

Midterm 2, Math 53
July 11th, 2008

Solutions of all problems must be accompanied by relevant explanations. Show your work, but not to others. Simplify your answers where possible.

Problem 1. Find the limit or prove that it doesn't exist

a)

$$\lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 + y^2)e^{xy} - 3 \sin(x^2 + y^2)}{x^3y + xy^3 + x^2 + y^2},$$

10 points

b)

$$\lim_{(x,y) \rightarrow (0,0)} \frac{(x + y)^4 - 14x^2y^2}{x^4 + y^4}.$$

10 points

Solution:

a) It does exist.

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 + y^2)e^{xy} - 3 \sin(x^2 + y^2)}{x^3y + xy^3 + x^2 + y^2} &= \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 + y^2)e^{xy} - 3 \sin(x^2 + y^2)}{(x^2 + y^2)xy + x^2 + y^2} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 + y^2)(e^{xy} - 3 \frac{\sin(x^2 + y^2)}{(x^2 + y^2)})}{(x^2 + y^2)(xy + 1)} = \lim_{(x,y) \rightarrow (0,0)} \frac{e^{xy} - 3 \frac{\sin(x^2 + y^2)}{(x^2 + y^2)}}{xy + 1} = \frac{1 - 3}{0 + 1} = -2. \end{aligned}$$

Here we use the fact that $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1$.

b) It does not exist. Send the point (x, y) to the origin along the line $y = x$, that is set $y = x$ and take the limit as $x \rightarrow 0$. We get

$$\lim_{x \rightarrow 0} \frac{(2x)^4 - 14x^4}{2x^4} = 1.$$

Now send the point along the line $y = -x$, that is set $y = -x$ and take the limit $x \rightarrow 0$ and get

$$\lim_{x \rightarrow 0} \frac{(x - x)^4 - 14x^4}{2x^4} = -7.$$

Since $1 \neq -7$ the limit does not exist.

Problem 2. Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(x, y) = (0, 0)$ if $z = e^{uv} + v^7$ and $u = \sin(\sin x)$ and $v = \cos x + y^2$. Find the directional derivative of the function $z(x, y)$ at the point $(x, y) = (0, 0)$ in the direction of the point $(-2, 0)$.

10 points

Solution:

Clearly we have to use the chain rule

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}.$$

Now we calculate $\frac{\partial z}{\partial u} = ve^{uv}$, $\frac{\partial z}{\partial v} = ue^{uv} + 7v^6$, $\frac{\partial u}{\partial x} = \cos(\sin x) \cos x$, $\frac{\partial u}{\partial y} = 0$, $\frac{\partial v}{\partial x} = -\sin x$, $\frac{\partial v}{\partial y} = 2y$. At the point $(x, y) = (0, 0)$ we get $u = 0$ and $v = 1$. By plugging in these values we get

$$\frac{\partial z}{\partial x}(0, 0) = 1 \cdot 1 + 0 \cdot 0 = 1 \quad \text{and} \quad \frac{\partial z}{\partial y}(0, 0) = 1 \cdot 0 + 0 \cdot 0 = 0$$

So the gradient of $z(x, y)$ at the point $(0, 0)$ is $\langle 1, 0 \rangle$. To find the directional derivative at the point $(0, 0)$ in the direction of the point $(-2, 0)$ we dot this vector with the normalized vector $\langle -2, 0 \rangle$ which is $\langle -1, 0 \rangle$. We get $\langle 1, 0 \rangle \cdot \langle -1, 0 \rangle = -1$.

Problem 3. Are there any points on the surface $x^3 + x + \sin(yz)e^z = 9$ where the tangent plane to this surface is parallel to the plane $z = 0$. Are there any points on this surface where the tangent plane is parallel to the plane $y = 0$. Justify your answers.

10 points

Solution:

There are no such points.

For the tangent plane at a point (x_0, y_0, z_0) to be parallel to the plane $z = 0$ we need the normal vector of the tangent plane and the normal vector of the plane $z = 0$ to be parallel. Similarly for the tangent plane to be parallel to the plane $y = 0$ we need the normal vector of the tangent plane to be parallel to the normal vector of the plane $y = 0$. The normal vector of the tangent plane at a point (x_0, y_0, z_0) is

$$\nabla F(x_0, y_0, z_0) = \langle 3x_0^2 + 1, z_0 e^{z_0} \cos(y_0 z_0), y_0 \cos(y_0 z_0) e^{z_0} + \sin(y_0 z_0) e^{z_0} \rangle$$

The normal vector of the plane $z = 0$ is $\langle 0, 0, 1 \rangle$ and the normal vector of the plane $y = 0$ is $\langle 0, 1, 0 \rangle$. So for the parallelism with the $z = 0$ plane we need

$$\langle 3x_0^2 + 1, z_0 e^{z_0} \cos(y_0 z_0), y_0 \cos(y_0 z_0) e^{z_0} + \sin(y_0 z_0) e^{z_0} \rangle = \lambda \langle 0, 0, 1 \rangle,$$

and for the parallelism with the $y = 0$ plane we need

$$\langle 3x_0^2 + 1, z_0 e^{z_0} \cos(y_0 z_0), y_0 \cos(y_0 z_0) e^{z_0} + \sin(y_0 z_0) e^{z_0} \rangle = \lambda \langle 0, 1, 0 \rangle.$$

In both cases we need $3x_0^2 + 1 = 0$ which has no solutions. Therefore there are no such points.

