SUPPLEMENT TO ORTHOSYMPLECTIC LIE SUPERALGEBRAS, KOSZUL DUALITY, AND A COMPLETE INTERSECTION ANALOGUE OF THE EAGON–NORTHCOTT COMPLEX

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This note is a supplement to [S2]. The main point is to prove Theorem 2.1.1, which is a special case of [S2, Theorem 3.3.6], without the use of Lie superalgebras or Koszul duality. The focus here is on the case of the symplectic group, though the proofs given here could be adapted to the other cases from [S2]. The proof we give here was found before the one in [S2] and we believe that it is of independent interest. We also include some combinatorial lemmas regarding the modification rules which played a prominent role in [S2] but which were ultimately not needed.

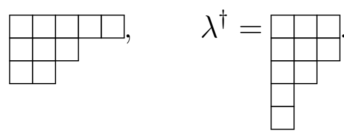
1. Preliminaries

Throughout, we work over a fixed field $\mathbf{k}$ of characteristic 0.

1.1. **Partitions.** A finite sequence of weakly decreasing integers $\lambda = (\lambda_1, \dots, \lambda_n)$ is a **partition**. We write $\ell(\lambda) = \max\{n \mid \lambda_n \neq 0\}$ and $|\lambda| = \sum_i \lambda_i$. If $i > \ell(\lambda)$, we use the convention that $\lambda_i = 0$. The **transpose partition** is $\lambda^\dagger$ and is defined by $\lambda^\dagger_i = \#\{j \mid \lambda_j \geq i\}$. This is best explained in terms of Young diagrams, which we define via an example.

Example 1.1.1. If $\lambda = (5, 3, 2)$, then $\lambda^\dagger = (3, 3, 2, 1, 1)$:

$\lambda =$, $\lambda^\dagger =$.



So $\ell(5, 3, 2) = 3$ and $|(5, 3, 2)| = 10$. □

The sum of two partitions is defined componentwise: $(\lambda + \mu)_i = \lambda_i + \mu_i$. The exponential notation (a^b) denotes the number a repeated b times. Its Young diagram is a rectangle, so we also denote this by $b \times a$. We say that $\lambda \subseteq \mu$ if $\lambda_i \leq \mu_i$ for all i. If $\lambda \subseteq b \times a$, then the notation $(b \times a) \setminus \lambda$ refers to the partition $(a - \lambda_b, \dots, a - \lambda_1)$, i.e., rotate the Young diagram of λ 180 degrees and remove it from the bottom-right corner of the $b \times a$ rectangle. We set $-\lambda^{\mathrm{op}} = (0, \dots, 0, -\lambda_{\ell(\lambda)}, \dots, -\lambda_2, -\lambda_1)$; this notation is used in the context when we consider integer sequences of a fixed length, in which case the number of 0's above is implied.

Let λ be a partition. Then we can define the **Schur functor** $\mathbf{S}_\lambda$ (denoted $L_{\lambda'}$ in [W, §2.1]). For a vector space E of dimension n, $\mathbf{S}_\lambda(E)$ is a representation of the general linear group $\mathbf{GL}(E)$ and $\mathbf{S}_\lambda(E) \neq 0$ if and only if $\ell(\lambda) \leq n$. If $\lambda = (1^n)$, then $\mathbf{S}_{(1^n)}(E) = \det E = \bigwedge^n(E)$. Furthermore, $\mathbf{S}_{\lambda+(1^n)}(E) = \mathbf{S}_\lambda(E) \otimes (\det E)$. Using this, we can define $\mathbf{S}_\lambda(E)$ for any weakly decreasing sequence of integers of length n: find N such that $\lambda + (1^N)$ is nonnegative, and define $\mathbf{S}_\lambda(E) = \mathbf{S}_{\lambda+(1^N)}(E) \otimes (\det E^*)^N$. This does not depend on the choice of N.

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Now let V be a symplectic or orthogonal space. Let λ be a partition with $2\ell(\lambda) \le \dim(V)$. Then we can define $\mathbf{S}_{[\lambda]}(V)$, which is a representation of either the symplectic or orthogonal group of V, respectively. It is irreducible when the base field has characteristic 0. We will refer the reader to [FH, §§17.3, 19.5] for the details in characteristic 0 and to [SW, §2] for a definition and basic properties in the general case.

Let Q_{-1} be the set of partitions with the following inductive definition. The empty partition belongs to Q_{-1}. A non-empty partition μ belongs to Q_{-1} if and only if the number of rows in μ is one more than the number of columns, i.e., $\ell(\mu) = \mu_1 + 1$, and the partition obtained by deleting the first row and column of μ, i.e., $(\mu_2 - 1, \dots, \mu_{\ell(\mu)} - 1)$, belongs to Q_{-1}. The first few partitions in Q_{-1} are 0, $(1,1)$, $(2,1,1)$, $(2,2,2)$. Define $Q_1 = \{\lambda \mid \lambda^\dagger \in Q_{-1}\}$. We write $Q_{-1}(2i)$ for the set of $\lambda \in Q_{-1}$ with $|\lambda| = 2i$, and similarly we define $Q_1(2i)$.

The significance of these sets are the following decompositions (see [M, I.A.7, Ex. 4,5]):

$$\bigwedge^i(\mathrm{Sym}^2(E)) = \bigoplus_{\mu \in Q_1(2i)} \mathbf{S}_\mu(E), \qquad \bigwedge^i(\bigwedge^2(E)) = \bigoplus_{\mu \in Q_{-1}(2i)} \mathbf{S}_\mu(E). \tag{1.1.2}$$

Lemma 1.1.3.
(a) *If $\mu \in Q_1$, then $\sum_i \max(0, \mu_i - i) = |\mu|/2$.*
(b) *If $\mu \in Q_{-1}$, then $\sum_i \max(0, \mu_i - i + 1) = |\mu|/2$.*

Proof. (a) Let $\nu_i = \mu_{i+1} - 1$. Then $\nu \in Q_1$ and $\sum_i \max(0, \nu_i - i) = |\nu|/2$. So

$$\sum_i \max(0, \mu_i - i) = (\mu_1 - 1) + \sum_i \max(0, \nu_i - i) = \mu_1 - 1 + |\nu|/2 = |\mu|/2.$$

