

SHARP L^q BOUNDS ON SPECTRAL CLUSTERS FOR HOLDER METRICS

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ABSTRACT. We establish L^q bounds on eigenfunctions, and more generally on spectrally localized functions (spectral clusters), associated to a self-adjoint elliptic operator on a compact manifold, under the assumption that the coefficients of the operator are of regularity C^s , where $0 \leq s \leq 1$. We also produce examples which show that these bounds are best possible for the case $q = \infty$, and for $2 \leq q \leq q_n$.

1. INTRODUCTION

Let M be a compact manifold without boundary, on which we fix a smooth volume form dx . Let a be a section of real, symmetric quadratic forms on $T^*(M)$, with associated linear transforms $a_x : T_x^*(M) \rightarrow T_x(M)$, and let ρ be a real valued function on M . We assume both a and ρ are strictly positive, with uniform bounds above and below.

Consider the eigenfunction problem, with d^* the transpose of d relative to dx ,

$$(1.1) \quad d^*(a df) + \lambda^2 \rho f = 0.$$

Under the condition that a and ρ are bounded, measurable, and uniformly bounded from below, there exists a complete orthonormal basis ϕ_j of eigenfunctions for $L^2(M, \rho dx)$ with frequencies $\lambda_j \rightarrow \infty$. In this paper we establish the following theorem.

Theorem 1.1. *Suppose that $a, \rho \in C^s(M)$, where $0 \leq s \leq 1$. Assume that the frequencies λ_j of f are contained in the interval $[\lambda, \lambda + \lambda^{1-s}]$, so that*

$$(1.2) \quad f = \sum_{j: \lambda_j \in [\lambda, \lambda + \lambda^{1-s}]} c_j \phi_j$$

Then for $2 \leq q \leq q_n = \frac{2(n+1)}{n-1}$

$$(1.3) \quad \|f\|_{L^q(M)} \leq C \lambda^{\left(\frac{2(n-1)}{2+s} + 1 - s\right)\left(\frac{1}{2} - \frac{1}{q}\right)} \|f\|_{L^2(M)}.$$

Furthermore

$$(1.4) \quad \|f\|_{L^\infty(M)} \leq C \lambda^{\frac{n-s}{2}} \|f\|_{L^2(M)}.$$

The constant C depends only on the C^s norm and lower bounds of a and ρ . In particular, C is uniform under small C^s perturbations of a and ρ .

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We also produce examples to show that, for general C^s metrics, (1.4) and (1.3) are best possible, in the latter case for all q in the given range. The examples are exponentially localized eigenfunctions on open sets, and show that the exponents in Theorem 1.1 cannot be improved in general even for functions f with frequency spread $O(\lambda^{-N})$. It is not known what the sharp bounds are for $q_n < q < \infty$.

We compare Theorem 1.1 to the bounds in the case a and ρ belong to C^2 , or more generally $C^{1,1}$, and where the frequencies of f lie in the interval $[\lambda, \lambda + 1]$. In that case, by [8],

$$\begin{aligned} \|f\|_{L^q(M)} &\leq C \lambda^{\frac{n-1}{2}(\frac{1}{2}-\frac{1}{q})} \|f\|_{L^2(M)}, & 2 \leq q \leq q_n, \\ \|f\|_{L^q(M)} &\leq C \lambda^{n(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}} \|f\|_{L^2(M)}, & q_n \leq q \leq \infty. \end{aligned}$$

These estimates were established by Sogge [12] for smooth a and ρ , and are best possible at all q for unit width spectral clusters. The case $q = \infty$ is related to the spectral counting remainder estimate; see Hörmander [5]. Recently, Bronstein and Ivrii [2] have obtained spectral counting remainder estimates in the case of Hölder coefficients. A direct corollary of their estimates is an upper bound of the form $O((\log \lambda)^\sigma \lambda^{n-s})$ for the number of frequencies λ_j (counting multiplicity) in the interval $[\lambda, \lambda + \lambda^{1-s}]$. A corollary of (1.4) is that this upper bound holds with $\sigma = 0$. Indeed, (1.4) implies pointwise bounds on the kernel $\chi_\lambda(x, y)$ of the spectral projection onto frequencies in the range $[\lambda, \lambda + \lambda^{1-s}]$,

$$|\chi_\lambda(x, y)| \leq C^2 \lambda^{n-s},$$

and integrating over the diagonal yields the desired trace bounds.

