Gauss-Bonnet and Uniformization

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Abstract

This paper was written for Prof. Peter Teichner's Algebraic Topology course (Math 215B) at UC Berkeley in Spring 2006. We review the Gauss-Bonnet and uniformization theorems for surfaces, briefly developing the techniques of riemannian and complex geometry necessary. Consequences and generalizations of both are also discussed.

1 Introduction

Two major trends in modern topology and geometry are results concerning the relation between local and global properties of manifolds, and results concerning the classification of manifolds. These ideas not only illustrate the deep relation between topology and geometry, but serve as motivation for other relations between different branches of mathematics. Arguably the most important such geometric-topological results are the Gauss-Bonnet theorem and the uniformization theorem. Together, these theorems reduce the study of surfaces to relatively simple cases.

The Gauss-Bonnet theorem is a classical demonstration of how the local geometric properties of a manifold can determine its global topological properties, and vice-versa. The theorem asserts a relationship between the Euler characteristic of a surface (a global topological invariant) and its curvature (a local geometric invariant). This is quite an amazing result, in that curvature and Euler characteristic are defined in very different ways, and there is no a priori reason why such a relation should hold. Just as interesting is the fact that any geometric structure on a surface determines the same topological structure.

Classification theorems are important not just for aesthetic reasons, but practical ones as well. Computations that would be impossible to carry out on all manifolds can be reduced to easier to manage cases. The uniformization theorem for surfaces illustrates such a result; it states that there are only three simply-connected Riemann surfaces. This theorem, combined with the Gauss-Bonnet theorem, describes the possible metrics a surface can admit. Again, this has practical implications for mathematical as well as non-mathematical (especially physical) applications.

The aim of this paper is to discuss the Gauss-Bonnet and uniformization theorems. We start by reviewing riemannian manifolds, connections, and cur-

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vature. This will lead us to the statement of the Gauss-Bonnet theorem. Then we review Riemann surfaces and uniformization. After this we discuss consequences of these two theorems. Finally, we briefly discuss generalizations of both Gauss-Bonnet and uniformization, providing references for further study.

To give a full treatment to the topics considered in this paper, one would need a deeper background in riemannian and complex geometry. We in no way attempt to give such a treament. Indeed, definitions and background we give will be brief, and only with our stated goal in mind. Where appropriate, we provide references for further results in riemannian and complex geometry.

2 Riemannian Manifolds and Curvature

We assume familiarity with the notions of smooth manifolds, tensors, and differential forms. Throughout this paper, M will denote a smooth manifold, TM its tangent bundle, and $\Gamma(TM)$ the space of smooth sections of TM; i.e. vector fields on M. For most of this section, we use [5] as a guide.

Definition 2.1. A (riemannian) metric on a smooth manifold M is a symmetric 2-tensor g on M; i.e. a choice of a smoothly varying positive definite inner product $g = \langle , \rangle$ on TM. A manifold with a choice of a metric is called a riemannian manifold.

A metric allows one to define the geometric notions of distance, geodesics, etc. For us, the notion of curvature will be most important. To define curvature, we need a few more preliminary definitions. We note that via partitions of unity one shows that every smooth manifold admits a metric.

The following definition will be useful when discussing uniformization later.

Definition 2.2. Two metrics g_1 and g_2 on a manifold M are said to be conformal if $g_1 = fg_2$ for some positive function $f \in C^\infty(M)$. A diffeomorphism $\phi : (M_1, g_1) \to (M_2, g_2)$ is said to be a conformal equivalence if $\phi^* g_2$ is conformal to g_1.

Thus conformal metrics give different values when measuring length, but the same values when measuring angles.

For us, riemannian 2-manifolds will be most important; henceforth the word surface will refer to such a manifold. We recall the following fact, classifying the topological properties of closed surfaces:

Theorem 2.1 (Classification of Surfaces)**.** *An oriented closed surface is homeomorphic to either the* 2*-sphere or the connected sum of g tori; g is called the genus of the surface (with the sphere having genus* 0*). A non-orientable closed surface is homeomorphic to the connected sum of g (the genus) copies of* $\mathbb{RP}^2$.

2.1 Connections

A basic notion in riemannian geometry, and indeed geometry in general, is that of a connection. On a manifold, a connection allows the differentiation of vector fields. In fact, the general notion of a connection allows differentiation of sections of any vector bundle, but we will not need this general notion. Connections also allow one to define the notion of differentiation along a curve, which leads to geodesics — the "straight lines" in a riemannian manifold.