(b) Let $\nu_i = \mu_{i+1} - 1$. Then $\nu \in Q_{-1}$ and $\sum_i \max(0, \nu_i - i + 1) = |\nu|/2$. So

$$\sum_i \max(0, \mu_i - i + 1) = \mu_1 + \sum_i \max(0, \nu_i - i + 1) = \mu_1 + |\nu|/2 = |\mu|/2. \qquad \square$$

Lemma 1.1.4.
(a) *If $\nu \in Q_1$ and $\ell(\nu) \le n$, then $(\nu_n - n, \dots, \nu_2 - 2, \nu_1 - 1)$ is a signed permutation of $(n, n-1, \dots, 1)$, i.e., there exists a permutation w and $\varepsilon_i \in \{\pm 1\}$ such that $\varepsilon_i w(i) = \nu_i - i$.*
(b) *If $\nu \in Q_{-1}$ and $\ell(\nu) \le n$, then $(\nu_n - (n-1), \dots, \nu_2 - 1, \nu_1)$ is a signed permutation of $(n-1, \dots, 1, 0)$, i.e., there exists a permutation w and $\varepsilon_i \in \{\pm 1\}$, such that $\varepsilon_i w(i) = \nu_i - (i-1)$.*

Proof. We just prove (a); the proof of (b) is similar. Pick $\nu \in Q_1$ with $\ell(\nu) \le n$. We may as well assume that $\ell(\nu) = n$ since having trailing zeros will not affect the validity of the statement. Set $\mu = (\nu_2 - 1, \dots, \nu_n - 1)$. By induction, there exists a permutation w' and signs ε'_i such that $\varepsilon'_i w'(i) = \mu_i - i = \nu_{i+1} - (i+1)$ for $i = 1, \dots, n-1$. Note that $\nu_1 = n+1$, so we define $w(1) = n$, $\varepsilon_1 = 1$, and $w(i) = w'(i-1)$ and $\varepsilon_i = \varepsilon'_{i-1}$ for $i = 2, \dots, n$. $\square$

1.2. **Type A Weyl group.** Let $\mathcal{U}$ be the set of all integer sequences $(a_1, a_2, \dots)$ which are eventually 0. We identify partitions with non-increasing sequences in $\mathcal{U}$ (such sequences are necessarily nonnegative). For $i \ge 1$, let s_i be the transposition which switches a_i and a_{i+1}, and let $\mathfrak{S}$ be the group of automorphisms of $\mathcal{U}$ generated by the s_i. The group $\mathfrak{S}$ is a Coxeter group (in fact, the infinite symmetric group), and admits a length function $\ell \colon \mathfrak{S} \to \mathbf{Z}_{\ge 0}$. The **length** of $w \in \mathfrak{S}$ is the minimum number $\ell(w)$ so that there exists an expression

$$w = s_{i_1} \cdots s_{i_{\ell(w)}}. \tag{1.2.1}$$

Alternatively, $\ell(w) = \#\{i < j \mid w(i) > w(j)\}$ is the number of inversions of w, interpreted as a permutation [BB, Proposition 1.5.2].

We define a second action of $\mathfrak{S}$ on $\mathcal{U}$, denoted $\bullet$, as follows. For $w \in \mathfrak{S}$ and $\lambda \in \mathcal{U}$ we put $w \bullet \lambda = w(\lambda + \rho) - \rho$, where $\rho = (-1, -2, \ldots)$. If we add the same constant c to each entry of ρ, then this action is unchanged, so we will do that if it simplifies notation. In terms of the generators, this action is:

$$s_i \bullet (\ldots, a_i, a_{i+1}, \ldots) = (\ldots, a_{i+1} - 1, a_i + 1, \ldots).$$

1.3. **Type C Weyl group.** Fix $k \geq 0$. We associate to a partition λ two quantities, $\iota^{\mathrm{C}}_{2k}(\lambda)$ and $\tau^{\mathrm{C}}_{2k}(\lambda)$, which will be of fundamental importance to the results of this paper. We give two equivalent definitions of these quantities, one via a Weyl group action and one via border strips. The presentation here follows [SSW, §3.5], where the reader can find further details and references.

We begin with the Weyl group definition. In §1.6 we defined automorphisms s_i of the set $\mathcal{U}$ of integer sequences, for $i \geq 1$. We now define an additional automorphism: s_0 negates a_1. We let W_∞ be the group generated by the s_i, for $i \geq 0$. Then W_∞ is a Coxeter group of type BC_∞, and, as such, is equipped with a length function $\ell \colon W_\infty \to \mathbf{Z}_{\geq 0}$, which is defined just as in (1.2.1). Let $\rho = (-(k+1), -(k+2), \ldots)$. Define a new action of W_∞ on $\mathcal{U}$ by $w \bullet \lambda = w(\lambda + \rho) - \rho$. The action of s_0 is given by

$$s_0 \bullet (a_1, a_2, \ldots) = (2k + 2 - a_1, a_2, \ldots).$$

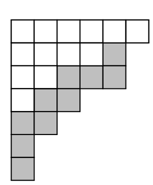
Given a partition $\lambda \in \mathcal{U}$, exactly one of the following two possibilities hold:

- There exists a unique element $w \in W_\infty$ such that $w \bullet \lambda^\dagger = \mu^\dagger$ is a partition and $\ell(\mu) \leq k$. We then put $\iota^{\mathrm{C}}_{2k}(\lambda) = \ell(w)$ and $\tau^{\mathrm{C}}_{2k}(\lambda) = \mu$.
- There exists a non-identity element $w \in W_\infty$ such that $w \bullet \lambda^\dagger = \lambda^\dagger$. We then put $\iota^{\mathrm{C}}_{2k}(\lambda) = \infty$ and leave $\tau^{\mathrm{C}}_{2k}(\lambda)$ undefined.

Note that if $\ell(\lambda) \leq k$, then we are in the first case with $w = 1$, and so $\iota^{\mathrm{C}}_{2k}(\lambda) = 0$ and $\tau^{\mathrm{C}}_{2k}(\lambda) = \lambda$.

