $\mathbb{Z}/2$ induces on $H_1(\mathbb{Z}/3;\mathbb{Z})$, we see by the Universal Coefficient Theorem [3] that it must give the same action on $H^2(\mathbb{Z}/3)$. Now, by an exercise in [2], we see that the ring structure on $H^*(\mathbb{Z}/3;\mathbb{Z})$ tells us that the generator of $H^{2k}(\mathbb{Z}/3;\mathbb{Z})$ is just k cup products of the generator of $H^2(\mathbb{Z}/3;\mathbb{Z})$ with itself. Therefore, if the action induced on $H^2(\mathbb{Z}/3;\mathbb{Z})$ is multiplication by 2, we see that the action on $H^4(\mathbb{Z}/3;\mathbb{Z})$ is simply multiplication by 4. However, since $H^4(\mathbb{Z}/3;\mathbb{Z}) \cong \mathbb{Z}/3$ and $2^2 \equiv 1 \pmod 3$, we see that this is really just multiplication by 1, or in other terms, the trivial action on H^4. This tells us the fixed point set is all of $H^4(\mathbb{Z}/3;\mathbb{Z})$, so the proof is complete. □

We observe that in the proof of the final claim, we made use of the fact that S_3 is an extension of the cyclic groups $\mathbb{Z}/2$ and $\mathbb{Z}/3$. This leads us to consider the question, which group extensions may act freely on homotopy spheres? While this general question may be difficult to solve, the case for a certain set of cyclic extensions is presented.

3. Classifying Cyclic Group Extensions which Act Freely on Homotopy Spheres

In order to generalize the case of S_3, we now consider the cyclic extensions that act freely on a CW-complex homotopy equivalent to S^{2k-1}. Swan's theorem tells us that this is the case if and only if the group has cohomology of period $2k$. In this section, we attempt to calculate the periods for certain extensions of cyclic groups. Solving this problem will also ultimately lead to a means of calculating the group structure of the cohomology rings of these cyclic extensions.

3.1. **Period of a Cyclic Extension.**

If we choose an exact sequence of groups

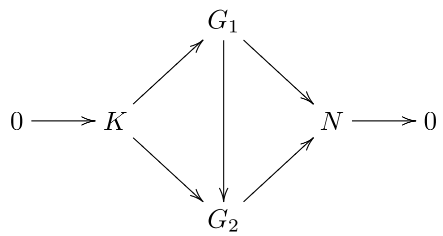
$$0 \to \mathbb{Z}/m \to G \to \mathbb{Z}/n \to 0$$

such that m and n are relatively prime, then we want to see the possible cohomology rings of G, and thus whether it can act freely on a homotopy sphere or not. The following theorem will greatly help us by narrowing down the possible choices of what G can be.

Theorem 6. *Let G be a finite group that satisfies the exact sequence given above. Then G must be of the form $\mathbb{Z}/m \rtimes \mathbb{Z}/n$.*

This can easily be seen as a consequence of the following sequence of theorems; we omit proofs since these are not needed for anything outside of showing the exact sequence splits. Again, the reader is referred to [1] for further details on these theorems.

We first say that two extensions, $0 \to K \to G_1 \to N \to 0$, $0 \to K \to G_2 \to N \to 0$ are equivalent if there is a map between G_1 and G_2 such that



forms a commutative diagram.

Theorem 7. *There is a bijection between $H^2(\mathbb{Z}/m, \mathbb{Z}/n)$ and the set of equivalence classes of extensions of $\mathbb{Z}/m$ corresponding to an action of it on $\mathbb{Z}/n$.*

Theorem 8. *Let G and H be finite groups. If $|G|$ is invertible in H, then $H^n(G, H) = 0$ for $n > 0$.*

Since m and n are relatively prime, the order of $\mathbb{Z}/m$ is invertible in $\mathbb{Z}/n$, so by combination of the two theorems, there can only be one extension up to equivalency. Since we know that the semi-direct product can represent an action of $\mathbb{Z}/m$ on $\mathbb{Z}/n$ forming an exact sequence $0 \to \mathbb{Z}/m \to \mathbb{Z}/m \rtimes \mathbb{Z}/n \to \mathbb{Z}/n \to 0$, it is clear that any such extension must be isomorphic to $\mathbb{Z}/m \rtimes \mathbb{Z}/n$.