Estimates for C^s metrics are derived from the C^2 result together with a frequency dependent scaling argument. Our work shows that there are two distinct spatial scales that enter into the estimates when $s < 1$,

$$R = \lambda^{-\frac{2-s}{2+s}}, \quad T = \lambda^{s-1}.$$

The scale R is the size of a ball on which, when working with solutions f with frequencies of magnitude λ , the coefficients a and ρ are well approximated by C^2 functions, in the sense that the errors can be absorbed as an appropriate source term. The larger scale T is the size of a ball on which the coefficients are well approximated by C^1 functions.

Compared to the C^2 case, there is in Theorem 1.1 a loss of $(TR^{-1})^{\frac{1}{q}} T^{-\frac{1}{2}}$ at the indices $q = q_n$ and $q = \infty$. (The sharp bounds for $2 \leq q \leq q_n$ are obtained by interpolation.) The loss of $T^{-\frac{1}{2}}$ arises in spatially localizing solutions to balls of size T in order to have energy flux bounds. The loss of $(TR^{-1})^{\frac{1}{q}}$ arises from the fact that we have good L^q bounds on sets of size R , and need to sum over a total of TR^{-1} sets to obtain bounds on sets of size T . For $1 \leq s \leq 2$ only the scale R enters, and the loss relative to the C^2 case is $R^{\frac{1}{q}}$ for $q = q_n$ and $q = \infty$, as shown by the second author in [9].

One can establish better bounds on the L^q norm of f over balls of size R . In that case, there is only the loss of $T^{-\frac{1}{2}}$ relative to the C^2 case. Precisely, in the process of proving Theorem 1.1 we also establish the following.

Theorem 1.2. *Let $B_R \subset M$ be a ball of radius $R = \lambda^{-\frac{2-s}{2+s}}$. Then under the same conditions as Theorem 1.1, and with a constant C uniform over such balls B_R ,*

$$(1.5) \quad \|f\|_{L^q(B_R)} \leq C \lambda^{n(\frac{1}{2}-\frac{1}{q})-\frac{s}{2}} \|f\|_{L^2(M)}, \quad q_n \leq q \leq \infty.$$

The examples we produce to show that (1.4) is sharp also show that (1.5) cannot be improved for any $q_n \leq q \leq \infty$. It is expected, though, that the bound obtained by interpolating (1.3) at $q = q_n$ with (1.4) is not sharp for $q_n < q < \infty$. That is, the additional loss of $(TR^{-1})^{\frac{1}{q}}$ which would be obtained by adding (1.5) over TR^{-1} disjoint sets is sharp only for q_n and ∞ .

Since the constant C in (1.5) is independent of the center of B_R , estimate (1.4) is an immediate consequence of the case $q = \infty$ of (1.5). We also remark that all cases of (1.5) for $q_n \leq q \leq \infty$ follow from the case $q = q_n$ of (1.5). This was noted in [9], using heat kernel estimates. Briefly, by Theorem 6.3 of Saloff-Coste [7], the heat kernel $h_\lambda(x, y)$ at time $t = \lambda^{-2}$ for the diffusion system associated to (1.1) satisfies

