Sketch the subset of $\mathbb{C}$ described by each equation or inequality in (1) through (5) below, and say (no explanation necessary) whether each is open in $\mathbb{C}$ (or not), close in $\mathbb{C}$ (or not), connected (or not), and compact (or not). (So you have to do 5 sketches, and answer 20 questions.)

Problem 1. $z\overline{z} = 4$

Solution. Not open, closed, connected, and compact. ☐

Problem 2. $|\mathrm{Re}\,(z)| \geq 3$

Solution. Not open, closed, connected, not compact. ☐

Problem 3. $3 < |z - 2i| \leq 4$

Solution. Not open, not closed, connected, not compact. ☐

Problem 4. $\mathrm{Im}\,((2+i)z) > 0$

Solution. open, not closed, connected, compact. ☐

Problem 5. $\mathrm{Re}\,(z) \geq |z|$

Solution. not open, closed, connected, not compact. ☐

Problem 6. *Let S be an open subset of $\mathbb{C}$, let $f : S \to \mathbb{C}$ be a continuous function, and let $a \in S$ be such that $f(a) \neq 0$. Prove that there is an open ball containing a such that $f(z) \neq 0$ for all z in the ball.*

Solution. Note that $U = \mathbb{C} - \{0\}$ is an open set. Therefore $f^{-1}(U) \subset S$ is open, since f is continuous and by definition preimage of open sets are open. Furthermore, $a \in f^{-1}(U)$, since $f(a) \neq 0$. By definition of openness, we can find $\epsilon > 0$ such that $B_\epsilon(a) \subset f^{-1}(U)$. Therefore for all $z \in B_\epsilon(a)$ we have $f(z) \in U$, which means that $f(z) \neq 0$.

Alternate solution: If we have $|f(z) - f(a)| < |f(a)|$ we get that

$$|f(a)| - |f(z)| \leq |f(a) - f(z)| < |f(a)|,$$

by reverse triangle inequality. (This follows immediately from triangle inequality: $|f(a)| = |(f(a) - f(z)) + f(z)| \leq |f(a) - f(z)| + |f(z)|$.) Therefore $|f(z)| > 0$, which means $f(z) \neq 0$. Since f is continuous, for $\epsilon = |f(a)|$ we can find $\delta > 0$ such that $|z - a| < \delta$ then $|f(z) - f(a)| < \epsilon$, which implies that $f(z) \neq 0$. Since S is open, we can find δ' small enough such that $|z - a| < \delta'$ implies $z \in S$. Therefore a ball of radius $\min(\delta, \delta')$ centered at a is satisfies the desired properties. (Note: We can also prove $|f(z)| - |f(a)| \leq |f(a) - f(z)|$ exactly the same way. Putting these two inequalities together, we get $||f(z)| - |f(a)|| \leq |f(a) - f(z)|$, which is the way reverse triangle inequality is usually presented.) ☐

Problem 7. *Give an explicit formula for a path $f : [0,4] \to \mathbb{C}$ such that $f([0,4])$ is the boundary of a square.*

Solution. The function

$$f(x) = \begin{cases} x & \text{if } x \in [0,1), \\ 1 + (x-1)i & \text{if } x \in [1,2), \\ (3-x) + i & \text{if } x \in [2,3), \\ (4-x)i & \text{if } x \in [3,4], \end{cases}$$

gives such a function. (This is by no mean the only solution, and there are many other valid functions.) ☐

Problem 8. *Give an explicit formula for a path* $f : [0,4] \to \mathbb{C}$ *such that* $f([0,4])$ *is the boundary of a semicircle.*

Solution. The function

$$f(x) = \begin{cases} e^{\frac{i\pi x}{2}} & \text{for } x \in [0,2), \\ x - 3 & \text{for } x \in [2,4], \end{cases}$$

is such a function. (Note: Some of you did not include the line on the diameter as part of the function. I did not deduct any marks for that.) ☐