Definition 2.3. A connection on the tangent bundle TM is a map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$, written $(X, Y) \mapsto \nabla_X Y$ such that

- ∇ is linear in X over $C^\infty(M)$
- ∇ is linear in Y over $\mathbb{R}$
- $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$ for all $X, Y \in \Gamma(TM)$ and $f \in C^\infty(M)$.

We call $\nabla_X Y$ the covariant derivative of Y along X.

We note that for the reader who is familiar with the notion of principle bundles, the definition of a connection given above can be equivalently described by the more general notion of a connection on a principle bundle — in this case the $O(n)$-principle bundle of orthonormal frames of TM.

Among all possible connections on the tangent bundle of a manifold, there is one which is especially useful for use in riemannian geometry; the Levi-Civita connection.

Theorem 2.2. *There exists a unique connection ∇ on the tangent bundle TM, called the Levi-Civita connection, such that*

- $\nabla_X \langle Y_1, Y_2 \rangle = \langle \nabla_X Y_1, Y_2 \rangle + \langle Y_1, \nabla_X Y_2 \rangle$ *for all* $X, Y_1, Y_2 \in \Gamma(TM)$
- $\nabla_X Y - \nabla_Y X - [X, Y] \equiv 0$.

A connection satisfying the first condition above is said to be compatible with the metric, and a connection satisfying the second condition is said to be torsion-free (or symmetric). Hence the theorem, known as the Fundamental Lemma of Riemannian Geometry, asserts that any riemannian manifold has a unique torsion-free connection compatible with the metric.

One may ask why we should desire a connection satisfying the properties above. Perhaps the simplest motivation comes from the standard Euclidean connection, which we define later. The Nash embedding theorem states that any riemannian manifold can be isometrically embedded into $\mathbb{R}^N$ for large enough N. Then the restriction of the standard Euclidean connection on $\mathbb{R}^N$ to M is in fact compatible with the metric and torsion-free. Thus it makes sense to have an intrinsic connection on M that also satisfies these properties.

From now on, we use ∇ to denote the Levi-Civita connection.

2.2 Curvature

As stated above, the notion of curvature will be of utmost importance in this discussion. Indeed, curvature is perhaps the most important geometric invariant of a riemannian manifold.

Definition 2.4. The curvature of ∇ is the map $\Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The curvature tensor of ∇ is the 4-tensor Rm given by

$$Rm(X,Y,Z,W) = \langle R(X,Y)Z, W \rangle.$$

The curvature then, in a sense, measures the noncommutativity of covariant derivatives. It is an important result that a riemannian manifold has vanishing curvature tensor if and only if it is locally isometric to Euclidean space. Further study of curvature involves the notions of Ricci curvature, scalar curvature, and sectional curvature (of which the below is a special case). We refer to [5] and [8] for further reference.

The language of differential forms provides another useful notion of curvature. Here, the curvature of a riemannian manifold M is described by the $\mathfrak{o}(n)$-valued 2-form Ω defined by

$$R(X,Y)Z = \Omega(X,Y)Z.$$

Here $\mathfrak{o}(n)$ is the Lie algebra of $O(n)$ — the structure group of TM. Since $\mathfrak{o}(n)$ consists of skew-symmetric matrices, Ω (called the curvature form) can be thought of as an $n \times n$ matrix whose entries are 2-forms. We leave the details to the references, but note that it is this viewpoint which will appear in generalizations of the Gauss-Bonnet theorem.

For surfaces, we have a simpler notion of curvature, which in fact determines the full curvature tensor.

Definition 2.5. For a 2-dimensional riemannian manifold M, the Gaussian curvature is the function given by

$$K(x) = \frac{Rm(X,Y,Y,X)}{|X|^2|Y|^2 - \langle X,Y \rangle^2},$$

where (X,Y) is any frame of T_xM.

The Gaussian curvature captures the essential properties of curvature in general, and indeed will be a main ingredient in the Gauss-Bonnet theorem.

2.3 Standard Examples

We now give three examples of surfaces and their curvatures, which in fact will be the most important examples from the point of view of uniformization.

Example 2.1. *The Euclidean metric on $\mathbb{R}^2$ is given by $g = dx^2 + dy^2$. This simply induces the standard inner product on each tangent space. The Levi-Civita connection is given by*

$$\nabla_X (Y^i \partial_i) = (XY^i)\partial_i,$$

in other words, the components of $\nabla_X Y$ are simply the ordinary derivatives of the components of Y in the direction of X. This has Gaussian curvature which vanishes identically.

