Delta and Simplicial Objects
and
Chain Complexes

Matthias Görner

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Contents

1 Introduction

Delta sets are a nice way to encode the information needed to specify a Delta complex. We will learn two different definitions of Delta sets and see that the category of Delta sets is equivalent to the category of Delta complexes.
In the following chapter, we introduce a very similar concept: simplicial sets or more generally simplicial objects. These sets are slightly more complicated as

they contain not just the simplices of a Delta complex but also the degenerations of these simplices. However, most of the facts about Delta sets have an analogous result and can be proven very similarly to the case of simplicial objects. Furthermore the equivalence of Delta sets and Delta complexes becomes a relation between the category of CW-complexes and Simplicial sets. Simplicial sets have a couple of advantages. One of them which we will show is that if a Simplicial Set comes with an Abelian group structure, it is equivalent to a chain complex.
As an outlook we describe how to build Eilenberg-MacLane spaces from Simplicial Abelian groups respectively chain complexes.

The literature used here and recommended for further reading is [May65], [Cur71], [Wala], [Walb] and [Wei94].
A nice application of simplicial sets is the computer program “Kenzo” which is able to compute homotopy groups of a lot of spaces. See [RS02].

2 Delta Objects

2.1 Definition

Recall the definition of a Delta complex from [Hat02]. The information needed to specify a Delta complex has two parts:

- the number of simplices in each dimension
- which of the n simplices forms the i-th face of an $n+1$-simplex. (Recall that all simplices have ordered vertices: hence the faces of the $n+1$-simplex are numbered and have an induced ordering of the vertices which has to be compatible with the ordering of the n-simplex)

The first part can be encoded by giving a sequence of sets $X_0, X_1, \dots$ with the elements in X_n being the n-simplices of the Delta complex. The second part can be encoded by maps $d_i : X_{n+1} \to X_n$ $(0 \leq i \leq n+1)$ such that $d_i(x)$ is the n simplex forming the i-th face of the $n+1$-simplex x. The X_n together with the maps d_i will be called the Delta set associated to the Delta complex. Notice that the d_i fullfill the relations

$$d_i d_j = d_{j-1} d_i \qquad \text{if } \ i < j.$$

Let’s have a look at the abstract definition of Delta set:

Definition 2.1. *Let $\hat{\Delta}$ denote the category which has as objects finite ordered sets $[n] = \{0, 1, 2, \dots, n\}$ and as morphisms maps $f : [n] \to [k]$ $(n < k)$ which are strictly order-preserving in the sense that $f(a) < f(b)$ if $a < b$.*
A **Delta-Object in the category** *$\mathcal{C}$ is a contravariant functor $\hat{\Delta}^{\mathrm{op}} \to \mathcal{C}$. If $\mathcal{C} =$* **Sets***, we say* **Delta-Set***.*
A morphism between two Delta-Objects is a natural transformation of the two functors.

To understand how this abstract definition relates to the statement above that Delta sets are a way of capturing the combinatorial information of a Delta complex, we can give an equivalent definition of a Delta object only involving face operators:

Definition 2.2. *In $\hat{\Delta}$, we call the morphism $d_i : [n] \to [n+1]$ $(0 \leq i \leq n)$ in $\hat{\Delta}$ given by*

$$d_i : (0, \ldots, i-1, i, \ldots, n) \mapsto (0, \ldots, i-1, i+1, \cdots, n+1)$$

the **i-th face operator***. It is uniquely determined by $i \notin \mathrm{im}(d_i)$, in other words d_i "skips i".*
For a simplicial object in $\mathcal{C}$, we call the image of this morphism in $\mathcal{C}$ the **i-th face operator***.*

Notice that each morphism in $\hat{\Delta}^{\mathrm{op}}$ is the composition of face operators[1]. For each morphism, there is even a canonical composition $d_{i_1} d_{i_2} \cdots d_{i_m}$ such that $i_1 < i_2 < \cdots < i_m$, making two morphisms equal if and only if they have the same canonical composition.
Furthermore the following relations

$$d_i d_j = d_{j-1} d_i \qquad \text{if } \ i < j \tag{1}$$

holds for the face operators in $\hat{\Delta}^{\mathrm{op}}$ and are enough to bring each composition of face operators into canonical form. Therefore, there can be no other relations than the ones derived from 1 and the above definition is equivalent to the following definition:

Definition 2.3. *A* **Delta-Object in $\mathcal{C}$** *is a sequence $X_0, X_1, \ldots$ of objects in $\mathcal{C}$ and a collection of morphisms $d_i : X_{n+1} \to X_n (0 \leq i \leq n+1)$ such that relation 1 holds.*

Recall that when encoding a Delta complex, the maps $d_i : X_{n+1} \to X_n$ map the $n+1$-simplex to the n-simplex which is its i-th face. One can check that these d_i fulfill condition 1 and hence they form together with the X_n a Delta set.