Problem 9. *Use the definition of continuous to prove that the function* $f(z) = \overline{z}$ *is continuous everywhere on* $\mathbb{C}$*. Also show that it is differentiable nowhere (in the complex sense).*

Solution. First we will show that f is continuous everywhere. Let $z_0 \in \mathbb{C}$. For any $\epsilon > 0$ let $\delta = \epsilon$. Then for any $z \in \mathbb{C}$ such that $|z - z_0| < \delta = \epsilon$ we have that

$$\begin{aligned} |f(z) - f(z_0)| &= |\overline{z} - \overline{z_0}| \\ &= |\overline{z - z_0}| \\ &= |z - z_0| \\ &< \epsilon. \end{aligned}$$

Therefore f is continuous at z_0. Since z_0 was arbitrary, f is continuous everywhere.

Now we will show that f is differentiable nowhere. Pick $z_0 \in \mathbb{C}$. If f is differentiable at z_0 then $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists, and it equals to the derivative of f at z_0. On the other hand

$$\begin{aligned} \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{z \to z_0} \frac{\overline{z - z_0}}{z - z_0} \\ &= \lim_{z \to 0} \frac{\overline{z}}{z}. \end{aligned}$$

If this limit exists, then it should be the same from every direction. Specifically, $\lim_{z \to 0} \frac{\overline{z}}{z} = \lim_{h \to 0} \frac{\overline{h}}{h} = 1$ were the second limit is over real h's. On the other hand, approaching zero on the imaginary axis we get $\lim_{z \to 0} \frac{\overline{z}}{z} = \lim_{h \to 0} \frac{\overline{ih}}{ih} = -1$. Therefore $\lim_{z \to 0} \frac{\overline{z}}{z}$ does not exist. Therefore f is not differentiable at z_0, which was an arbitrary point in $\mathbb{C}$. Therefore f is not differentiable anywhere. $\square$

For the rest of the problems, we let $f(x+iy) = U(x+iy) + iV(x+iy)$ were U and V are functions from complex numbers to reals.

Problem 10. *Same question for* $f(z) = |z|$.

Solution. To show continuous, we use reverse triangle inequality, which says (as we showed in question 6) that $||a| - |b|| \leq |a - b|$. Then for $\epsilon > 0$ pick $\delta = \epsilon > 0$, we have whenever $|z - z_0| < \delta = \epsilon$ for $z, z_0 \in \mathbb{C}$ then

$$\begin{aligned} |f(z) - f(z_0)| &= ||z| - |z_0|| \\ &\leq |z - z_0| < \epsilon. \end{aligned}$$

Therefore f is continuous. (Note: we have proved something slightly stronger, namely f is uniformly continuous, since δ is independant at the point were we want to show the function is continuous.)

To show that f is not differentiable, we use Cauchy-Riemann's equation. Recall that if $f(x+iy) = U(x+iy) + iV(x+iy)$ then f is differentiable if and only if U and V are differentiable in both x and y variable, and

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}\ \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}.$$

We have $U(x+iy) = \sqrt{x^2+y^2}$ and $V(x+iy) = 0$. We have $\frac{\partial U}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$, while $\frac{\partial V}{\partial y} = 0$. Therefore for $\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$ to hold, we need $x = 0$. On the other hand, we have $\frac{\partial U}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{|y|}$, while $\frac{\partial V}{\partial y} = 0$. Therefore for $\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$ to hold, we need $y = 0$. Therefore, the only possible place for f to be differentiable is at 0. But evaluating the derivative at 0 we get that

$$\begin{aligned} f'(0) &= \lim_{z \to 0} \frac{f(z) - f(0)}{z} \\ &= \lim_{z \to 0} \frac{|z|}{z}, \end{aligned}$$

which is not well defined, since approaching 0 from different angles we get different answers. (Note: This last step is necessary for a complete solution, although I did not deduct marks off from people who missed it.) $\square$

Problem 11. *Find all points in* $\mathbb{C}$ *where the function* $f(z) = |z|^2$ *is differentiable (in the complex sense).*