Here $\mathbb{Z}/n$ maps into $\mathrm{Aut}(\mathbb{Z}/m)$ by taking $1 \in \mathbb{Z}/n$ to the automorphism determined by $1 \mapsto r$, where r, m are relatively prime. We observe that a homomorphism from $\mathbb{Z}/n$ to $\mathrm{Aut}(\mathbb{Z}/m)$ exists if and only $r^n \equiv 1 \pmod{m}$. Because $\mathbb{Z}/m \lhd G$, [1] tells us that there is a Serre spectral sequence with terms $H^i(\mathbb{Z}/n; H^j(\mathbb{Z}/m; \mathbb{Z}))$ for $E_2^{i,j}$ that converge to $H^*(G; \mathbb{Z})$, where the coefficients for $H^j(\mathbb{Z}/m; \mathbb{Z})$ are twisted.

We use this spectral sequence to prove our cohomology classification, which will tell us the period of cohomology for such a cyclic extension.

Theorem 9. *Let G be a finite group of the form given above. Then G has cohomology period $2j$ where j is the smallest positive integer such that $r^j \equiv 1 \pmod{m}$.*

The use of spectral sequences here is very effective, since we will see that each differential is 0 and all of the entries where neither p nor q are 0 must be trivial. Therefore the E_∞ page will be the E_2 page, and the only nontrivial groups will be in the first column or row. From this, the cohomology group which we are looking to be isomorphic to $\mathbb{Z}/|G|$ can easily be read from the sequence, and the period will be calculated.

We therefore show the above by a series of simple steps.

Step 1. *$E_2^{i,j} = 0$ when i, j are both nonzero.*

Proof. By universal coefficients we have the exact sequence

$$0 \to \mathrm{Ext}(H_{i-1}(\mathbb{Z}/n), H^j(\mathbb{Z}/m)) \to E_2^{i,j} \to \mathrm{Hom}(H_i(\mathbb{Z}/n), H^j(\mathbb{Z}/m)) \to 0.$$

$H_{i-1}(\mathbb{Z}/n)$ is either 0, $\mathbb{Z}$, or $\mathbb{Z}/n$ for $i \neq 0$, while $H^j(\mathbb{Z}/m) \cong 0$ or $\mathbb{Z}/m$ for $j \neq 0$. Because m, n are relatively prime, the Ext term always vanishes. By the same argument, the Hom must always be trivial as well, since $i \neq 0$ implies that $H_i(\mathbb{Z}/m)$ is not $\mathbb{Z}$. Therefore by exactness, $E_2^{i,j}$ must be 0. □

It is easily observed through universal coefficents that

$$H^i(\mathbb{Z}/n; H^0(\mathbb{Z}/m)) \cong H^i(\mathbb{Z}/n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0 \\ 0 & \text{for } i \text{ odd} \\ \mathbb{Z}/n & \text{for } i > 0 \text{ even} \end{cases}$$

and similarly $H^0(\mathbb{Z}/n; H^j(\mathbb{Z}/m)) \cong 0$ when j is odd, since $H^j(\mathbb{Z}/m)$ is trivial for j odd. This leads us to our next step.

Step 2. *Each differential, d_i, for $i \geq 2$, is the trivial map.*

Proof. We begin with $i = 2$. Since $d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$. Clearly, outside the first quadrant, the second differential is always zero. We consider the following cases:

Case 1: $(p \neq 0, q \neq 0)$
$E_2^{p,q} = 0$ by Step 1. Clearly d_2 must be zero.

Case 2: $(q = 0)$
The image of the differential must be outside of the first quadrant, which we know each entry here to be the trivial group. Thus, d_2 is trivial as well.

Case 3: $(p = 0, q \text{ odd})$
Since $E_2^{p,q} \cong H^0(\mathbb{Z}/n; H^q(\mathbb{Z}/m; \mathbb{Z})) \cong H^q(\mathbb{Z}/m; \mathbb{Z})$, $E_2^{p,q}$ is 0.

Case 4: $(p = 0, q \text{ even})$
We have that the image of d_2 must be in $E_2^{p+2,q-1}$. We have already considered the case where q is 0. Otherwise, since $q \geq 2$, the image must be in some $E_2^{s,t}$, where s, t are both nonzero. Step 1 tells us that d_2 is mapping into the trivial group, so it must be 0.

Therefore d_2 is always 0, so the E_3 page must be the same as the E_2 page. We now consider general $i \geq 2$. This is shown by a strong induction.