We now give the border strip definition. If $\ell(\lambda) \leq k$ we put $\iota^{\mathrm{C}}_{2k}(\lambda) = 0$ and $\tau^{\mathrm{C}}_{2k}(\lambda) = \lambda$. Suppose $\ell(\lambda) > k$. A **border strip** is a connected skew Young diagram containing no 2×2 square. Let R_λ be the connected border strip of length $2(\ell(\lambda) - k - 1)$ which starts at the first box in the final row of λ, if it exists. If R_λ exists, is non-empty and $\lambda \setminus R_\lambda$ is a partition, then we put $\iota^{\mathrm{C}}_{2k}(\lambda) = c(R_\lambda) + \iota^{\mathrm{C}}_{2k}(\lambda \setminus R_\lambda)$ and $\tau^{\mathrm{C}}_{2k}(\lambda) = \tau^{\mathrm{C}}_{2k}(\lambda \setminus R_\lambda)$, where $c(R_\lambda)$ denotes the number of columns that R_λ occupies; otherwise we put $\iota^{\mathrm{C}}_{2k}(\lambda) = \infty$ and leave $\tau^{\mathrm{C}}_{2k}(\lambda)$ undefined.

Example 1.3.1. Set $k = 1$ and $\lambda = (6, 5, 5, 3, 2, 1, 1)$. Then $2(\ell(\lambda) - k - 1) = 10$. We shade in the border strip R_λ of length 10 in the Young diagram of λ:



In this case $c(R_\lambda) = 5$. □

For the proof that these two definitions are equivalent, we refer to [SSW, Proposition 3.5] (with a small gap corrected in [S1, Lemma 2.7]).

Let $\le$ denote the Bruhat order on W_∞ [BB, §2]. Recall that a **transposition** $t \in W_\infty$ is an element conjugate to some s_i.

Proposition 1.3.2. *Fix λ with $\ell(\lambda) \le k$. Given μ such that $\tau^{\mathrm{C}}_{2k}(\mu) = \lambda$, let $w_\mu \in W_\infty$ be the unique element such that $w_\mu \bullet \mu^\dagger = \lambda^\dagger$. Then $w_\mu \le w_\nu$ if and only if $\mu \subseteq \nu$.*

Proof. Suppose that $w_\mu \le w_\nu$. We can factor this relation into those of the form $w_\mu < w_\mu t$ for some transposition t [BB, Definition 2.1.1], so it is enough to assume that $w_\nu = t w_\mu$ (Bruhat order is preserved under the map $w \mapsto w^{-1}$ [BB, Corollary 2.2.5]). The only possibilities are that t negates the ith entry for some i or swaps the ith and jth entries for $i < j$ and negates them (if it just swapped two positions then ν would not be a partition). We treat them simultaneously by allowing $i = j$. Under the transformation $(a_1, a_2, \dots) \mapsto (\dots, -a_2 + k, -a_1 + k)$, the action of W_∞ becomes the standard action of the Weyl group of type BC_∞ and ρ becomes $\rho' = (\dots, 2, 1)$. So $tw_\mu > w_\mu$ if and only if $\mu^\dagger_i + \mu^\dagger_j \le 2k$. Note that $\nu^\dagger_i = 2k + i + j - \mu^\dagger_j$ and $\nu^\dagger_j = 2k + i + j - \mu^\dagger_i$ and $\nu^\dagger_{j'} = \mu^\dagger_{j'}$ for $j' \ne i, j$. Hence $\nu^\dagger_i > \mu^\dagger_i$ and $\nu^\dagger_j > \mu^\dagger_j$ (if not, then $\nu^\dagger_i + \nu^\dagger_j \le \mu^\dagger_i + \mu^\dagger_j$ implies $i + j \le 0$), so $\mu \subseteq \nu$.

Now suppose that $\mu \subseteq \nu$. Let $\mu = \mu^0 \supset \mu^1 \supset \cdots \supset \mu^r = \lambda$ and $\nu = \nu^0 \supset \nu^1 \supset \cdots \supset \nu^s = \lambda$ be the sequence of partitions obtained by removing successive border strips as described in the modification rule and let $c(i)$ and $d(i)$ be the number of columns of μ^i/μ^{i+1} and ν^i/ν^{i+1}, respectively. Consider the following two conditions:

(a_i) $(\mu^i)^\dagger_j \le (\nu^i)^\dagger_j$ for $j = 1, \dots, c(i)$.

(b_i) $c(i) \le d(i)$.

We will prove them below, so assume that this has been done. Set $u_i = s_{i-1} \cdots s_1 s_0$. By [SSW, Proof of Proposition 3.5], we have reduced expressions $w_\mu = u_{c(r-1)} \cdots u_{c(1)} u_{c(0)}$ and $w_\nu = u_{d(s-1)} \cdots u_{d(1)} u_{d(0)}$. If $r > s$, then $\nu^s = \lambda$ and $c(s) > 0$, which implies that $(\mu^s)^\dagger_1 > k \ge \lambda^\dagger_1$, which contradicts (a_s). Since $c(i) \le d(i)$, we have that $u_{c(i)}$ is a subword of $u_{d(i)}$, and hence $w_\mu \le w_\nu$ [BB, Theorem 2.2.2].

Now we prove that (a_i) implies (b_i) and that (a_i) and (b_i) together imply (a_{i+1}). Note that (a_0) holds by the assumption that $\mu \subseteq \nu$.

Assume that (a_i) holds. Using the Weyl group definition of the modification rule, to remove a border strip, we start with the sequences $(\mu^i)^\dagger + \rho$ and $(\nu^i)^\dagger + \rho$, and apply $s_{c(i)-1} \cdots s_1 s_0$ and $s_{d(i)-1} \cdots s_1 s_0$, respectively, to them [SSW, Proof of Proposition 3.5]. Call the resulting sequences α and β, which are both strictly decreasing. Then

$$-\rho_1 - (\nu^i)^\dagger_1 = \beta_{d(i)} > \beta_{d(i)+1} = (\nu^i)^\dagger_{d(i)+1} + \rho_{d(i)+1}.$$

If $c(i) > d(i)$, then we have

$$(\mu^i)^\dagger_{d(i)+1} + \rho_{d(i)+1} = \alpha_{d(i)} > \alpha_{c(i)} = -\rho_1 - (\mu^i)^\dagger_1.$$

Using (a_i), we get $(\nu^i)^\dagger_{d(i)+1} \ge (\mu^i)^\dagger_{d(i)+1}$. Combining these inequalities implies that $(\mu^i)^\dagger_1 > (\nu^i)^\dagger_1$, which contradicts (a_i). Hence (a_i) implies (b_i).