Problem 4. Find all local minimums, local maximums and saddle points of the function $f(x, y) = x^4 + 2x^2 - 2y^3 + 3y^2 + 7$.

10 points

Solution:

First differentiate $f_x(x, y) = 4x^3 + 4x$ and $f_y(x, y) = -6y^2 + 6y$. Now solve the equations

$4x^3 + 4x = 0$ and $-6y^2 + 6y = 0$. From the first equation we get $4x(x^2 + 1) = 0$ so $x = 0$ and from the second $y = 0$ and $y = 1$. So the solutions are $(0, 0)$ and $(0, 1)$. Next find the second partial derivatives and get $f_{xx}(x, y) = 12x^2 + 4$, $f_{xy}(x, y) = 0$ and $f_{yy}(x, y) = -12y + 6$. So $f_{xx}(0, 0) = 4$, $f_{xy}(0, 0) = 0$ and $f_{yy}(0, 0) = 6$. Therefore $D(0, 0) = \begin{vmatrix} 4 & 0 \\ 0 & 6 \end{vmatrix} = 24$. So since both $f_{xx}(0, 0)$ and $D(0, 0)$ are positive at the point $(0, 0)$ we have a local minimum and $f(0, 0) = 7$. Next we calculate $f_{xx}(0, 1) = 4$, $f_{xy}(0, 1) = 0$ and $f_{yy}(0, 1) = -6$. So $D(0, 1) = \begin{vmatrix} 4 & 0 \\ 0 & -6 \end{vmatrix} = -24$. Since $D(0, 1)$ is negative the point $(0, 1)$ is a saddle point and $f(0, 1) = 8$. There are not local maximums.

Problem 5. Find the absolute minimum and the absolute maximum (and all the points at which these values are attained) of the function $f(x, y) = x^2 + y^2 - 4y + 7$ inside the circle of radius $\frac{1}{\sqrt{2}}$, centered at $(1, 1)$.

10 points

Solution:

First we find the critical points inside the circle $(x - 1)^2 + (y - 1)^2 = \frac{1}{2}$. We get $f_x(x, y) = 2x$ and $f_y(x, y) = 2y - 4$. The critical points are obtained by solving $2x = 0$ and $2y - 4 = 0$. So the only solution (and so the only critical point) is $(0, 2)$. But this point lies outside the circle (one way to check it is to plug in $(x, y) = (0, 2)$ into $(x - 1)^2 + (y - 1)^2$ - we get 2 which is greater than $\frac{1}{2}$). Now we find extremes on the boundary. A good way to do that it to use the Lagrange multiplier method. We need to solve the system

$$2x = 2\lambda(x - 1), \quad 2y - 4 = 2\lambda(y - 1) \quad \text{and} \quad (x - 1)^2 + (y - 1)^2 = \frac{1}{2}.$$

From the first two equations we get $x = \frac{\lambda}{\lambda - 1}$ and $y = \frac{\lambda - 2}{\lambda - 1}$ which we can then insert in the third one to obtain $\frac{2}{(\lambda - 1)^2} = \frac{1}{2}$. Thus we get $\lambda - 1 = \pm 2$ and so $\lambda = -1$ or $\lambda = 3$. The solution $\lambda = -1$ gives $(x, y) = (\frac{1}{2}, \frac{3}{2})$ and the solution $\lambda = 3$ gives $(x, y) = (\frac{3}{2}, \frac{1}{2})$. Now we just evaluate $f(\frac{1}{2}, \frac{3}{2}) = \frac{7}{2}$ which is the minimum and $f(\frac{3}{2}, \frac{1}{2}) = \frac{15}{2}$ which is the maximum.

Problem 6. a) Find the average value of the function $f(x, y) = x - 2y$ over the quadrilateral whose vertices are $(0, 0)$, $(1, -1)$, $(2, 0)$, and $(1, 1)$.