Example 2.2. *The 2-sphere S^2 inherits the Euclidean metric on $\mathbb{R}^3$: $g = dx^2 + dy^2 + dz^2$. This has constant Gaussian curvature 1.*

Example 2.3. *The 2-dimensional hyperbolic plane $\mathbb{H}^2$ is the upper half plane with metric given by*

$$g = \frac{dx^2 + dy^2}{y^2}.$$

This has constant Gaussian curvature -1. Equivalently, $\mathbb{H}^2$ is the open unit disk Δ^2 with metric

$$g = 4\frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}.$$

These examples all have direct higher dimensional analogues.

3 The Gauss-Bonnet Theorem

At this point we recall the definition of Euler characteristic:

Definition 3.1. The Euler characteristic $\chi(M)$ of a manifold M is defined to be

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} H_i(M).$$

Note that a surface admits a smooth triangulation, which is a 2-dimensional CW-complex structure. Hence the Euler characteristic is equivalently given by

$$\chi(M) = v - e + f$$

where v is the number of 0-cells, e the number of 1-cells, and f the number of 2-cells. Indeed, there are other equivalent definitions of Euler characteristic.

We now come to the statement of the Gauss-Bonnet theorem. The power of this theorem lies in its implications for the geometry and topology of a given surface.

Theorem 3.1 (Gauss-Bonnet)**.** *Let M be a compact, oriented surface, and let K be its Gaussian curvature. Then*

$$\int_M K\, dA = 2\pi\chi(M),$$

where dA is the area form on M and $\chi(M)$ is the Euler characteristic of M.

Thus, while curvature itself is only a geometric and non-topological invariant, the total curvature $\int_M K\,dA$ is a topological invariant!

We list some simple consequences of this theorem. Recall that an oriented closed surface of genus g has Euler characteristic $2-2g$. If an oriented closed surface S has positive Gaussian curvature everywhere, then by the Gauss-Bonnet theorem its Euler characteristic must be positive, and hence 2. Thus S has to be the 2-sphere. Hence here, the geometry of a surface determines its topology. Further consequences will be drawn after the discussion of uniformization.

We also make a few comments about the proof of the Gauss-Bonnet theorem. There are two standard proofs given in the literature. One relies on a study of curved polygons in a surface and uses triangulations. Another, however, proceeds by showing that the integrand $K\,dA$ actually represents the Euler class of the manifold, so that the standard homology-cohomology pairing between it and the fundamental class (this pairing in the language of differential forms is exactly integration) gives the Euler characteristic.

4 Riemann Surfaces

The uniformization theorem for surfaces classically belongs to the realm of complex analysis and the study of Riemann surfaces. Hence we begin with a brief review of complex manifolds and Riemann surfaces.

Definition 4.1. A complex n-dimensional manifold is a smooth $2n$-dimensional real manifold, locally homeomorphic to $\mathbb{C}^n$, where the transition maps $\phi_{ij} : \mathbb{C}^n \to \mathbb{C}^n$ are holomorphic, i.e. complex analytic.

Complex manifolds require a more rigid structure than do smooth manifolds. By definition, every complex manifold is smooth, but the converse is false — for example, odd dimensional smooth manifolds can never be complex..

Definition 4.2. A Riemann surface is a 1-dimensional (and hence real 2-dimensional) complex manifold.

The question of which smooth manifolds admit the structure of a complex manifold is settled by the following theorem. First, we recall that an almost complex structure on a smooth manifold M is a bundle map $J : TM \to TM$ such that $J^2 = -1$. Clearly, any complex manifold admits such a map (multiplication by i), and the existence of such a map is therefore a necessary condition for a smooth manifold to be complex. Given such a structure, its Nijenhuis tensor is given by

$$\mathcal{N}_J(X,Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y].$$

An almost complex structure on a smooth manifold M is said to be integrable when it arises from a complex manifold structure on M.

Theorem 4.1 (Newlander-Nirenberg)**.** *An almost complex structure J is integrable if and only if its Nijenhuis tensor $\mathcal{N}_J$ vanishes.*

We note that an orientable surface always satisfies this condition. Indeed, an orientable surface always admits an almost complex structure (since the choice of a riemannian metric reduces the structure group to $O(1)$, and the choice of orientation further reduces it to $SO(2) = U(1)$) and the Nijenhuis tensor must vanish by dimensionality reasons. Thus orientable surfaces are Riemann surfaces. In fact, there is a one-to-one correspondence between complex structures and conformal classes of metrics on a surface. The standard surfaces given above: $\mathbb{R}^2$, S^2, and $\mathbb{H}^2$, correspond respectively to the Riemann surfaces $\mathbb{C}$, $\mathbb{C} \cup \{\infty\}$, and Δ^2.

A further development of Riemann surfaces can be found in [9] and [3].