There is another way to think about the Delta set associated to a Delta complex (which has a linear structure on each simplex).
If the objects $[n]$ in the category $\hat{\Delta}$ are considered as the set of vertices of the standard n-simplex Δ^n in $\mathbb{R}^{n+1}$, then a morphism $[n] \to [n']$ can be identified with an affine map $\tau : \Delta^n \to \Delta^{n'}$ between the standard simplices mapping vertices to vertices in a strictly order-preserving way (such a map is a **strictly-order preserving simplicial map**).
Now let X_n be the set of linear maps from Δ^n to an n-simplex in the Delta complex mapping vertices in a strictly order-preserving way to vertices.
If $\tau : \Delta^n \to \Delta^{n'}$ is the linear map identified with a morphism of $\hat{\Delta}$, an element in

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[1] We regard the identity map as composition of zero face operators.

$X_{n'}$ can be composed with τ to get an element in X_n. Hence to each morphism of $\hat{\Delta}$ can be assigned a morphism $X_{n'} \to X_n$. This assignment is a contravariant functor $\hat{\Delta}^{\mathrm{op}} \to$ **Sets**, hence a Delta set. It is isomorphic to the Delta set associated to a Delta complex defined in the beginning.

2.2 Geometric realization

The next definition gives us a way to recover the Delta complex from a Delta set.

Definition 2.4. *For a Delta set X, define the* **geometric realization** *RX as the quotient of the disjoint union of standard simplices Δ^n over a certain equivalence relation*

$$RX = \frac{\bigsqcup_n X_n \times \Delta^n}{\sim}.$$

This equivalence relation is given by pointwise identifying for each $x \in X_{n+1}$ the i-th face of the simplex $\{x\} \times \Delta^{n+1}$ with the simplex $\{d_i(x)\} \times \Delta^n$. More precisely, $(x, \partial_i(p)) \sim (d_i(x), p)$ where $x \in X_{n+1}, p \in \Delta^n$ and $\partial_i : \Delta^n \to \Delta^{n+1}$ is the inclusion map of Δ^n onto the i-th face of Δ^{n+1}.

Note that the geometric realization is functorial.

Theorem 2.5. *If X is the Delta set associated to a Delta complex, then RX is homeomorphic to the Delta complex.*

Note that for any Delta set X the geometric realization RX has a Delta complex structure. Hence we get a bijective correspondence between Delta sets and Delta complexes. However, it might be necessary to subdivide the Delta structure to get a simplicial complex (see [Hat02] for the difference of Delta and simplicial complex).

Proof. As explained above, an element in X_n can be considered as a map from Δ^n to the Delta complex. Hence we get a map from $\bigsqcup_n X_n \times \Delta^n$ to the Delta complex which is a homeomorphism when restricted to one simplex. Two points are identified by the equivalence relation if and only if their image under this map is the same. Hence we get an induced homeomorphism on the quotient.
For the case of an infinite dimensional Delta complex, notice that a Delta set is equipped with the weak topology, i.e. a set is open if and only if the intersection with each simplex of the Delta set is open. The disjoint union $\bigsqcup_n X_n \times \Delta^n$ also has the weak topology and hence so does the quotient of it by $\sim$. □

In fact, the categories of Delta-complexes (with simplicial maps preserving the ordering of the vertices) and Delta-sets are equivalent

2.3 Examples

- Consider a 2-sphere. Put three vertices v_0, v_1, v_2, i.e. 0-simplices, on the equator such that the equator is split into three edges e_0, e_1, e_2 and thus the equator splits the 2-sphere into two triangles f_0, f_1 (see picture 1 in appendix).
  This gives a Delta complex structure on the 2-sphere and the associated Delta set has $X_0 = \{v_0, v_1, v_2\}, X_1 = \{e_0, e_1, e_2\}, X_2 = \{f_0, f_1\}, X_3 = X_4 = \cdots = \emptyset$. d_0, d_1, d_2 would map f_0 respectively f_1 to e_0, e_1, e_2. The edge e_2 would be mapped to v_2 by d_0.
- Picture 2 in appendix shows a Delta complex which is obtained from a triangle by identifying two of its edges. Notice that this is not a simplicial complex.
- The Delta set with one 0-simplex and one 1-simplex yields a circle as geometric realization. For higher dimensions however, collapsing the boundary of an n-ball to a point to obtain an n-sphere won't yield a Delta complex structure.
- Define $\hat{\Delta}^n$ as the following Delta set:
  $$(\hat{\Delta}^n)_m = \{(a_0, \ldots, a_m) : 0 \leq a_0 < \cdots < a_m \leq n\}$$
  with $d_i : (\hat{\Delta}^n)_m \to (\hat{\Delta}^n)_{m-1}$ mapping $(a_0, \ldots, a_m)$ to $(a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m)$. The geometric realization of $\hat{\Delta}^n$ is in fact the standard simplex with the canonical Delta complex structure.
- The boundary of the standard simplex can be constructed from $\partial\hat{\Delta}^n$ which is obtained by deleting the simplex $(0, 1, \ldots, n)$ from $(\hat{\Delta}^n)_n$.