$$|h_\lambda(x, y)| \leq C \lambda^n \exp(-c \lambda^2 d(x, y)^2).$$

By Young's inequality, then for $q_n \leq q \leq \infty$

$$\begin{aligned} \|f\|_{L^q(B_R)} &\leq C \lambda^{n(\frac{1}{q_n} - \frac{1}{q})} \|H_\lambda^{-1} f\|_{L^{q_n}(B_R^*)} + C_N \lambda^{-N} \|H_\lambda^{-1} f\|_{L^2(M \setminus B_R^*)} \\ &\leq C \lambda^{n(\frac{1}{2} - \frac{1}{q}) - \frac{N}{2}} \|f\|_{L^2(M)} \end{aligned}$$

where we use (1.5) at $q = q_n$ with B_R replaced by its double B_R^* , and the fact that $\|H_\lambda^{-1} f\|_{L^2} \approx \|f\|_{L^2}$ since $\exp(\lambda_j^2/\lambda^2) \approx 1$ for $\lambda_j \in [\lambda, \lambda + 1]$.

Since (1.3) follows from the case $q = q_n$ (by interpolating with the trivial case $q = 2$) we are thus reduced to establishing the estimates (1.3) and (1.5) at $q = q_n$.

Notation. By the C^s norm on \mathbb{R}^n for $0 < s \leq 1$ we mean

$$\|f\|_{C^s} = \|f\|_{L^\infty(\mathbb{R}^n)} + \sup_{h \in \mathbb{R}^n} |h|^{-s} \|f(\cdot + h) - f(\cdot)\|_{L^\infty(\mathbb{R}^n)}.$$

Thus, C^s coincides with Lipschitz for $s = 1$.

We use d to denote the differential taking functions to covector fields, and d^* its adjoint with respect to dx . When working on \mathbb{R}^n , $d = (\partial_1, \dots, \partial_n)$, and d^* is the standard divergence operator.

The notation $A \lesssim B$ means $A \leq C B$, where C is a constant that depends only on the C^s norm of a and ρ , as well as on universally fixed quantities, such as the manifold M and the non-degeneracy of a and ρ . In particular, C will depend continuously on a and ρ in the C^s norm.

2. PROOF OF THEOREMS 1.1 AND 1.2

For $s = 0$ both theorems are a result of Sobolev embedding, for example using heat kernel estimates, so we restrict to the case $s > 0$. Let ϕ_T denote a smooth cutoff to a ball $B_T \subset M$ of diameter $T = \lambda^{s-1}$. We then write

$$d^*(a d(\varphi_T f)) + \lambda^2 \rho(\varphi_T f) = T^{-1} (d^* g_1 + g_2)$$

where g_1, g_2 are supported in B_T^* (the double of B_T), with

$$g_1 = a(T d \varphi_T) f \quad g_2 = a(T d \varphi_T, df) + T \varphi_T (d^*(a df) + \lambda^2 \rho f)$$

and hence

$$\|g_1\|_{L^2} + \lambda^{-1} \|g_2\|_{L^2} \lesssim \|f\|_{L^2(B_T^*)} + \lambda^{-1} \|df\|_{L^2(B_T^*)} + \lambda^{s-2} \|d^*(a df) + \lambda^2 \rho f\|_{L^2(B_T^*)}$$

Note that if f is of the form (1.2), and $\lambda \geq 1$, then

$$\begin{aligned} \|d^*(a df) + \lambda^2 \rho f\|_{L^2(M)} &\lesssim \lambda^{2-s} \|f\|_{L^2(M)}, \\ \|df\|_{L^2(M)} &\lesssim \lambda \|f\|_{L^2(M)}. \end{aligned}$$