Solution. If $f(z)$ is differentiable, then it must satisfy the Cauchy-Riemann equations. Recall that Cauchy-Riemann equations state $\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$ and $\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$ were $U(x+iy) = x^2+y^2$ and $V(x+iy) = 0$. Therefore $\frac{\partial U}{\partial x} = 2x = 0$ and $\frac{\partial V}{\partial y} = 2y = 0$. Therefore the only place the function can be differentiable is at 0. In fact, since all partial derivatives exist at 0 we get that f is differentiable at 0. $\square$

Problem 12. *Find all holomorphic functions* $f(z)$ *on* $\mathbb{C}$ *such that* $\operatorname{Re} f(x+yi) = x^3 - 3xy^2 + 4xy$.

Solution. Using Cauchy-Riemann equations we have that

$$\begin{aligned} \frac{\partial V}{\partial y} &= \frac{\partial U}{\partial x} \\ &= 3x^2 - 3y^2 + 4y \\ \Rightarrow V &= 3x^2y - y^3 + 2y^2 + g(x), \end{aligned}$$

were g is a function of x, and

$$\begin{aligned} \frac{\partial V}{\partial x} &= -\frac{\partial U}{\partial y} \\ &= 6xy - 4x \\ \Rightarrow V &= 3x^2y - 2x^2 + h(y), \end{aligned}$$

were h is a function of y. Putting these together we get that $V = 3x^2y - y^3 + 2y^2 - 2x^2 + C$ were C is any real constant. Therefore

$$\begin{aligned} f(x+iy) &= (x^3 - 3xy^2 + 4xy) + i(3x^2y - y^3 + 2y^2 - 2x^2 + C) \\ &= (x+iy)^3 - 2i(x+iy)^2 + iC, \end{aligned}$$

for any real number C. □

Problem 13. *Let* S *be open in* $\mathbb{C}$, *and let* $f : S \to \mathbb{C}$ *be a function. Write*

$$f(r(\cos\theta + i\sin\theta)) = u(r,\theta) + v(r,\theta)i,$$

where u *and* v *are real-valued functions (defined on the set of* (r,θ) *such that* $re^{i\theta} \in S$*). Prove that* f *satisfies the Cauchy-Riemann equations at a nonzero point* $re^{i\theta}$ *if and only if the equations*

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

hold at (r,θ).

Solution. Note that we have $x = r\cos\theta$ and $y = r\sin\theta$. Therefore, using chain rule, we get

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\ &= \frac{\partial u}{\partial x}(\cos\theta) + \frac{\partial u}{\partial y}(\sin\theta), \\ \frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} \\ &= \frac{\partial v}{\partial x}(\cos\theta) + \frac{\partial v}{\partial y}(\sin\theta), \end{aligned}$$

and

$$\begin{aligned}
\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \\
&= \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta), \\
\frac{\partial v}{\partial \theta} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} \\
&= \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta).
\end{aligned}$$

Therefore, if the Cauchy-Riemann eqautions $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ were satisfied, then we get

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial v}{\partial y}(\cos\theta) + -\frac{\partial v}{\partial x}(\sin\theta) \\
&= \frac{1}{r}\frac{\partial v}{\partial \theta}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial v}{\partial r} &= -\frac{\partial u}{\partial y}(\cos\theta) + \frac{\partial u}{\partial x}(\sin\theta) \\
&= -\frac{1}{r}\frac{\partial u}{\partial \theta}.
\end{aligned}$$

On the other hand if $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ then

$$\begin{aligned}
\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta &= \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta \\
\Rightarrow \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\cos\theta &+ \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\sin\theta,
\end{aligned}$$

and similarly

$$\begin{aligned}
\frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta &= -\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta \\
\Rightarrow \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\cos\theta &+ \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)\sin\theta.
\end{aligned}$$

Since the matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is invertible, this system of equation has a unique solution, given by $\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} = 0$ and $\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$, which are the Cauchy-Riemann equations. $\square$