3 Simplicial Objects

3.1 Definition

Simplicial objects are similar to Delta objects, except that we allow instead of only strictly order-preserving maps all order-preserving maps. When encoding a Delta complex by a simplicial set, this implies that there will appear a lot more elements in each X_n corresponding to degenerate simplices, i.e. simplices where two vertices coincide. This extra structure makes simplicial sets a bit harder, but has the great advantage of making homotopy theory possible.

Definition 3.1. *Let Δ denote the category which has the same objects as $\hat{\Delta}$ but has morphisms that are order-preserving maps $f : [n] \to [k]$ such that $f(a) \leq f(b)$ if $a < b$.*
Besides the i-th **face operator** $d_i : [n] \to [n+1]$ $(0 \leq i \leq n)$, *we have the i-th* **degeneracy operator** $s_i : [n+1] \to [n]$ $(0 \leq i \leq n)$ *given by*

$$s_i : (0, \ldots, i, i+1, \ldots, n+1) \mapsto (0, \ldots, i, i, \ldots n).$$

It is uniquely determined by surjectivity and $|s_i^{-1}(i)| = 2$*, i.e.* s_i *"repeats* i*".*

Analogously to Delta objects, we define:

Definition 3.2. *A* **simplicial object** *in the category* $\mathcal{C}$ *is a contravariant functor* $\Delta^{\text{op}} \to \mathcal{C}$*. A morphism between simplicial objects is a natural transformation of the functors. The category of simplicial objects will be called* $s\mathcal{C}$*. If* $\mathcal{C} = \mathbf{Sets}$ *is the category of sets, we call the simplicial object a* **simplicial set***.*

Similar to the case of the sub-category $\hat{\Delta}^{\text{op}}$, we can factor each morphism in the category Δ^{op} uniquely as $s_{i_1} \cdots s_{i_m} d_{j_1} \cdots d_{j_{m'}}$ with $i_1 > \cdots > i_m$ and $j_1 < \cdots < j_{m'}$ and therefore we get analogously another definition of a simplicial object equivalent to the one above:

Definition 3.3. *A* **simplicial object in the category** $\mathcal{C}$ *is a sequence of objects* $X_0, X_1, \ldots$ *in* $\mathcal{C}$ *together with morphisms in* $\mathcal{C}$

$$d_i : X_n \to X_{n-1}$$

$$s_i : X_n \to X_{n+1}$$

for each $0 \leq i \leq n$ *such that the following conditions hold:*

$$\begin{array}{rcll} d_i d_j = & d_{j-1} d_i & \text{for} & i < j \\ d_i s_j = & s_{j-1} d_i & \text{for} & i < j \\ d_i s_j = & \text{id} & \text{for} & i = j \text{ or } i = j+1 \\ d_i s_j = & s_j d_{i-1} & \text{for} & i > j+1 \\ s_i s_j = & s_{j+1} s_i & \text{for} & i \leq j \end{array}$$

Similar to the case of $\hat{\Delta}$, we can think of the category Δ as having as objects the standard n-simplices Δ^n and the morphisms being the simplicial maps induced by the $[n] \to [k]$, i.e. linear maps $\tau : \Delta^n \to \Delta^k$ mapping the $n+1$ vertices to the $k+1$ vertices of the standard simplices in the same way the morphism would map the elements in $[n]$ to $[k]$. We can associate a simplicial set to a Delta-complex in the same fashion we associated a Delta-set to it:

Definition 3.4. *The* **simplicial set** $\Delta^{\text{op}} \to$ **Sets associated to a Delta-complex** *has as objects* X_n *the set of all linear maps* x *mapping* Δ^n *to a simplex of the Delta-complex such that the vertices are mapped to vertices in an order-preserving way. A morphism* $\tau : \Delta^n \to \Delta^k$ *in* Δ *is assigned the following morphism in* **Sets***:*

$$X_k \to X_n, x \mapsto x \circ \tau.$$

3.2 Geometric realization

In case of standard simplices, the i-th degeneracy operator $\sigma_i : \Delta^n \to \Delta^{n-1}$ is a simplicial map collapsing the i-th and $i+1$-st vertex.