Consequently, if we prove that for f satisfying

$$(2.1) \quad d^*(a df) + \lambda^2 \rho = T^{-1}(d^*g_1 + g_2)$$

we have

$$(2.2) \quad \|f\|_{L^{q_n}(B_T)} \lesssim \lambda^{\frac{1}{q_n}} (TR^{-1})^{\frac{1}{q_n}} T^{-\frac{1}{2}} (\|f\|_{L^2} + \lambda^{-1} \|df\|_{L^2} + \|g_1\|_{L^2} + \lambda^{-1} \|g_2\|_{L^2})$$

as well as

$$(2.3) \quad \|f\|_{L^{q_n}(B_R)} \lesssim \lambda^{\frac{1}{q_n}} T^{-\frac{1}{2}} (\|f\|_{L^2} + \lambda^{-1} \|df\|_{L^2} + \|g_1\|_{L^2} + \lambda^{-1} \|g_2\|_{L^2})$$

then summing over a cover of M by balls B_T with bounded overlap yields (1.3) and (1.5) at $q = q_n$, and hence all cases of Theorems 1.1 and 1.2 as remarked above.

By choosing local coordinates we may assume we are working with an equation of the form (2.1) on \mathbb{R}^n with Lebesgue measure, and with f supported in a ball of radius T . After making a linear change of coordinates and multiplying ρ by a harmless constant, we may additionally assume (for $s > 0$) that

$$(2.4) \quad \|a - I\|_{C^s(\mathbb{R}^n)} + \|\rho - 1\|_{C^s(\mathbb{R}^n)} \leq c_0,$$

for c_0 a suitably small constant to be fixed depending on the dimension n .

Let $S_r = S_r(D)$ denote a smooth cutoff on the Fourier transform side to frequencies of size $|\xi| \leq r$. Let $a_\lambda = S_{c\lambda} a$, for c to be chosen suitably small. Then

$$\|(a - a_\lambda)df\|_{L^2} \lesssim \lambda^{-1} T^{-1} \|df\|_{L^2}, \quad \lambda^2 \|(\rho - \rho_\lambda)f\|_{L^2} \lesssim \lambda T^{-1} \|f\|_{L^2},$$

and thus we may replace a and ρ by a_λ and ρ_λ at the expense of absorbing the above two terms into g_1 and g_2 , which does not change the size of the right hand side of (2.2) and (2.3).

Next, let $f_{<\lambda} = S_{c\lambda} f$. Then

$$(2.5) \quad \|[S_{c\lambda}, a_\lambda]\|_{L^2 \rightarrow L^2} \lesssim \lambda^{-1} \|a_\lambda\|_{C^1} \lesssim \lambda^{-1} T^{-1},$$

and similarly for $[S_{c\lambda}, \rho_\lambda]$, hence we can absorb the commutator terms into g_1 and g_2 , and since all terms are localized to frequencies less than λ we can write

$$(2.6) \quad d^*(a_\lambda df_{<\lambda}) + \lambda^2 \rho_\lambda f_{<\lambda} = T^{-1} g_{<\lambda},$$

where

$$\|g_{<\lambda}\|_{L^2} \lesssim \lambda \|f\|_{L^2} + \|df\|_{L^2} + \lambda \|g_1\|_{L^2} + \|g_2\|_{L^2}.$$

Since $\|d^*(a_\lambda df_{<\lambda})\|_{L^2} \lesssim (c\lambda)^2 \|f_{<\lambda}\|_{L^2}$, for c suitably small the L^2 norm of the left hand side of (2.6) is comparable to $\lambda^2 \|f_{<\lambda}\|_{L^2}$, hence we have

$$\|f_{<\lambda}\|_{L^2} \lesssim \lambda^{-1} T^{-1} (\|f\|_{L^2} + \lambda^{-1} \|df\|_{L^2} + \|g_1\|_{L^2} + \lambda^{-1} \|g_2\|_{L^2}).$$

Since $\frac{1}{q_n} = n(\frac{1}{2} - \frac{1}{q_n}) - \frac{1}{2}$, Sobolev embedding implies

$$(2.7) \quad \|f_{<\lambda}\|_{L^{q_n}} \lesssim \lambda^{\frac{1}{q_n} - \frac{1}{2}} T^{-1} (\|f\|_{L^2} + \lambda^{-1} \|df\|_{L^2} + \|g_1\|_{L^2} + \lambda^{-1} \|g_2\|_{L^2}),$$

which implies (2.2) and (2.3) for this term since $R, T \geq 1$ and $\lambda^{-\frac{1}{2}} \leq T^{\frac{1}{2}}$.

