Midterm 1, Math 53
July 11th, 2008

Solutions of all problems must by accompanied by relevant explanations. Show your work, but not to others. Simplify your answers where possible.

Problem 1. Consider the curve given by parametric equations

$$x = t + \sin t, \quad y = t + \sin t \cos t, \quad -1 \leq t \leq 4.$$

Find all values of parameter t which corespond to the point $(0, 0)$ and find the equation of the tangent line at this point. Find all points where this tangent line intersects (or touches) the curve.

Problem 2. Find the lenght of the part of the curve given by parametric equations

$$x = e^{3t} - e^t, \quad y = \sqrt{3}(9 - e^{2t})$$

that lies in the first quadrant.

Problem 3. Let a and b be given real numbers. Show that the curve given by polar equation $r = a\cos\theta + b\sin\theta$ is a cricle that passes through the origin. Find its center and radius. Give a strict proof, not just a "proof by sketch".

Problem 4. Find the area of the region that lies inside the curve $r = 4 - \sin^2\theta$ and outside the curve $r = 2 + \sin^2\theta$.

Problem 5. Let $\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Find the vector $\mathbf{a}$, orthogonal to $\mathbf{b}$ which satisfies $\mathbf{a} \times \mathbf{b} = \mathbf{i} - \mathbf{j}$.

Problem 6. Find parametric equations of the line which is contained in the plane $x + y + z = 0$, intersects the line $\dfrac{x+1}{2} = \dfrac{y}{1} = \dfrac{z+4}{2}$ and is orthogonal to this line.

Problem 7. Consider the tetrahedron $ABCD$ whose vertices have coordinates $A(0, 0, 0)$, $B(1, 0, 0)$, $C(0, 1, 0)$ and $D(0, 0, 1)$. Let E be the midpoint of the segment AD, F be the midpoint of the segment AC and G be the midpoint of the segment BC.

i) Show that the lines EF and FG are orthogonal.

ii) Find the area of the triangle EFG.

Problem 8. Consider the curve in the 3-dimensional space defined by the vector function $\mathbf{r}(t) = (t^3 - 3t)\mathbf{i} + (t^2 + 4t)\mathbf{j} + (2t - 7)\mathbf{k}$. Find the Cartesian coordinates of all points where the tangent line to this curve is parallel to the

i) xy plane,

ii) yz plane,

iii) xz plane.