Definition 3.5. *The* **geometric realization** RX **of a simplicial set** X *(also denoted by* $|X|$*) is the quotient of the disjoint union of standard simplices* Δ^n *over a certain equivalence relation*

$$RX = \frac{\coprod_n X_n \times \Delta^n}{\sim}.$$

This equivalence relation is given by

- $(x, \partial_i(p)) \sim (d_i(x), p)$ *where* $x \in X_{n+1}, p \in \Delta^n$,
i.e. pointwise identifying for each $x \in X_{n+1}$ *the* i*-th face of the corresponding simplex* $\{x\} \times \Delta^{n+1}$ *with the simplex* $\{d_i(x)\} \times \Delta^n$
- $(x, \sigma_i(p)) \sim (s_i(x), p)$ *where* $x \in X_{n-1}, p \in \Delta^n$,
this means that the quotient will collapse the n*-simplex* $s_i(x)$ *onto the* $n-1$*-simplex* x.

Notice that if one of the X_n has an element, then the s_0 has to map this element to some element in X_{n+1}, so X_{n+1} is also non-empty, yielding that a non-trivial simplicial set has elements in infinitely many dimensions. However the identification via the maps s_i makes it possible to have finite CW-complexes if there are only finitely many non-degenerate simplices:

Definition 3.6. *If* X *is a simplicial set, we call* $x \in X_{n+1}$ *a* **degenerate simplex** *if* $x = s_i x'$ *for some* $x' \in X_n$ *and some* $0 \le i \le n$.

If $x' \in X_n$ is also a degenerate simplex, we can express x' as $s_j(x'')$ and so on, finally ending up with a (unique) non-degenerate simplex. We can thus write each degenerate simplex uniquely as

$$x = s_{i_1} s_{i_2} \cdots s_{i_m} x' \qquad \text{where } i_1 > i_2 > \cdots > i_m \text{ and } x' \text{ non-degenerate.}$$

Theorem 3.7. *The geometric realization* RX *of a simplicial set* X *is a CW-complex with* n*-skeleton*

$$RX^n = \frac{\coprod_{m \le n} X_m \times \Delta^m}{\sim} \subset RX.$$

Moreover, the n*-cells of* RX *are in 1-1 correspondence with the simplices* $x \in \tilde{X}_n$ *which are in* X_n *and non-degenerate.*

Proof. Clearly RX^0 is a 0-skeleton. Induction on n: First, notice that if $x \in X_{n+1}$ is a degenerate simplex, then RX^n already contains it, so we don't need to consider them. Consider the non-degenerate simplices $x \in \tilde{X}_{n+1}$:

$$RX^{n+1} = \frac{RX^n \coprod \tilde{X}_{n+1} \times \Delta^{n+1}}{\sim}$$

The relation $\sim$ will identify a point on the boundary of a Δ^{n+1} with a point in RX^n given by assembling the maps induced by the d_i. The relation $\sim$ doesn't

identify points of the interior of Δ^{n+1}. In other words, the Δ^{n+1} are the $n+1$-cells with the attaching maps induced by the d_i.
As RX is the union of $RX^0, RX^1, \ldots$ with the weak topology, it is a CW-complex. $\square$

In contrast to Delta-sets, a simplicial set doesn't necessarily induce a Delta-complex structure on RX. The construction of the n-sphere below as a quotient $\Delta^n/\partial\Delta^n$ will demonstrate that: The CW-complex associated to this quotient will have only a 0-cell and an n-cell and hence cannot be given a Delta-complex structure.

3.3 Examples

- Define Δ^n as the following simplicial set:

  $$(\Delta^n)_m = \{(a_0, \ldots, a_m) : 0 \le a_0 \le a_1 \le \cdots \le a_m \le n\}$$

  with morphisms order-preserving maps, e.g. the morphism $d_i : (\Delta^n)_m \to (\Delta^n)_{m-1}$ maps $(a_0, \ldots, a_m) \mapsto (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m)$. The geometric realization of Δ^n is in fact the standard n-simplex.

- Let i_n be the element $(0, 1, \ldots, n)$ in $(\Delta^n)_n$. The boundary $\partial\Delta^n$ is the simplicial subset of elements of the form $s_{i_1} \cdots s_{i_m} d_{j_1} \cdots d_{j_{m'}}$ with at least one face operator. In other words, the sub-complex is generated by $d_0 i_n, \ldots, d_n i_n$. The sub-complex is also defined by

  $$(\partial\Delta^n)_m = \{(a_0, \ldots, a_m) : a_0 \le \cdots \le a_m, \{a_0, \ldots, a_m\} \subsetneq \{0, \ldots, n\}\}$$

  The geometric realization of this is the $n-1$-sphere as a boundary of the standard n-simplex.

- There is another way to get the n-sphere as the geometric realization of a simplicial set: Define the simplicial set S^n as the quotient $\Delta^n/\partial\Delta^n$. The quotient is well-defined: the images under all the d_i and s_i of an element in $\partial\Delta^n$ are also in $\partial\Delta^n$. More generally, quotients of simplicial sets are well-defined, unlike for Delta sets.