If we let $f_{>\lambda} = f - S_{c^{-1}\lambda}f$, then similar arguments let us write

$$(2.8) \quad d^*(a_\lambda df_{>\lambda}) + \lambda^2 \rho_\lambda f_{>\lambda} = T^{-1} d^* g_{>\lambda}$$

where now $g_{>\lambda}$, like $f_{>\lambda}$, is frequency localized to frequencies larger than $c^{-1}\lambda$, and

$$\|g_{>\lambda}\|_{L^2} \lesssim \|f\|_{L^2} + \lambda^{-1} \|df\|_{L^2} + \|g_1\|_{L^2} + \lambda^{-1} \|g_2\|_{L^2}.$$

Taking the inner product of both sides of (2.8) against $f_{>\lambda}$ yields

$$\|df_{>\lambda}\|_{L^2}^2 - 4\lambda^2 \|f_{>\lambda}\|_{L^2}^2 \lesssim T^{-1} \|g_{>\lambda}\|_{L^2} \|df_{>\lambda}\|_{L^2},$$

and by the frequency localization of $f_{>\lambda}$ we obtain

$$\|f_{>\lambda}\|_{H^1} \lesssim T^{-1} (\|f\|_{L^2} + \lambda^{-1} \|df\|_{L^2} + \|g_1\|_{L^2} + \lambda^{-1} \|g_2\|_{L^2}).$$

Since $n(\frac{1}{2} - \frac{1}{q_n}) = \frac{1}{q_n} + \frac{1}{2} \leq 1$, Sobolev embedding yields (2.7) for the term $f_{>\lambda}$.

It remains to establish (2.2) and (2.3) for $f_\lambda = S_{c^{-1}\lambda}f - S_{c\lambda}f$. We decompose f_λ using a partition of unity in the Fourier transform variable ξ to cones of small angle. We may thus assume that f_λ is frequency localized to a small cone about the ξ_1 axis. Since the localization is by means of an order 0 multiplier at frequency λ , the commutator satisfies the same bounds (2.5), and we may write

$$(2.9) \quad d^*(a_\lambda df_\lambda) + \lambda^2 \rho_\lambda f_\lambda = T^{-1} g_\lambda$$

where

$$\|g_\lambda\|_{L^2} \lesssim \lambda \|f\|_{L^2} + \|df\|_{L^2} + \lambda \|g_1\|_{L^2} + \|g_2\|_{L^2}.$$

The functions a_λ and ρ_λ satisfy the condition (2.4). By Corollary 2.4 of [9], we thus have the localized estimate, uniformly over cubes Q_R of sidelength R ,

$$\|f_\lambda\|_{L^{q_n}(Q_R)} \lesssim R^{-\frac{1}{2}} \lambda^{\frac{1}{q_n}} (\|f_\lambda\|_{L^2(Q_R^*)} + \lambda^{-1} \|df_\lambda\|_{L^2(Q_R^*)} + RT^{-1} \lambda^{-1} \|g_\lambda\|_{L^2(Q_R^*)}).$$

Summing over disjoint cubes contained in a slab S_R of the form $\{x \in \mathbb{R}^n : |x_1 - c| \leq R\}$ yields

$$\|f_\lambda\|_{L^{q_n}(S_R)} \lesssim R^{-\frac{1}{2}} \lambda^{\frac{1}{q_n}} (\|f_\lambda\|_{L^2(S_R^*)} + \lambda^{-1} \|df_\lambda\|_{L^2(S_R^*)} + RT^{-1} \lambda^{-1} \|g_\lambda\|_{L^2(S_R^*)}).$$