Now suppose that (a_i) and (b_i) hold. We have $c(i+1) < c(i)$ by [S1, Lemma 2.7]. So $(\mu^{i+1})^\dagger_j = (\mu^i)^\dagger_{j+1} - 1$ for $j = 1, \dots, c(i+1)$ and by (b_i) we also have $(\nu^{i+1})^\dagger_j = (\nu^i)^\dagger_{j+1} - 1$ for $j = 1, \dots, c(i+1)$. Now use (a_i) to get (a_{i+1}). ☐

Lemma 1.3.3. *Fix k and pick two partitions $\mu \subseteq \nu$ with $\ell(\nu) \le 2k$ such that $\tau^{\mathrm{C}}_{2k}(\mu) = \tau^{\mathrm{C}}_{2k}(\nu) = \lambda$, and $\iota^{\mathrm{C}}_{2k}(\mu) + 1 = \iota^{\mathrm{C}}_{2k}(\nu)$. Then $\mu_d - \mu_{2k+1-d} \ge \nu_d - \nu_{2k+1-d}$ for all $d = 1, \dots, k$.*

Proof. By Proposition 1.3.2, there exist $w_\mu, w_\nu \in W_\infty$ such that $w_\mu \le w_\nu$, $\ell(w_\mu)+1 = \ell(w_\nu)$, and

$$w_\mu(\lambda^\dagger + \rho) - \rho = \mu^\dagger, \quad w_\nu(\lambda^\dagger + \rho) - \rho = \nu^\dagger.$$

As explained in the proof of Proposition 1.3.2, there is a transposition t such that $w_\nu = tw_\mu$ and there exists $i \le j$ so that t swaps the ith and jth positions and negates them. Then $\nu^\dagger_i = 2k+i+j-\mu^\dagger_j$, $\nu^\dagger_j = 2k+i+j-\mu^\dagger_i$, and $\nu^\dagger_{j'} = \mu^\dagger_{j'}$ for $j' \ne i,j$. So $\nu \supset \mu$ implies

- $\nu_e = \mu_e + 1$ for $e = \mu^\dagger_j + 1, \dots, \mu^\dagger_i$,
- $\nu_e = \mu_e + 2$ for $e = \mu^\dagger_i + 1, \dots, 2k+i+j-\mu^\dagger_i$,
- $\nu_e = \mu_e + 1$ for $e = 2k+i+j-\mu^\dagger_i + 1, \dots, 2k+i+j-\mu^\dagger_j$,
- $\nu_e = \mu_e$ otherwise.

This allows us to conclude:

- If $d = 1, \dots, \mu^\dagger_j$, then $\nu_e = \mu_e$, so the inequality holds.
- If $\mu^\dagger_j < d \le \min(\mu^\dagger_i, k)$, then $\nu_d = \mu_d + 1$. Also $\mu^\dagger_j < 2k+1-d \le 2k+i+j-\mu^\dagger_j$, so $\nu_{2k+1-d} \ge \mu_{2k+1-d} + 1$, and the inequality holds.
- If $\mu^\dagger_i < d \le k$, then $\mu^\dagger_i < 2k+1-d < 2k+i+j-\mu^\dagger_i$, so $\nu_{2k+1+d} = \mu_{2k+1+d} + 2$, so the inequality holds. ∎

Lemma 1.3.4. *Pick a partition μ such that $\tau^{\mathrm{C}}_{2k}(\mu)$ is well-defined and $\iota^{\mathrm{C}}_{2k}(\mu) > 0$. Then $|\mu| - (\mu_1 + \cdots + \mu_k) \ge \iota^{\mathrm{C}}_{2k}(\mu) + 1$. In case of equality, we have $\mu = (\mu_1, \dots, \mu_k, 1, 1)$, and in particular, $\iota^{\mathrm{C}}_{2k}(\mu) = 1$.*

Proof. Write $|\alpha_{\le k}| = \alpha_1 + \cdots + \alpha_k$ for any partition α. Let R_μ be the border strip of length $2\ell(\mu) - 2k - 2$ that we remove from μ to follow the modification rule and set $\nu = \mu \setminus R_\mu$. Let c be the number of columns that R_μ occupies. We separate the cases $\nu^\dagger_c \le k$ and $\nu^\dagger_c > k$.

First assume that $\nu^\dagger_c \le k$. Let $i \ge 1$ be defined by the property $\nu^\dagger_{i-1} > k$ and $\nu^\dagger_i \le k$. Then $i \le c$. Note that $\nu^\dagger_j + 1 = \mu^\dagger_{j+1}$ for $j = 1, \dots, c-1$ and $\nu^\dagger_j = \mu^\dagger_j$ for $j > c$. So

$$2\ell(\mu) - 2k - 2 = |R_\mu| = |\mu| - |\nu| = \ell(\mu) - \nu^\dagger_c + c - 1.$$

Rewrite this as $\ell(\mu) - k = k - \nu^\dagger_c + c + 1$. If we remove all boxes in the first k rows from R_μ, we get a border strip that occupies $\ell(\mu) - k = k - \nu^\dagger_c + c + 1$ rows and $\ge i$ columns. So

$$|\mu| - |\mu_{\le k}| \ge |\nu| - |\nu_{\le k}| + (i + k - \nu^\dagger_c + c) \ge |\nu| - |\nu_{\le k}| + (i + c).$$

If $i = 1$, then $\ell(\nu) \le k$, so $|\nu| = |\nu_{\le k}|$, and $c = \iota^{\mathrm{C}}_{2k}(\mu)$. Hence we get $|\mu| - |\mu_{\le k}| \ge \iota^{\mathrm{C}}_{2k}(\mu) + 1$. Going back, we see that this can only be an equality if $\nu^\dagger_c = k$, and then $\mu = (\mu_1, \dots, \mu_k, c, 1^c)$. Then $2c = |\mu| - |\mu_{\le k}| = c + 1$, so $c = 1$ and we are done in this case. Otherwise, if $i > 1$, i.e., $\ell(\nu) > k$, then by induction, we get $|\nu| - |\nu_{\le k}| \ge 1 + \iota^{\mathrm{C}}_{2k}(\nu)$, and hence $|\mu| - |\mu_{\le k}| \ge i + \iota^{\mathrm{C}}_{2k}(\mu) + 1 > \iota^{\mathrm{C}}_{2k}(\mu) + 1$.