5 Uniformization of Surfaces

As stated above, the uniformization theorem is a classification theorem; it provides a relatively simple description of all Riemann surfaces. For the reader familiar with complex analysis, we note the similarity between this theorem and the Riemann mapping theorem, which indeed can be used as part of a proof. The full proof (or one possible proof) uses a deeper study of harmonic functions; again we refer to [3] and [9] for details. We now state the theorem:

Theorem 5.1 (Uniformization Theorem)**.** *Any simply-connected Riemann surface is biholomorphically equivalent to* $\mathbb{C} \cup \{\infty\}$*,* $\mathbb{C}$*, or* Δ^2*.*

As stated here, this is a result in complex geometry and the theory of Riemann surfaces. However, noting the realizations of the standard surfaces mentioned above as Riemann surfaces, we can give a (weaker) more geometric-topological statement:

Theorem 5.2 (Uniformization Theorem - Topological Version)**.** *Any simply-connected surface is diffeomorphic to* S^2 *or* $\mathbb{R}^2$*.*

The uniformization theorem implies that the universal cover of any surface must be one of $\mathbb{R}^2$, S^2, or $\mathbb{H}^2$. Recalling the relation between the covering space of a topological space and its fundamental group, we have the following corollary:

Corollary 5.1. *Any surface (orientable or not) is diffeomorphic to the quotient of* S^2*,* $\mathbb{R}^2$*, or* $\mathbb{H}^2$ *by a discrete group of isometries, which is isomorphic to the fundamental group of the surface.*

A surface with universal cover S^2 is called elliptic, one with universal cover $\mathbb{R}^2$ is parabolic, and those with universal covers $\mathbb{H}^2$ are called hyperbolic. Since we know that the three simply connected Riemann surfaces have metrics of constant curvature, we get:

Corollary 5.2. *Any surface admits a complete metric of constant Gaussian curvature.*

The above corollaries are in fact equivalent to the uniformization theorem. We make a note about the last corollary: the metric on a surface given as the quotient of a simply connected surface Σ by a group G comes from the standard metric on Σ — this descends to the base since G acts on Σ by isometries.

6 Constant Curvature Surfaces

The Gauss-Bonnet theorem, combined with the uniformization theorem, restricts the possible metrics a surface can admit. The main result along these lines, which easily follows from the two theorems, is the following.

Theorem 6.1. *A surface is elliptic (resp. parabolic, hyperbolic) if and only if it admits a metric of constant positive (resp. zero, negative) Gaussian curvature.*

Hence the topology of a surface completely determines the possible geometric (in the sense of metrics) structures in can admit.

We now turn to specific examples. Any surface admitting a constant positive Gaussian curvature metric must have universal cover S^2 and positive genus, so the only examples are S^2 and $\mathbb{RP}^2$. Any surface admitting a zero Gaussian curvature metric must have universal cover $\mathbb{R}^2$ and vanishing Euler characteristic. The only examples are then $\mathbb{R}^2$, cylinders, tori $\mathbb{C}/\Lambda$, the Möbius strip, and the Klein bottle. Since most surfaces have genus $g > 1$, and hence negative Euler characteristic, it follows that "almost all" surfaces are hyperbolic, and hence cannot admit metrics of constant positive or zero Gaussian curvature.

We note that a deeper study of curvature leads to similar results on constant curvature metrics for higher dimensional manifolds. Further relations between curvature and uniformization can be found in [6] and [7].

7 The Chern-Gauss-Bonnet Theorem

The question can now be asked as to what is the analog, if any, of the Gauss-Bonnet theorem in higher dimensions. Such a result was first proved for submanifolds of Euclidean space by Hopf, and then by Chern for abstract manifolds [2]. Again, this theorem, the Chern-Gauss-Bonnet theorem, relates the geometry of a manifold to its topology. The integrand in this case however is more mysterious than is Gaussian curvature. Nevertheless, it provides a satisfactory generalization of the classical Gauss-Bonnet theorem.

Let M be an oriented, compact $2n$-dimensional riemannian manifold. Here we consider the curvature from the point of view of differential forms, as alluded to earlier. Let Ω be the curvature form of M, which we recall is a $2n \times 2n$ matrix whose entries are 2-forms.

Definition 7.1. The Pfaffian $\mathrm{Pf}(A)$ of a $2n \times 2n$ skew-symmetric matrix is the polynomial in the entries of A such that

$$\det(A) = (\mathrm{Pf}(A))^2.$$

It is a basic result in linear algebra that such a polynomial always exist.