- The k-th horn Λ^n_k is defined as the sub-complex of $\partial\Delta^n$ generated by

  $$d_0 i_n, \ldots, d_{k-1} i_n, d_{k+1} i_n, \ldots, d_n i_n.$$

  The geometric realization corresponds to the boundary of the standard n-simplex without the k-th face.

4 Simplicial sets and topological spaces

We have seen that the category of Delta complexes and Delta sets are equivalent. Notice however that in order to relate the category of CW-complexes and $s\mathbf{Sets}$, we no longer have a Delta complex structure and hence no notion of a simplicial map. The solution is to get a simplicial set by allowing all continuous mappings from the standard n-simplex Δ^n to a CW-complex:

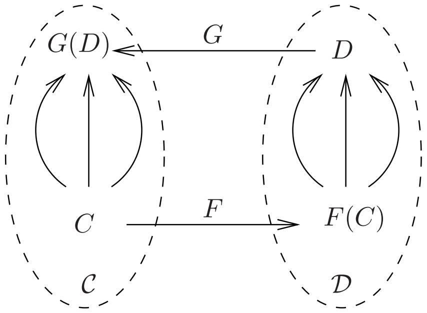
Definition 4.1. *The* **singular set** $S(Y)$ *of a CW-complex/topological space* Y *has as objects* $S(Y)_n = \mathrm{Hom}(\Delta^n, Y)$*, i.e. the set of all continuous maps from* Δ^n *to* Y*. Regarding a morphism* Δ *as a map* $\Delta^n \to \Delta^k$*, such a morphism is assigned the induced morphism*

$$S(Y)_k = \mathrm{Hom}(\Delta^k, Y) \to S(Y)_n = \mathrm{Hom}(\Delta^n, Y)$$

by the functor $S(Y) : \Delta^{\mathrm{op}} \to \mathbf{Sets}$*.*

Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is left-adjoint to a functor $G : \mathcal{D} \to \mathcal{C}$ if there is a natural (in the sense explained below) bijection

$$\mathrm{Hom}_{\mathcal{C}}(C, G(D)) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{D}}(F(C), D) \tag{2}$$



for each $C \in \mathcal{C}, D \in \mathcal{D}$.
The bijection in 2 needs to be a natural isomorphism between functors. Here, the two functors are $\mathrm{Hom}_{\mathcal{D}}(F(-), -)$ and $\mathrm{Hom}_{\mathcal{C}}(-, G(-))$ which are functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Sets}$.
If F is left-adjoint to G, we write

$$F : \mathcal{C} \Leftrightarrow \mathcal{D} : G.$$

Example: Let $\mathcal{C} = \mathbf{Sets}$ and $\mathcal{D}$ be the category of Abelian groups. The free group functor F sends a set to the free Abelian group with generators the elements of that set. Let G be the forgetful functor sending each Abelian group to its underlying set. Notice that giving a morphism from a free Abelian group to a group is the same as specifying where the generators of the free Abelian group are mapped to. This means that F is left-adjoint to G.

Theorem 4.2. *The geometric realization functor is left-adjoint to the singular set functor:*

$$R : s\mathbf{Sets} \Leftrightarrow \mathbf{Top} : S$$

Proof. Let's regard the geometric realization RC of a simplicial set C. Recall from the proof of theorem 3.7 that RC is a CW-complex with an n-cell being a standard n-simplex Δ^n for each non-degenerate n-simplex of C. To describe a map from RC to a topological space D, we can specify where each cell of RC goes to in D, i.e. we specify a map $\Delta^n \to D$ for each non-degenerate n-simplex in C. These maps $\Delta^n \to D$ have to fulfill certain compatibility conditions coming from the d_i and s_i in C. Notice that giving such a map $\Delta^n \to D$ is the same as giving the image of the non-degenerate n-simplex of C in SD. The way we can map the non-degenerate simplices of C into SD is subject to the same conditions as the maps $\Delta^n \to D$ were. Hence we can construct from each map RC to D a map C to SD and vice versa. □

Let X be a topological space. Notice that an n-cell Δ^n in RSX corresponds to a non-degenerate simplex in SX. But such a simplex in SX is a map $\Delta^n \to X$. So each cell of RSX comes with a map to X and the simplicial structure of SX makes these maps on the cells compatible to each other, so that they can be glued together to a map called the **evaluation map** $RSX \to X$. The map is natural in the sense that it is a natural transformation between the functor $RS-$ and the identity functor $-$. For a CW-complex X, the map $RSX \to X$ turns out to be a homotopy equivalence. If X is a topological space, $RSX \to X$ is still a weak homotopy equivalence and it is a CW-approximation. It is a **canonical CW-approximation** in the sense that a continuous map $X_1 \to X_2$ induces a map $RSX_1 \to RSX_2$ commuting with the CW-approximations $RSX_i \to X_i$.