Now suppose that $\nu^\dagger_c > k$. Then R_μ does not contain any boxes in the first k rows, so $\mu_i = \nu_i$ for $i = 1, \dots, k$ and hence $|\mu| - |\mu_{\le k}| = |R_\mu| + |\nu| - |\nu_{\le k}|$. Since $\ell(\nu) > k$, we get $|\nu| - |\nu_{\le k}| \ge \iota^{\mathrm{C}}_{2k}(\nu) + 1$ by induction. If this is an equality, then $\nu = (\nu_1, \dots, \nu_k, 1, 1)$ and $|R_\mu| > c$. Otherwise, if it is strict, we can at least say that $|R_\mu| \ge c$. In either case, we get $|\mu| - |\mu_{\le k}| > c + \iota^{\mathrm{C}}_{2k}(\nu) + 1 = \iota^{\mathrm{C}}_{2k}(\mu) + 1$. ∎

1.4. **Type D Weyl group.** Now pick $k \in \frac{1}{2}\mathbf{Z}$ with $k \geq 0$. We now associate to a partition λ two quantities $\iota^{\mathrm{D}}_{2k}(\lambda)$ and $\tau^{\mathrm{D}}_{2k}(\lambda)$. We again give two equivalent definitions. The presentation here follows [SSW, §4.4], where more details and references can be found.

We begin with the Weyl group definition. Let s_0 be the automorphism of the set $\mathcal{U}$ which negates and swaps the first and second entries, and let W_∞ be the group generated by the s_i with $i \geq 0$. This is a Coxeter group of type D_∞. Let $\ell\colon W_\infty \to \mathbf{Z}_{\geq 0}$ be the length function, which is defined just as in (1.2.1). Let $\rho = (-k, -k-1, \dots)$. Define a new action of W_∞ on $\mathcal{U}$ by $w \bullet \lambda = w(\lambda+\rho) - \rho$. The action of s_0 is given by

$$s_0 \bullet (a_1, a_2, a_3, \dots) = (2k+1-a_2, 2k+1-a_1, a_3, \dots).$$

We say that a partition λ is **admissible** if $\lambda^\dagger_1 + \lambda^\dagger_2 \leq 2k$. For an admissible partition λ, define another admissible partition λ^σ by $(\lambda^\sigma)^\dagger_1 = 2k - \lambda^\dagger_1$ and $(\lambda^\sigma)^\dagger_i = \lambda^\dagger_i$ for $i > 1$.

Given a partition $\lambda \in \mathcal{U}$, exactly one of the following two possibilities hold:

- There exists a unique element $w \in W_\infty$ such that $w \bullet \lambda^\dagger = \mu^\dagger$ is an admissible partition. We then put $\iota^{\mathrm{D}}_{2k}(\lambda) = \ell(w)$ and $\tau^{\mathrm{D}}_{2k}(\lambda) = \mu$.
- There exists a non-identity element $w \in W_\infty$ such that $w \bullet \lambda^\dagger = \lambda^\dagger$. We then put $\iota^{\mathrm{D}}_{2k}(\lambda) = \infty$ and leave $\tau^{\mathrm{D}}_{2k}(\lambda)$ undefined.

Note that if λ is admissible, then we are in the first case with $w = 1$, and so $\iota^{\mathrm{D}}_{2k}(\lambda) = 0$ and $\tau^{\mathrm{D}}_{2k}(\lambda) = \lambda$.

The border strip definition is the same as the one given in §1.3, except for three differences

(D1) the border strip R_λ has length $2\ell(\lambda) - 2k$,
(D2) in the definition of $\iota_{2k}(\lambda)$, we use $c(R_\lambda) - 1$ instead of $c(R_\lambda)$, and
(D3) if the total number of border strips removed is odd, then replace the result μ with μ^σ.

Let $\leq$ denote the Bruhat order on W_∞ [BB, §2].

Proposition 1.4.1. *Fix an admissible partition λ. Given μ such that $\tau^{\mathrm{D}}_{2k}(\mu) = \lambda$, let $w_\mu \in W_\infty$ be the unique element such that $w_\mu \bullet \mu^\dagger = \lambda^\dagger$. Then $w_\mu \leq w_\nu$ if and only if $\mu \subseteq \nu$.*

Proof. Same as Proposition 1.3.2. ☐

1.5. **Geometric technique for free resolutions.** Let X be a projective variety. Let

$$0 \to \xi \to \varepsilon \to \eta \to 0$$

be an exact sequence of vector bundles on X, with ε trivial, and let $\mathcal{V}$ be another vector bundle on X. Put

$$A = \mathrm{H}^0(X; \mathrm{Sym}(\varepsilon)) = \mathrm{Sym}(\mathrm{H}^0(X;\varepsilon)), \qquad M^{(i)}(\mathcal{V}) = \mathrm{H}^i(X; \mathrm{Sym}(\eta) \otimes \mathcal{V}).$$

Then each $M^{(i)}(\mathcal{V})$ is an A-module. Let $A(-i)$ denote the module A with a grading shift: $A(-i)_d = A_{d-i}$. For the following, see [W, §5.1].

Theorem 1.5.1. *There is a minimal graded A-free complex $\mathbf{F}_\bullet$ with terms*

$$\mathbf{F}_i = \bigoplus_{j \geq 0} \mathrm{H}^j(X; \textstyle\bigwedge^{i+j}(\xi) \otimes \mathcal{V}) \otimes A(-i-j)$$

with the property that for all $i \geq 0$, we have $\mathrm{H}_{-i}(\mathbf{F}_\bullet) = M^{(i)}(\mathcal{V})$ and $\mathrm{H}_j(\mathbf{F}_\bullet) = 0$ for $j > 0$. In fact, $\mathbf{F}_\bullet$ is quasi-isomorphic to the derived pushforward of the twisted Koszul complex $\bigwedge^\bullet(\xi) \otimes \mathcal{V}$ along the projection $\varepsilon \to \mathrm{Spec}(A)$.