We claim that for each $i \geq 2$, $d_i = 0$. The base case has been shown above. Assume that $d_i = 0$ for $2 \leq i < p$. We show d_p is the trivial map. Thus the E_2 page must be the same as the E_p page. Now look at d_p. It maps from $E_p^{s,t}$ to $E_p^{s+p+1,t-p}$. If t is odd, then the above cases show that $E_p^{s,t}$ is zero for all t, so d_p must be trivial as well. We therefore consider the case when t is even. If $t = 0$, we are clearly done. Now when t is nonzero, we need only be concerned with the map from where $p = t$, so the differential is mapping onto the row $t = 0$, since these are the only nonzero groups with s positive. Also, we see that for the differential to map from a nonzero group, we must have $s = 0$. Therefore, d_p maps from $E_p^{0,t}$ to $E_p^{p+1,0}$. But $p+1$ is now odd, so since $H^k(\mathbb{Z}/n; H^0(\mathbb{Z}/m; \mathbb{Z}))$ is 0 for k odd, d_p will once again be mapping into a trivial group. Therefore d_p is the trivial map, and the induction is complete. □

This immediately tells us that the E_2 page is the E_∞ page, so we may now read off the cohomology groups from this page.

Step 3. $H^k(G; \mathbb{Z}) \cong E_\infty^{0,k} \oplus E_\infty^{k,0}$

Proof. We know that the $E_\infty^{p,k-p}$ correspond to F_p^k/F_{p-1}^k where we have a filtration of groups $0 \leq F_0^k \leq F_1^k \cdots \leq F_k^k = H^k(G;\mathbb{Z})$. However, for $1 \leq p < k$, $E_\infty^{p,k-p} = 0$ by Step 1. Therefore $F_0^k = F_1^k = \cdots = F_{k-1}^k = H^0(\mathbb{Z}/n; H^k(\mathbb{Z}/m;\mathbb{Z}))$ and thus $F_k^k = H^k(\mathbb{Z}/n; H^0(\mathbb{Z}/m;\mathbb{Z})) \oplus H^0(\mathbb{Z}/n; H^k(\mathbb{Z}/m;\mathbb{Z}))$. We know this is a direct sum since these two groups are cyclic subgroups of $\mathbb{Z}/n$ and $\mathbb{Z}/m$ respectively, and thus their orders are relatively prime. □

Therefore, we see that the cohomology period for G is exactly 4 if $E_\infty^{0,4} \cong \mathbb{Z}/m$ and G is not cyclic. This now leads us to prove the main result, which will classify the given cyclic extensions of any positive period.

Proof of Theorem 6. We now calculate $H^0(\mathbb{Z}/n; H^k(\mathbb{Z}/m;\mathbb{Z}))$. Since this group is isomorphic to $H^k(\mathbb{Z}/m;\mathbb{Z})^{\mathbb{Z}/n}$, where the coefficients on H^k are twisted, we need the fixed-point set of $H^{2j}(\mathbb{Z}/m;\mathbb{Z})$, untwisted, under the action of $\mathbb{Z}/n$. Since $H^k(\mathbb{Z}/m;\mathbb{Z})) = 0$ for k odd, $E_\infty^{0,k} = 0$, so we consider the case where $k = 2j, j > 0$. From our computation of $H^2(S_3;\mathbb{Z})$ in the preceding section, we saw that the action of $\mathbb{Z}/2$ on $\mathbb{Z}/3$ was multiplication by 2; then on H^{2k}, the induced action was multiplication by 2^k. Therefore, it is easy to see by the ring structure on $H^*(\mathbb{Z}/m;\mathbb{Z})$ that the action on H^{2j} is multiplication by r^j. It is also clear that all the points of $H^{2j}(\mathbb{Z}/m;\mathbb{Z}) \cong \mathbb{Z}/m$ are fixed if and only if the homomorphism from $\mathbb{Z}/n$ into $\mathrm{Aut}(H^4(\mathbb{Z}/m;\mathbb{Z}))$ is the trivial homomorphism. This can only occur if $r^j \equiv 1 \pmod{m}$. □

From this we are able to see that a group of the form $\mathbb{Z}/m \rtimes \mathbb{Z}/n$, where m, n are relatively prime, always has periodic cohomology, since all that is necessary is that the equation, $r^k \equiv 1 \pmod{m}$, has a solution; if r is a unit of $\mathbb{Z}/m$, such a k will always exist (if k is odd, we can just take $2k$ as the period). This in turn says that there will always be a homotopy sphere of odd dimension that a cyclic extension of this form will act freely on.