Hence the Pfaffian of Ω is also defined, and is a $2n$-form on M. We can thus integrate this form over M, and the result is:

Theorem 7.1 (Chern-Gauss-Bonnet). *Let M be a compact, oriented $2n$-dimensional manifold, and let Ω be its curvature form. Then*

$$\int_M \mathrm{Pf}(\Omega) = (2\pi)^n \chi(M)$$

where $\chi(M)$ is the Euler characteristic of M.

The reader may guess that in the case of a surface, $\mathrm{Pf}(\Omega) = K\,dA$ where K is the Gaussian curvature. Hence the Chern-Gauss-Bonnet theorem is a direct generalization of the classical Gauss-Bonnet theorem. However, the Pfaffian of Ω does not in general have a nice interpretation in terms of sectional curvature, so the geometric implications of this general theorem are not as clear as in the classical case.

8 Uniformization in Higher Dimensions

The goal of uniformization in higher dimensions is, as in the two dimensional case, to classify manifolds and their geometric structures. In general, there are not as many results in higher dimensions as there are for surfaces. Here we mention two areas along these lines.

Thurston's Geometrization Conjecture is perhaps one of the most exciting uniformization conjectures. Indeed, this conjecture implies the Poincaré conjecture. A full statement of the conjecture would require one to know deep results in the theory of 3-manifolds; in particular, the prime decomposition (which splits a 3-manifold up along 2-spheres into prime 3-manifolds) and the JSJ decomposition (which decomposes a prime 3-manifold into further components along tori). Here, we just state the conjecture and leave the full development to other references.

Conjecture 8.1 (Geometrization Conjecture). *The components appearing after applying the prime and JSJ decompositions to a 3-manifold each admit one of the following geometries: Euclidean geometry, hyperbolic geometry, spherical geometry, the geometry of $S^2 \times \mathbb{R}$, the geometry of $\mathbb{H}^2 \times \mathbb{R}$, the geometry of the universal cover of $SL_2(\mathbb{R})$, nil geometry, or sol geometry.*

There is a growing consensus that the geometrization conjecture has in fact been proved by Perelman, using a method involving the notion of Ricci flow. For further details, we refer to [1], which contains references for Perelman's original papers.

Another topic in higher dimensional uniformization is the Yamabe problem: Given a compact riemannian manifold (M, g), find a metric h of constant scalar curvature which is conformal to g. That such a metric exists was conjectured by Yamabe, and proved by Schoen. An important aspect of this, and similar problems, is the use of Einstein metrics — which are metrics whose Ricci tensor is a scalar multiple of the metric itself. Einstein metrics are useful because they are exactly the critical points of the total scalar curvature functional:

$$\mathcal{A} = \int_M S\, dV.$$

Hence, in some sense, Einstein metrics form an "optimal" class of metrics. For more details the reader can consult [4].

9 Conclusion

The Gauss-Bonnet and uniformization theorems for surfaces occupy different areas of mathematics — one belonging to riemannian geometry and the other to complex geometry. Even so, there is a strong relationship between the two, which leads to important consequences. Among these are the restrictions the geometry of a surface places on its topology, and vice-versa.

The relations between geometry and topology run quite deep. Here we have presented a few such connections (pun intended!), but there are many others. We hope the interested reader will find the given references useful, and will seek out further references.

References

[1] Michael T. Anderson, *Geometrization of 3-manifolds via the Ricci flow*, Notices of the AMS **51** (2004), no. 2, 184–193.

[2] Shiing-Shen Chern, *A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds*, The Annals of Mathematics **45** (1944), no. 4, 747–752.

[3] Hershel M. Farkas and Irwin Kra, *Riemann surfaces*, Springer-Verlag, Graduate Texts in Mathematics, 1992.

[4] J.M. Lee and T. Parker, *The Yamabe Problem*, Bull Am. Math. Soc. **17** (1987), no. 1.

[5] John M. Lee, *Riemannian manifolds: An introduction to curvature*, Springer-Verlag, Graduate Texts in Mathematics, 1997.

[6] Rafe Mazzeo and Michael Taylor, *Curvature and uniformization* (2001), available at `math.DG/0105016`.

[7] Ngaiming Mok, *The uniformization theorem for compact Kahler manifolds of nonnegative holomorphic bisectional curvature*, J. Differential Geom. **27** (1988), no. 2, 179–214.

[8] M.M. Postnikov, *Geometry VI: Riemannian Geometry*, Vol. 91, Springer, Encyclopaedia of Mathematical Sciences, 2001.

[9] Mika Seppala and Tuomas Sorvali, *Geometry of Riemann surfaces and Teichmuller spaces*, Elsevier Science Publishers, 1992.