In particular, $\mathbf{F}_i = 0$ for all $i < 0$ if and only if $M^{(j)}(\mathcal{V}) = 0$ for all $j > 0$. In this case, $\mathbf{F}_\bullet$ is a minimal free resolution of $M^{(0)}(\mathcal{V})$.

1.6. **Borel–Weil–Bott theorem.** Let E be a vector space and let $\mathbf{Gr}(n, E)$ be the Grassmannian of n-dimensional subspaces of E. We have a tautological sequence on $\mathbf{Gr}(n, E)$

$$0 \to \mathcal{R} \to E \otimes \mathcal{O}_{\mathbf{Gr}(n,E)} \to \mathcal{Q} \to 0,$$

where $\mathcal{R}$ has rank n. Recall the notation $\mathfrak{S}$ and ℓ from §1.2. The Borel–Weil–Bott theorem [W, §4.1] is then:

Theorem 1.6.1 (Borel–Weil–Bott)**.** *Let λ be a partition with at most n parts, let μ be any partition and let $\mathcal{V}$ be the vector bundle $\mathbf{S}_\lambda(\mathcal{R}^*) \otimes \mathbf{S}_\mu(\mathcal{Q}^*)$ on $\mathbf{Gr}(n, E)$. Set $\beta = (\lambda_1, \dots, \lambda_n, \mu_1, \mu_2, \dots)$. Exactly one of the following happens:*

- *There is a unique $w \in \mathfrak{S}$ such that $w \bullet \beta = \alpha$ is a partition. Then $\mathrm{H}^{\ell(w)}(\mathbf{Gr}(n, V); \mathcal{V}) = \mathbf{S}_\alpha(E^*)$ and all other cohomology groups vanish.*
- *There is a non-identity $w \in \mathfrak{S}$ such that $w \bullet \beta = \beta$. Then all cohomology of $\mathcal{V}$ vanishes.*

Now let V be a symplectic vector space of dimension $2n$ with symplectic form ω_V. For $m \le n$, $\mathbf{IGr}(m, V)$ is the subvariety of $\mathbf{Gr}(m, V)$ consisting of subspaces W such that $\omega_V|_W \equiv 0$. Let $\mathcal{R}$ be the tautological subbundle on $\mathbf{Gr}(m, V)$ restricted to $\mathbf{IGr}(m, V)$. We will need an analogue of Theorem 1.6.1 for $\mathbf{IGr}(m, V)$. We only need the case $m = n$, so we only state it in that case. See [W, Corollary 4.3.4] for the case of general m.

We consider sequences of length n and define $\rho = (n, n-1, \dots, 1)$. The simple reflections are: for $1 \le i \le n-1$, we have s_i that swaps positions i and $i+1$, and s_n negates the last entry. The group generated by the s_i is the Weyl group W of type BC_n and we define the length function of an element $w \in W$ as in (1.2.1) (for some details and references, see [S1, §2.3]). We define the dotted action of W on sequences analogously.

Theorem 1.6.2 (Borel–Weil–Bott)**.** *Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a weakly decreasing sequence of integers. Exactly one of the following happens:*

- *There is a unique $w \in W$ such that $w \bullet \lambda = \alpha$ is a partition. In this case, $\mathrm{H}^{\ell(w)}(\mathbf{IGr}(n, V); \mathbf{S}_\lambda(\mathcal{R}^*)) = \mathbf{S}_{[\alpha]}(V)$ and all other cohomology groups vanish.*
- *There is a non-identity $w \in W$ such that $w \bullet \lambda = \lambda$. Then all cohomology of $\mathbf{S}_\lambda(\mathcal{R}^*)$ vanishes.*

Lemma 1.6.3. *Assume $\lambda \subseteq n \times (n + 2k + 1)$.*
(a) $\mathrm{H}^\bullet(\mathbf{IGr}(n, V); \mathbf{S}_\lambda(\mathcal{R}) \otimes (\det \mathcal{R}^*)^k) \ne 0$ *if and only if $\tau^{\mathrm{C}}_{2k}(\lambda^\dagger)$ is well-defined.*
(b) *Let $\alpha = n \times k \setminus \tau^{\mathrm{C}}_{2k}(\lambda^\dagger)^\dagger$. Then*

$$\mathrm{H}^{\iota^{\mathrm{C}}_{2k}(\lambda^\dagger)}(\mathbf{IGr}(n, V); \mathbf{S}_\lambda(\mathcal{R}) \otimes (\det \mathcal{R}^*)^k) = \mathbf{S}_{[\alpha]}(V)$$

and the cohomology vanishes in all other degrees.

Proof. In the notation of Theorem 1.6.2, the vector bundle $\mathbf{S}_\lambda(\mathcal{R}) \otimes (\det \mathcal{R}^*)^k$ corresponds to the weight $\mu = (k - \lambda_n, \dots, k - \lambda_1)$. All cohomology vanishes for this vector bundle if and only if the absolute values of two entries of $\mu + \rho$ agree, where $\rho = (n, n-1, \dots, 2, 1)$. To calculate $\tau^{\mathrm{C}}_{2k}(\lambda^\dagger)$, we perform a dotted Weyl group action on λ with the group of infinite signed permutations with $\rho' = (-(k+1), -(k+2), -(k+3), \dots)$. The first n entries of $\lambda + \rho'$ are the negatives of those of $\mu + \rho$. Since $\lambda_1 \le n + 2k + 1$, none of the first n entries of $\lambda + \rho'$ are equal in absolute value to the remaining entries. So the absolute values of the entries of $\mu + \rho$ are distinct if and only if $\tau^{\mathrm{C}}_{2k}(\lambda^\dagger)$ is well-defined, which proves (a).