4.1 A category theoretic definition of the geometric realization

There is another way to explain the geometric realization. Remember that we had constructed a functor $F : \Delta \to \{\Delta^n\}$ which sends each $[n]$ to the standard n-simplex Δ^n and a morphism $[n] \to [k]$ to the induced linear maps sending the vertices of Δ^n in the predescribed way to the vertices of Δ^k.

Given a simplicial set X, construct a diagram[2] where the objects are a copy of the standard n-simplex Δ^n for each simplex $x \in X_n$. For a relation $m(x) = y$ where $x \in X_n, y \in X_k$ and m a morphism in Δ^{op}, add the affine map $F(m)$ between the copy of Δ^k and Δ^n corresponding to y and x.
For example if $x \in X_n, y \in X_{n-1}$ and $d_0(x) = y$, we would get a copy of Δ^n and Δ^{n-1} as objects and as morphism the map sending Δ^{n-1} onto the 0-th face of Δ^n.
If we take the colimit of this diagram, it will be the geometric realization.

In category theory terms, this idea can be expressed as follows: the category $\Delta \downarrow X$ has as objects maps $\Delta^n \to X$ (map in the sense of a morphism between two simplicial sets) and as morphisms commutative triangles where the map $\Delta^n \to \Delta^k$ is induced by a map $[n] \to [k]$. Notice that a map $\Delta^n \to X$ is

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[2] To be precise, we are constructing a functor from the small-category $\Delta \downarrow X$ to topological spaces.

uniquely determined by the images of i_n in X_n.
Then the geometric realization can be defined as

$$RX = \lim_{\overrightarrow{\Delta\downarrow X}} F.$$

See [DS95] for the definition of colimit. For category theory proofs of the adjointness of the geometric realization functor R and the singular simplex functor S, see [GJ99] or [Zak06].

5 Simplicial Sets and Delta Sets

Notice that $\hat{\Delta}$ is a sub-category of Δ, hence every Simplicial Object $\Delta \to \mathcal{C}$ is also a Delta Object $\hat{\Delta} \to \mathcal{C}$. This corresponds to removing all degeneracy operators from the Simplicial Objects and therefore we will call the functor sending a Simplicial Object to the underlying Delta Object the **forgetful functor**.
As we have seen, the geometric realization of a Delta set yields a Delta complex whereas this is not the case in general for Simplicial sets. After applying the forgetful functor to a Simplicial set, we can't detect degenerate simplices anymore. Hence we cannot expect an inverse functor to the forgetful functor. However there is an adjoint functor:

Theorem 5.1. *The functor sending a Delta-set Y to the simplicial set X associated to its geometric realization RY (as defined in 2.4) is left-adjoint to the forgetful functor.*

Proof. Notice that the categories of Delta-sets and Delta-complexes are equivalent.
We go from a simplicial sets P to Delta-complexes by first applying the forgetful functor and then taking the geometric realization Q. Each n-simplex of the simplicial set P, whether degenerate or not, will appear as n-simplex in the Delta-complex Q. A map from a Delta-complex X to Q is determined by where a simplex of X is going to and this information has to be compatible with the boundary relations of X. A simplex of X appears as non-degenerate simplex in the simplicial set Y associated to X and a map from Y to P is determined by saying where a non-degenerate simplex of Y is going to in P, this information being again subject to the same boundary relations. We see that the maps from X to Q and Y to P are in bijective correspondence. ☐

Theorem 5.2. *The left-adjoint functor of the forgetful functor is sending a Delta set Y to a simplicial set, say X, given by:*

- $X_n = \{(\sigma, y) : \sigma : [n] \to [k]$ *surjective morphism of* Δ *and* $y \in Y_k\}$
- *The morphism $\tau^* : X_n \to X_j$ corresponding to a morphism $\tau : [j] \to [n]$ sends*

$$\tau^* : (\sigma, y) \to (\sigma', \delta^*(y))$$

such that $\sigma \circ \tau = \delta \circ \sigma'$ *where* σ' *is a surjective map of* Δ *and* δ *is an injective map of* $\hat{\Delta} \subset \Delta$.

Notice that each morphism in Δ factors uniquely into a surjective and injective morphism respectively the other way around. Furthermore each injective (respectively surjective) morphism factors through the face $d_{i_m} \cdots d_{i_0}$ (respectively degeneracy operators $s_{i_m} \cdots s_{i_0}$). Recall that this factorization is unique if we require certain inequalities on the i_j.
X has as non-degenerate simplices the simplices of Y and we add degenerate simplices by adding "formal images" (σ, y) of non-degenerate simplices under degeneracy operators. If we want to know what τ does to such a formal image, we have to determine what $\sigma \circ \tau$ (simplicial sets are a contravariant functor) does to the non-degenerate simplex y. The answer has to be of the form (σ', y'). Hence we have to refactor $\sigma \circ \tau$ and the above formula results.