10 points

b) Evaluate the iterated integral

$$\int_{-1}^1 \int_{|x|}^1 e^{y^2} dy dx$$

10 points

Solution:

- a) See the Figure 1 in the figure pdf file. It's important to figure out the equations of the edges of this quadrilateral. Points $(0, 0)$ and $(1, 1)$ lie on the line $y = x$, points $(0, 0)$ and $(1, -1)$ lie on the line $y = -x$, points $(1, -1)$ and $(2, 0)$ lie on $y = x - 2$ and $(1, 1)$ and $(2, 0)$ lie on $y = 2 - x$. We can divide the quadrilateral into two triangles T_1 with vertices $(0, 0)$, $(1, -1)$ and $(1, 1)$ and T_2 with vertices $(1, -1)$, $(2, 0)$ and $(1, 1)$. Both of these triangles are type 1 regions so the integrals over these regions can be easily evaluated.

$$\iint_{T_1} x - 2y \, dA = \int_0^1 \int_{-x}^x x - 2y \, dy dx = \int_0^1 [xy - y^2]_{-x}^x dx = \int_0^1 2x^2 dx = \left[\frac{2}{3}x^3 \right]_0^1 = \frac{2}{3}$$

$$\begin{aligned} \iint_{T_2} x - 2y \, dA &= \int_1^2 \int_{x-2}^{2-x} x - 2y \, dy dx = \int_1^2 [xy - y^2]_{x-2}^{2-x} dx \\ &= \int_1^2 x(2-x) - (2-x)^2 - x(x-2) + (x-2)^2 dx = \int_1^2 4x - 2x^2 dx \\ &= \left[2x^2 - \frac{2}{3}x^3 \right]_1^2 = \frac{4}{3}. \end{aligned}$$

So the total integral is

$$\iint_{T_1} x - 2y \, dA + \iint_{T_2} x - 2y \, dA = \frac{2}{3} + \frac{4}{3} = 2.$$

To find the average we need to divide this by the area of the quadrilateral which is obviously equal to 2. So the average is 1.

- b) See the figure 2 in the figure pdf file. Notice that $\int_{-1}^1 \int_{|x|}^1 e^{y^2} dy dx = \iint_D e^{y^2} dA$, where $D = \{(x, y) \mid -1 \leq x \leq 1, |x| \leq y \leq 1\}$. A good idea is to change the order of the integration. Thus we express D as $D = \{(x, y) \mid 0 \leq y \leq 1, -y \leq x \leq y\}$ and so

$$\begin{aligned} \int_{-1}^1 \int_{|x|}^1 e^{y^2} dy dx &= \iint_D e^{y^2} dA = \int_0^1 \int_{-y}^y e^{y^2} dx dy = \int_0^1 [xe^{y^2}]_{x=-y}^{x=y} dy \\ &= \int_0^1 2ye^{y^2} dy = [e^{y^2}]_0^1 = e - 1. \end{aligned}$$

Problem 7. Evaluate the integral

$$\iint_D x \, dA,$$

where D is the region bounded by the parabola $y = x^2$ from below and by $y = 4 + 2x - x^2$ from above.

10 points

Solution:

See the Figure 3 in the figure pdf file. First we find the intersection of these parabolas. We set

$x^2 = 4 + 2x - x^2$ to get the equation $x^2 - x - 2 = 0$ which has the solutions $x = -1$ and $x = 2$. So the integral can be evaluated as follows

$$\begin{aligned} \iint_D x dA &= \int_{-1}^2 \int_{x^2}^{4+2x-x^2} x dy dx = \int_{-1}^2 x((4+2x-x^2) - x^2) dx = \int_{-1}^2 4x + 2x^2 - 2x^3 dx \\ &= \left[2x^2 + \frac{2}{3}x^3 - \frac{1}{2}x^4 \right]_{-1}^2 = \frac{9}{2}. \end{aligned}$$

Problem 8. Evaluate the integral

$$\iint_D (x^2 + y^2) dA,$$

where D is the region in the first quadrant located inside the circle $x^2 + y^2 = 2$ and below the line $y = 1$.

10 points

Solution:

An easy way to solve this problem is to divide the region D into two parts D_1 and D_2 (see the Figure 4 in the figure pdf file), where D_1 can be written in polar coordinates as $D_1 = \{(r, \theta) | 0 \leq \theta \leq \frac{\pi}{4}, 0 \leq r \leq \sqrt{2}\}$ and D_2 can be written in Cartesian coordinates as $D_2 = \{(x, y) | 0 \leq x \leq 1, x \leq y \leq 1\}$. Then

$$\iint_{D_1} x^2 + y^2 dA = \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} r^2 r dr d\theta = \int_0^{\frac{\pi}{4}} \left[\frac{r^4}{4} \right]_0^{\sqrt{2}} d\theta = \int_0^{\frac{\pi}{4}} 1 d\theta = \frac{\pi}{4}.$$

Moreover

$$\iint_{D_2} x^2 + y^2 dA = \int_0^1 \int_x^1 x^2 + y^2 dy dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_x^1 dx = \int_0^1 \frac{1}{3} + x^2 - \frac{4}{3}x^3 dx = \left[\frac{x}{3} + \frac{x^3}{3} - \frac{x^4}{3} \right]_0^1 = \frac{1}{3}.$$

Therefore

$$\iint_D x^2 + y^2 dA = \iint_{D_1} x^2 + y^2 dA + \iint_{D_2} x^2 + y^2 dA = \frac{\pi}{4} + \frac{1}{3}.$$