Now assume that $\tau^{\mathrm{C}}_{2k}(\lambda^\dagger)$ is well-defined. If w is the signed permutation such that $w(\lambda + \rho') - \rho' = \tau^{\mathrm{C}}_{2k}(\lambda^\dagger)^\dagger$, then there is a signed permutation v with $\ell(v) = \ell(w) = \iota^{\mathrm{C}}_{2k}(\lambda^\dagger)$ such that $v(-\lambda^{\mathrm{op}}+\rho+(k^n))-\rho-(k^n)=-(\tau^{\mathrm{C}}_{2k}(\lambda^\dagger)^\dagger)^{\mathrm{op}}$ (we are reversing and negating sequences to get v from w). Since $-\lambda^{\mathrm{op}}+(k^n)=\mu$, we see that the value of our cohomology is $n\times k\setminus\tau^{\mathrm{C}}_{2k}(\lambda^\dagger)^\dagger$. □

2. Main result

2.1. **Statement and outline.** Let V be a vector space of dimension $2n+2d$ equipped with a symplectic form. Let $\mathbf{IGr}(n,V)$ be the variety of n-dimensional isotropic subspaces of V and let $\mathcal{R}\subset V\times\mathbf{IGr}(n,V)$ be the tautological subbundle. With respect to the symplectic form, let $\mathcal{R}^\perp$ be its orthogonal complement. Define the following vector bundle on $\mathbf{IGr}(n,V)$:

$$\mathcal{E}_\lambda=\mathbf{S}_\lambda(\mathcal{R}^\perp)\otimes(\det\mathcal{R}^*)^k.$$

In the notation of [S2], $\mathcal{E}_\lambda$ is called $\mathcal{E}^{(k^d)}_\lambda$.

Theorem 2.1.1. *Let $\lambda\subseteq n\times(n+2d)$ be a partition.*
(a) *If $k\geq\ell(\lambda)$, then $\mathrm{H}^i(\mathbf{IGr}(n,V);\mathcal{E}_{\lambda^\dagger})=0$ for $i>0$ and $\mathrm{H}^0(\mathbf{IGr}(n,V);\mathcal{E}_{\lambda^\dagger})\neq 0$.*
(b) *The cohomology of $\mathcal{E}_{\lambda^\dagger}$ vanishes unless $\tau^{\mathrm{C}}_{2k}(\lambda)=\mu$ is defined, in which case the cohomology is nonzero only in degree $\iota^{\mathrm{C}}_{2k}(\lambda)=i$, and we have an $\mathbf{Sp}(V)$-equivariant isomorphism*

$$\mathrm{H}^i(\mathbf{IGr}(n,V);\mathcal{E}_{\lambda^\dagger})\cong\mathrm{H}^0(\mathbf{IGr}(n,V);\mathcal{E}_{\mu^\dagger}).$$

We use the notation $\mathcal{L}=\det\mathcal{R}^*$, where $\mathcal{R}$ is the rank n tautological subbundle of either $\mathbf{Gr}(n,V)$ or $\mathbf{IGr}(n,V)$. Before proceeding with the proof of Theorem 2.1.1, we give its outline.

(1) The bundle $\mathcal{E}_{\mu^\dagger}$ is not an irreducible homogeneous bundle as soon as $d>0$, so we cannot apply the usual Borel–Weil–Bott theorem (Theorem 1.6.2). Instead, we will resolve $\mathcal{E}_{\mu^\dagger}$ by irreducible homogeneous bundles in a larger homogeneous space (2.1.2) and analyze the resulting spectral sequence.
(2) The first step is to show that the terms of the spectral sequence for $\mu^\dagger$ and $\lambda^\dagger$ match as representations of $\mathbf{Sp}(V)$ (up to simultaneous homological shift), which is done in Proposition 2.2.4 by using Theorem 1.6.1 and careful combinatorial analysis.
(3) The spectral sequence comes from a double complex and we need to calculate the homology of its total complex. The action of $\mathbf{Sp}(V)$ does not provide enough structure to compare the two complexes. Instead, we lift these complexes to minimal complexes of free modules over $\mathrm{Sym}(\bigwedge^2(V))$ such that the original complexes are specializations of the lifted complexes (Proposition 2.3.1). This is inspired by the equivalence between the categories of modules of $\mathbf{Sp}(\infty)$ and $\mathrm{Sym}(\bigwedge^2(\mathbf{C}^\infty))$ in [SS, Theorem 4.3.2].
(4) The lifted complexes have more structure and allow us to reduce to the case that $\mathcal{R}^\perp$ is irreducible (i.e., $d=0$). They have enough structure so that we can show that they are isomorphic (Proposition 2.3.4), which implies the desired properties. A priori, they contain more information than just the cohomology of $\mathcal{E}_{\mu^\dagger}$ since the specialization mentioned above kills modules whose support is not all of $\mathrm{Sym}(\bigwedge^2(V))$.

Now we begin the proof.

Embed $\mathbf{IGr}(n,V)$ into $\mathbf{Gr}(n,V)$. Then $\mathcal{R}^\perp$ is the restriction of $\mathcal{Q}^*$. By [W, Proposition 4.3.6], we have a locally free resolution on $\mathbf{Gr}(n,V)$:

$$0\to\bigwedge^{\binom{n}{2}}(\bigwedge^2(\mathcal{R}))\to\cdots\to\bigwedge^2(\bigwedge^2(\mathcal{R}))\to\bigwedge^2(\mathcal{R})\to\mathcal{O}_{\mathbf{Gr}(n,V)}\to\mathcal{O}_{\mathbf{IGr}(n,V)}\to 0. \tag{2.1.2}$$

Tensoring this locally free resolution with $\mathbf{S}_\nu(\mathcal{Q}^*) \otimes \mathcal{L}^k$ gives us a spectral sequence

$$\mathrm{E}_1^{-p,q} = \mathrm{H}^q(\mathbf{Gr}(n,V); \textstyle\bigwedge^p(\bigwedge^2(\mathcal{R})) \otimes \mathbf{S}_\nu(\mathcal{Q}^*) \otimes \mathcal{L}^k) \Rightarrow \mathrm{H}^{q-p}(\mathbf{IGr}(n,V); \mathcal{E}_\nu). \tag{2.1.3}$$

By (1.1.2), we have $\bigwedge^i(\bigwedge^2(\mathcal{R})) = \bigoplus_{\theta \in Q_{-1}(2i)} \mathbf{S}_\theta(\mathcal{R})$.

Proof of Theorem 2.1.1. (a) is a special case of (b).