As an example of what the left-adjoint functor does, notice that the simplicial set Δ^n and $\partial\Delta^n$ are the images of $\hat{\Delta}^n$ and $\partial\hat{\Delta}^n$ under this functor. However the quotient $\Delta^n/\partial\Delta^n$ is never image of the left-adjoint functor.

6 Simplicial Abelian groups and chain complexes

A simplicial Abelian group is a simplicial object in the category of Abelian groups. We will see that the category of simplicial Abelian groups and of chain complexes are equivalent.

Definition 6.1. *If G is a simplicial Abelian group, define the chain complex $A(G)$ by*

$$A(G)_n = G_n, \qquad \partial = \sum_{i=0}^{n} (-1)^i d_i$$

and define the **Moore chain complex**[3] $N(G) \subset A(G)$ *by*

$$N(G)_n = G_n \cap \ker d_0 \cap \cdots \cap \ker d_{n-1}, \partial = (-1)^n d_n.$$

For a chain complex C, define a simplicial Abelian group $K(C)$ by

- $(KC)_n = \bigoplus_{\text{all surjective } \sigma:[n]\to[k]} C_k$
  We write (σ, y) for the element in $(KC)_n$ which is zero in each summand except for one, σ, where it is y. $(KC)_n$ is generated by all (σ, y).
- *The morphism $\tau^* : (KC)_n \to (KC)_j$ corresponding to a morphism $\tau : [j] \to [n]$ sends*
  $$\tau^* : (\sigma, y) \to (\sigma', (-1)^k \partial(y))$$

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[3] [Cur71] defines this as $G_n \cap \ker d_1 \cap \cdots \cap \ker d_n$ to remove the sign $(-1)^n$ from ∂

if there is a surjective $\sigma' : [n] \to [k]$ *of* Δ *such that* $\sigma \circ \tau = d_k \circ \sigma'$,

$$\tau^* : (\sigma, y) \to (\sigma', y)$$

if there is a surjective σ' *such that* $\sigma \circ \tau = \sigma'$ *and otherwise*

$$\tau^* : (\sigma, y) \to (\sigma, 0)$$

All the definitions are functorial. The definition of the functor K becomes clearer when we first define a Delta Abelian group from a chain complex:

Definition 6.2. *If* C *is a chain complex, define the Delta Abelian group* $\hat{K}C$ *by*

$$(\hat{K}C)_n = C_n$$

with

$$d_i : (\hat{K}C)_n \to (\hat{K}C)_{n+1}, d_i = \begin{cases} 0 & if \quad i < n \\ (-1)^n \partial & if \quad i = n. \end{cases}$$

Notice that the construction in theorem 5.2 can be done for groups. Instead of letting X_n be the set of pairs (σ, y), i.e. letting X_n be the disjoint union of Y_k, we have to take the coproduct in the category of Abelian groups, the direct sum

$$X_n = \bigoplus_{\text{all surjective } \sigma:[n]\to[k]} Y_k.$$

Thus we have a functor from Delta Abelian groups to Simplicial Abelian groups. Composing this functor with $\hat{K}$ gives the functor K.

Theorem 6.3 (Dold-Kan correspondence)**.** *The functors* K *and* N *provide a category equivalence of the category of Simplicial Abelian Groups and Chain complexes.*

There is no corresponding result for Delta Abelian Groups: after applying the functor A or N to an Delta Abelian Group, there is not enough information left to reconstruct the Delta Abelian Group.

Lemma 6.4. *Let* DG *denote the chain complex in* AG *generated by all degenerate simplices of* G*. Then*

$$AG = NG \oplus DG.$$

Proof. Notice that $NG \subset AG$ and define a map

$$f : AG \to NG$$

by

$$f : x \mapsto f^{n-1} \circ \cdots \circ f^0(x) \text{ if } x \in AG_n$$

where each f^k is defined by

$$f^k : x \mapsto x - s_k(d_k(x)),$$

and therefore ensures that d_k of x is zero after applying f^k. Hence f really maps into AG_n, and if x already has d_k equal to zero, then $f^k(x) = x$, i.e. f acts as identity on NG.