Returning to the special case where we considered a free action on a homotopy three-sphere, it should be noted that all finite cyclic groups act freely on a complex homotopy equivalent to S^3, even though we know their period is only 2. This is easily seen by taking the manifold S^3 as the Lie group of unit Quaternions, so we let the sphere be represented as

$$S^3 = \{(x_1 + x_2\hat{\imath} + x_3\hat{\jmath} + x_4\hat{k} : (x_1, x_2, x_3, x_4) \in \mathbb{R}^4, \|(x_1, x_2, x_3, x_4)\| = 1\}.$$

Therefore, we choose an action on S^3 by $\mathbb{Z}/n$ in the following way. It is clear that $\mathbb{Z}/n$ acts freely on S^1, regarded as the complex unit circle, by multiplication by $e^{2\pi i/n}$. We transfer this to the 3-sphere by simply taking a pair (x_1, x_2, x_3, x_4), and regarding it as a pair of complex numbers, $(x_1 + ix_2, x_3 + ix_4)$, and then acting on each complex number as before. This gives us a fixed-point free action on S^3, exemplifying how the necessary period for a group to act freely on a homotopy sphere of dimension $2k-1$ must divide $2k$, not to necessarily have minimal period $2k$. This shows us that for any $k \geq 1$, there exists a homotopy sphere of dimension $4k-1$ that admits a free action by S_3 (clearly, such spaces exist for any finite group with cohomological period 4). The above classification of periods now allows for free actions of these groups on homotopy spheres of a multitude of dimensions.

3.2. **Cohomology Ring of a Cyclic Extension.** While the periodicity has told us a great deal of information about how finite groups act on spheres, this question has actually allowed us to calculate the structure of the group cohomology, given such a cyclic extension, by simply following the development of the spectral sequence above. By Step 3, we obtain the following equation,

$$H^k(G;\mathbb{Z}) \cong H^k(\mathbb{Z}/n;\mathbb{Z}) \oplus FP_{\mathbb{Z}/n}(H^k(\mathbb{Z}/m;\mathbb{Z})),$$

where $FP_H(G)$ is the set of fixed points of the action of H on G.

This immediately allows us to see that for k odd, $H^k(G;\mathbb{Z}) = 0$, which gives us an improvement on Theorem 4. Once again, looking at our example of S_3, we know that the action of $\mathbb{Z}/2$ on $\mathbb{Z}/3$ is multiplication by 2, so this corresponds to an action of multiplication by 2 on $H^2(\mathbb{Z}/3;\mathbb{Z})$. Thus, the fixed point set will be exactly the 0 element. From Claim 1, we know that the fixed point set under the action of $\mathbb{Z}/2$ on H^4 is the entire group, so by Step 3, $H^4(S_3;\mathbb{Z}) \cong \mathbb{Z}/3 \oplus \mathbb{Z}/2 \cong \mathbb{Z}/6$, agreeing with our calculations in the previous section. By this method, we have now very quickly calculated the cohomology of S_3 to be

$$H^k(S_3;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{if } k \equiv 1,3 \pmod 4 \\ \mathbb{Z}/2 & \text{if } k \equiv 2 \pmod 4 \\ \mathbb{Z}/6 & \text{if } k \equiv 0 \pmod 4, k \neq 0 \end{cases} \qquad (\text{for } k \geq 0)$$

From this it is easy to see that for any $G = \mathbb{Z}/m \rtimes \mathbb{Z}/n$, for m, n relatively prime, every even, nonzero cohomology group contains $\mathbb{Z}/n$ as a subgroup. These properties have shown us that the underlying structure of each of the groups that compose G is exposed in its cohomology.

While we initially set out to distinguish between spaces homotopy equivalent and homeomorphic to the sphere, these calculations explained much more. We were able to determine a great deal of information about the group by using the topological concepts that were intrinsically tied into the structure of the group. Hopefully by utilizing these methods, all possible actions of finite groups on homotopy spheres can be determined, simply by means of the calculation of a few cohomology groups, compared to those which act on an actual sphere. Therefore, utilizing powerful tools of the past, such as the Serre spectral sequence, one could hope to gain more insight into new questions about groups, such as progressing on the classification of group extensions themselves.

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