We will show that

$$(2.10) \quad \|f_\lambda\|_{L^2(S_R)} + \lambda^{-1} \|df_\lambda\|_{L^2(S_R)} \lesssim R^{\frac{1}{2}} T^{-\frac{1}{2}} (\|f_\lambda\|_{L^2} + \lambda^{-1} \|df_\lambda\|_{L^2} + \lambda^{-1} \|g_\lambda\|_{L^2}).$$

Since $R^{\frac{1}{2}} T^{-1} \leq T^{-\frac{1}{2}}$, this yields

$$\|f\|_{L^{q_n}(S_R)} \lesssim \lambda^{\frac{1}{q_n}} T^{-\frac{1}{2}} (\|f\|_{L^2} + \lambda^{-1} \|df\|_{L^2} + \|g_1\|_{L^2} + \lambda^{-1} \|g_2\|_{L^2}).$$

The estimate (2.3) follows immediately; the estimate (2.2) follows after summing over the TR^{-1} disjoint slabs S_R that intersect B_T .

The bound (2.10) follows by energy methods. Let V denote the vector field

$$V = 2(\partial_1 f_\lambda) a_\lambda df_\lambda + (\lambda^2 \rho_\lambda f_\lambda^2 - \langle a_\lambda df_\lambda, df_\lambda \rangle) \vec{e}_1.$$

Then

$$d^*V = 2T^{-1}(\partial_1 f_\lambda) g_\lambda + \lambda^2(\partial_1 \rho_\lambda) f_\lambda^2 - \langle (\partial_1 a_\lambda) df_\lambda, df_\lambda \rangle.$$

Since $|\partial_1 a_\lambda| + |\partial_1 \rho_\lambda| \lesssim T^{-1}$, applying the divergence theorem on the set $x_1 \leq r$ yields

$$\int_{x_1=r} V_1 dx' \lesssim T^{-1} (\lambda^2 \|f_\lambda\|_{L^2(\mathbb{R}^n)}^2 + \|df_\lambda\|_{L^2(\mathbb{R}^n)}^2 + \|g_\lambda\|_{L^2(\mathbb{R}^n)}^2).$$

Since a_λ and ρ are pointwise close to the flat metric, we have pointwise that

$$V_1 \geq \frac{3}{4} |\partial_1 f_\lambda|^2 + \frac{3}{4} \lambda^2 |f_\lambda|^2 - |\partial_{x'} f_\lambda|^2.$$

The frequency localization of \widehat{f}_λ to $|\xi'| \leq c\lambda$ yields

$$\int_{x_1=r} V_1 dx' \geq \frac{1}{2} \int_{x_1=r} |df_\lambda|^2 + \lambda^2 |f_\lambda|^2 dx'.$$

Integrating this over r in an interval of size R yields (2.10). \square

3. EXAMPLES TO SHOW (1.4) AND (1.5) ARE SHARP

In this section we produce examples of Hölder metrics and associated eigenfunctions which show that the estimates (1.4) of Theorem 1.1 and (1.5) of Theorem 1.2 are best possible. Our example is a radial version of the 1-dimensional example of Castro-Zuazua [3], which in turn is based on a calculation of Colombini-Spagnolo [4]. In our example the metric depends on the frequency λ (with uniform bounds on its Hölder norm), but because of the exponential localization of the eigenfunctions one may easily cut and paste together a sequence of examples to produce a metric for which these estimates fail for a sequence of λ tending to ∞ , as in [3]. Our example is also global on \mathbb{R}^n , but again by its exponential localization it may be truncated and placed on a compact manifold, with the truncation errors small enough to show that the estimates are still best possible for spectral clusters, even those of spectral width $O(\lambda^{-N})$.