Now $AG = NG \oplus \ker f$, so it is left to compute $\ker f$. Let's compute $\ker f^k$: If $x \in \ker f^k$, then $x - s_k(d_k(x)) = 0$, so $x = s_k(d_k(x))$, so x is the image of a simplex under s_k. The other direction is also true: if x is the image of a simplex $s_k(y)$, then $x - s_k(d_k(x)) = s_k(y) - s_k(d_k(s_k(y))) = 0$, so $x \in \ker f^k$. We conclude that $\ker f^k$ is $\mathrm{Im} s_k$, so for each n

$$\ker f|_{AG_n} = \oplus \ker f^k|_{AG_n} = \oplus \mathrm{Im} s_k|_{AG_n} = DG_n$$

and hence

$$\ker f = DG.$$

□

Proof of Theorem. First construct the natural isomorphism C to NKC: For the sets underlying C, KC and NKC we have maps

$$\begin{array}{ccccc} C_n & \hookrightarrow & (KC)_n & \hookleftarrow & (NKC)_n \\ c & \mapsto & (\mathrm{id}, c) & \leftarrow & (\mathrm{id}, c) \end{array}$$

The map $C_n \to (KC)_n$ is injective, so it is left to show that every element in $(NKC)_n$ is in the image of C_n, i.e. is of the form (id, c). This means that there is no $\sum(\sigma, c) \in (KC)_n$ which is in $\ker d_0 \cap \cdots \cap \ker d_{n-1}$ with all $\sigma \neq \mathrm{id}$ and some $c \neq 0$.
Pick a $(\sigma, c) \in (KC)_n$ with $\sigma \neq \mathrm{id}$. Then for $\sigma : [n] \to [k]$ with $n > k$ there exists a d_i with $i < n$ such that $\sigma \circ d_i = \sigma'$ is another surjection, different for different σ. Hence d_i maps (σ, c) to (σ', c) and therefore (σ, c) is not in the kernel of d_i if $c \neq 0$. For $\sum(\sigma, c)$, each summand has a different σ, hence a different σ' and so all (σ', c) will be linearly independent and $\sum(\sigma', c)$ is not in $\bigcap_{i \neq n} \ker d_i$.

Next, construct the natural isomorphism from KNG to G. It is given by

$$\gamma : KNG \to G, \sum(\sigma, y) \mapsto \sum \sigma^* y$$

where $y \in NG$, i.e. $y \in G_n$ such that $d_i(y) = 0$ for $i < n$. This map is simplicial: the map of KNG respectively G corresponding to $\tau : [n] \to [n']$ maps $\sigma^* y$ to $\tau\sigma^* y$ respectively (σ, y) to $(\sigma', \delta(y))$, so $\tau \circ \sigma = \sigma' \circ \delta$ ensures that γ commutes with the τ^*.

This map is injective: by induction assume that $\gamma(x) = 0$ implies $x = 0$ if x is in $(KNG)_0, \ldots, (KNG)_n$.
Now assume $\gamma(x) = 0$ with $x \in (KNG)_{n+1}$. Then also $\gamma(d_i(x)) = 0$ for all i and but by the induction hypothesis, this means that $d_i(x) = 0$ for all i which implies $x \in NG \subset KNG$. But we also have $\gamma|_{NG} = \mathrm{id}$ and $NG \subset G$, so $x = 0$.

The map γ is surjective: again assume by induction that for each $x \in G_0, \dots G_n$ there exists a $x' \in KNG$ such that $\gamma(x') = x$.
Pick $x \in G_{n+1}$, then by lemma 6.4

$$x = x_1 + x_2 = x_1 + \sum s_i^*(y_i)$$

where $x_1 \in NG \subset KNG$ and $x_2 \in DG$, hence x_2 is the sum of degenerate elements. As all $y_i \in G_n$, by induction hypothesis each y_i has a preimage $y_i' \in KNG$ with $\gamma(y_i') = y_i$. Setting $x' = x_1 + \sum s_i^*(y_i')$ (where s_i^* is the morphism in KNG) gives the result. □

7 Outlook

It is possible to define homotopy on simplicial sets with certain extra conditions, called Kan-conditions, which for example all simplicial groups or singular sets $S(Y)$ fullfill. The last theorem can then be extended to say that the homotopy groups of a simplicial group are equal to the homology groups of the corresponding chain complex. Hence the chain complex

$$\cdots \longleftarrow 0 \longleftarrow \overset{\text{dimension } n}{G} \longleftarrow 0 \longleftarrow \cdots$$

corresponds to the simplicial group with $\pi_n = G$ and all other $\pi_i = 0$, so this gives a $K(G, n)$ in the category of simplicial groups. The geometric realization therefore gives a canonical Eilenberg-MacLane space $K(G, n)$, i.e. a functor from groups to topological groups. It is surprising how easy the construction of $K(G, n)$ is in the category of simplicial groups.
In a similar fashion, other spaces, like loop spaces, can be constructed explicitly in the category of simplicial groups, hence one application of simplicial sets and groups is to compute homotopy groups of spaces by looking at the corresponding chain complexes which are easier to understand.

One can also show that the category of simplicial sets can be given a model category structure, see [Zak06].

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