We start by producing smooth radial functions Φ, q_1, q_2 , all of which vanish near 0, with $\Phi(r) = r + O(1)$, such that for all $\varepsilon > 0$:

$$(\Delta + 1 + \varepsilon q_1 + \varepsilon^2 q_2) e^{-\varepsilon \Phi} \widehat{d\sigma} = 0,$$

where $d\sigma$ is surface measure on the unit sphere S^{n-1} . Furthermore, Φ', q_1, q_2 are globally bounded, together with their derivatives of all order.

For this, we write

$$\widehat{d\sigma}(r) = r^{\frac{1-n}{2}} F_n(r).$$

Then (see Stein p. 338–348)

$$F_n(r) = a_n \cos(r - \frac{n-1}{4} \pi) + b_n r^{-1} \sin(r - \frac{n-1}{4} \pi) + O(r^{-2}).$$

We normalize $d\sigma$ so $a_n = 1$, and set

$$\Phi(r) = 2 \int_0^r h(s) F_n(s)^2 ds,$$

where h is a smooth non-negative function which vanishes near 0 and equals 1 for $s > \frac{1}{2}$. By the asymptotics of F_n , we have $\Phi(r) = r + O(1)$.

Consider the radial Laplacian $\Delta = \frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr}$. Then

$$r^{\frac{n-1}{2}} \Delta r^{\frac{1-n}{2}} = \frac{d^2}{dr^2} - \frac{(n-1)(n-3)}{4r^2}$$

and hence we may expand $\Delta(e^{-\varepsilon \Phi} \widehat{d\sigma})$ as

$$r^{\frac{1-n}{2}} \left[F_n'' - \frac{(n-1)(n-3)}{4r^2} F_n - 4\varepsilon F_n' h F_n^2 - 2\varepsilon F_n (h F_n^2)' + 4\varepsilon^2 F_n h^2 F_n^4 \right] e^{-\varepsilon \Phi}$$

where $'$ denotes $\frac{d}{dr}$. This in turn may be written as

$$-(1 + \varepsilon q_1 + \varepsilon^2 q_2) e^{-\varepsilon \Phi} \widehat{d\sigma}$$

with

$$q_1 = 4hF_n F_n' + 2(hF_n^2) ', \quad q_2 = -4h^2 F_n^4.$$

These functions are smooth since h vanishes near 0, and the global boundedness of their derivatives follows easily from boundedness of $F_n^{(k)}$ for r bounded away from 0.

To construct the example, we set $\varepsilon = \lambda^{-s}$ and change variables $r \rightarrow \lambda r$, and let

$$\psi_\lambda(r) = e^{-\lambda^{-s} \Phi(\lambda r)} \widehat{d\sigma}(\lambda r), \quad \rho_\lambda(r) = 1 + \lambda^{-s} q_1(\lambda r) + \lambda^{-2s} q_2(\lambda r),$$

so that

$$\Delta \psi_\lambda + \lambda^2 \rho_\lambda \psi_\lambda = 0.$$

Note that $\rho_\lambda \in C^s$, since

$$\|1 - \rho_\lambda\|_{L^\infty} \leq \lambda^{-s}, \quad \|1 - \rho_\lambda\|_{\text{Lip}} \leq \lambda^{1-s},$$

and $\min(\lambda^{-s}, \lambda^{1-s}|x-y|) \leq |x-y|^s$. On the other hand,

$$|\psi_\lambda(r)| \approx e^{-\lambda^{1-s} r} |\widehat{d\sigma}|(\lambda r).$$

Precisely,

$$\|\psi_\lambda\|_{L^2}^2 = \lambda^{1-n} \int_0^\infty |F_n(\lambda r)|^2 e^{-2\lambda^{-s} \Phi(\lambda r)} dr \approx \lambda^{s-n} \int_0^\infty |F_n(\lambda^s r)|^2 e^{-2r} dr \approx \lambda^{s-n}.$$

The maximum of $|\widehat{d\sigma}|$ occurs at $r = 0$, so $\|\psi_\lambda\|_{L^\infty}$ is independent of λ . Thus

$$\frac{\|\psi_\lambda\|_{L^\infty}}{\|\psi_\lambda\|_{L^2}} \approx \lambda^{\frac{n-s}{2}}$$

showing that (1.4) is sharp.

We also note that $|\psi_\lambda| \approx 1$ for $r \leq \lambda^{-1}$. Hence $\|\psi_\lambda\|_{L^q(B_{1/\lambda})} \gtrsim \lambda^{-\frac{n}{q}}$. Hence

$$\frac{\|\psi_\lambda\|_{L^q(B_{1/\lambda})}}{\|\psi_\lambda\|_{L^2}} \gtrsim \lambda^{n(\frac{1}{2} - \frac{1}{q}) - \frac{s}{2}}$$

showing that (1.5) is sharp. \square

4. EXAMPLES TO SHOW (1.3) IS SHARP

In this section we produce examples to show that (1.3) is similarly sharp, in the range $2 \leq q \leq q_n$. These examples are exponentially localized to a tube of diameter $\lambda^{\frac{2}{2+s}}$, and of length λ^{s-1} . They are essentially a product of the examples of Smith-Sogge [10], where the metric depends on $n-1$ variables, with the 1-dimensional example of Castro-Zuazua [3].

For the examples, fix $0 < s < 1$. Let (x, y) denote variables on \mathbb{R}^n with $x \in \mathbb{R}$ and $y \in \mathbb{R}^{n-1}$. We take $A(y)$ to be the ground state solution of

$$-\Delta_y A(y) + |y|^s A(y) = cA(y).$$

Here $c > 0$ and $A(y)$ is radial and of class $C^{2,s}$, with

$$|A(y)| \lesssim e^{-N|y|}$$

for all $N > 0$. This may be seen, for example, by Theorem XIII.47 of [6] and Theorems 4.1 and 5.1 of [1]. The regularity follows by examining the ordinary differential equation in r near 0.

Given $\kappa > 0$, let $\lambda = \lambda(\kappa)$ solve

$$\lambda^2 - \kappa^2 = c\lambda^{2\delta}, \quad \delta = \frac{2}{2+s}.$$

For $s > 0$ there is a unique positive solution for large κ , which satisfies $\lambda \approx \kappa$.

We then have

$$(\Delta_y + \lambda^2(1 - |y|^s))A(\lambda^\delta y) = \kappa^2 A(\lambda^\delta y).$$

Next consider Φ, q_1, q_2 , as in the preceding section, with $n = 1$ and $F_1 = \cos(x)$. We set

$$\rho_\kappa(x) = 1 + \kappa^{-s}q_1(\kappa|x|) + \kappa^{-2s}q_2(\kappa|x|).$$

Then

$$(\partial_x^2 + \kappa^2\rho_\kappa(x))e^{-\kappa^{-s}\Phi(\kappa|x|)}\cos(\kappa x) = 0,$$

and consequently

$$\left(d_x^2 + \rho_\kappa(x)\Delta_y + \lambda^2(1 - |y|^s)\rho_\kappa(x)\right)e^{-\kappa^{-s}\Phi(\kappa|x|)}A(\lambda^\delta y)\cos(\kappa x) = 0.$$

This equation takes the form (1.1). Since $\kappa \approx \lambda$, the eigenfunction is exponentially concentrated in the set

$$|y| \leq \lambda^{-\frac{2}{2+s}}, \quad |x| \leq \lambda^{s-1}.$$

By Hölder's inequality, this implies that

$$\frac{\|\psi_\lambda\|_{L^p}}{\|\psi_\lambda\|_{L^2}} \gtrsim \lambda^{\left(\frac{2}{2+s}(n-1)+(1-s)\right)\left(\frac{1}{2}-\frac{1}{p}\right)}$$

showing that (1.3) is sharp. □

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