(b) If $\tau^{\mathrm{C}}_{2k}(\lambda)$ is not defined, then we set $\alpha = \lambda^\dagger$ and use Proposition 2.2.1 to conclude that $\mathrm{H}^i(\mathbf{IGr}(n,V); \mathcal{E}_{\lambda^\dagger}) = 0$ for all i. So suppose that $\tau^{\mathrm{C}}_{2k}(\lambda) = \mu$ is defined. Then the result follows from Propositions 2.3.1 and 2.3.7. □

2.2. **Combinatorial lemmas.**

Proposition 2.2.1. *Let α be a partition with $\alpha_1 \le n$ and suppose that $\tau^{\mathrm{C}}_{2k}(\alpha^\dagger)$ is not defined. Then for all $\theta \in Q_{-1}$ and all $i \ge 0$, we have $\mathrm{H}^i(\mathbf{Gr}(n,V); \mathbf{S}_\alpha(\mathcal{Q}^*) \otimes \mathcal{L}^k \otimes \mathbf{S}_\theta(\mathcal{R})) = 0$. In particular, for all $i \ge 0$, we have $\mathrm{H}^i(\mathbf{IGr}(n,V); \mathcal{E}_\alpha) = 0$.*

Proof. Since $\alpha^\dagger$ is a partition, all of the values $\alpha_i - (k+i)$ are distinct. So if $\tau^{\mathrm{C}}_{2k}(\alpha^\dagger)$ is not defined, then there exist $i \le j$ such that $\alpha_i - (k+i) = -(\alpha_j - (k+j))$. Set $D = \alpha_i - (k+i)$. Since $\alpha_i - (k+i) \ge \alpha_j - (k+j)$, we conclude that $D \ge 0$. Also $D \le n-1$ because $\alpha_1 \le n$. Since $\theta \in Q_{-1}$, Lemma 1.1.4 implies that $-\theta_i + i - 1 = \pm D$ for some i with $1 \le i \le n$. So the sequence $(-\theta_n + n - 1, \dots, -\theta_1, \alpha_1 - k - 1, \alpha_2 - k - 2, \dots)$ has a duplicate entry since $\{\alpha_i - k - i, \alpha_j - k - j\} = \{D, -D\}$. To calculate the cohomology of $\mathbf{S}_\alpha(\mathcal{Q}^*) \otimes \mathcal{L}^k \otimes \mathbf{S}_\theta(\mathcal{R})$, we apply Theorem 1.6.1 to the sequence $(k-\theta_n, \dots k-\theta_1, \alpha_1, \alpha_2, \dots)$. The sequence above has a duplicate, so all cohomology vanishes. The last statement is now an immediate consequence of (2.1.3). □

Each partition $\nu \in Q_{-1}$ can be written uniquely as the union of hook partitions of the form $(i, 1^i)$ so that no i is used more than once. We will use the notation $[i_1, \dots, i_d]$ for the partition ν that is the union of the hooks $(i_1, 1^{i_1}), \dots, (i_d, 1^{i_d})$ (for convenience, we allow the values to be unsorted; the ordering does not affect the definition of the partition).

Lemma 2.2.2. *Let α be a partition with $\ell(\alpha) \le k$. Then there is a set $\{b_1, \dots, b_e\}$ so that $\mathbf{S}_\theta(\mathcal{R}) \otimes \mathcal{L}^k \otimes \mathbf{S}_{\alpha^\dagger}(\mathcal{Q}^*)$ ($\theta \in Q_{-1}$) has nonzero cohomology if and only if $\theta = [b_{i_1}, \dots, b_{i_r}]$ for some subset of $\{b_1, \dots, b_e\}$.*

Proof. Let $b_1 > \cdots > b_e$ be any set of values so that $\mathbf{S}_{b_i, 1^{b_i}}(\mathcal{R}) \otimes \mathcal{L}^k \otimes \mathbf{S}_{\alpha^\dagger}(\mathcal{Q}^*)$ has nonzero cohomology for $i = 1, \dots, e$. This corresponds to the weight $(k^{n-b_i-1}, (k-1)^{b_i}, k-b_i, \alpha^\dagger_1, \alpha^\dagger_2, \dots)$; since $k \ge \ell(\alpha)$, this is equivalent to $k - b_i + 1 \notin \{\alpha^\dagger_j - j + 1 \mid j \ge 1\}$. Let $\theta = [b_1, \dots, b_e]$. We claim that $\mathbf{S}_\theta(\mathcal{R}) \otimes \mathcal{L}^k \otimes \mathbf{S}_{\alpha^\dagger}(\mathcal{Q}^*)$ has nonzero cohomology. This is equivalent to showing that the set $\{k - \theta_i + i\}$ is disjoint from $\{\alpha^\dagger_j - j + 1\}$. If $i \le e$, then we have $\theta_i = b_i + i - 1$, so $k - \theta_i + i = k - b_i + 1$, in which case we use the assumption. If $i > e$, then $\theta_i \le e$, so $k - \theta_i + i \ge k - e + i > k$, which is disjoint since $\ell(\alpha) \le k$.

Conversely, assume that $\theta = [b_1, \dots, b_e]$ and that $\mathbf{S}_\theta(\mathcal{R}) \otimes \mathcal{L}^k \otimes \mathbf{S}_{\alpha^\dagger}(\mathcal{Q}^*)$ has nonzero cohomology. The above shows that this is equivalent to $k - b_i + 1 \notin \{\alpha^\dagger_j - j + 1 \mid j \ge 1\}$ for all i, and hence $\mathbf{S}_{b_i, 1^{b_i}}(\mathcal{R}) \otimes \mathcal{L}^k \otimes \mathbf{S}_{\alpha^\dagger}(\mathcal{Q}^*)$ has nonzero cohomology. □

Lemma 2.2.3. *Let $\theta = [b_1, \dots, b_e]$ and pick ν so that $\mathbf{S}_\theta(\mathcal{R}) \otimes \mathcal{L}^k \otimes \mathbf{S}_\nu(\mathcal{Q}^*)$ has nonzero cohomology (in a single degree). Assume also that $\nu^\dagger$ is obtained from $\eta^\dagger$ by removing a border strip of length $2\eta_1 - 2k - 2 \le 2n - 2$. Then $\eta_1 - k - 1 \notin \{b_1, \dots, b_e\}$; set $\theta' = [b_1, \dots, b_e, \eta_1 -$*

$k-1]$. *Then ignoring cohomological degree,* $\mathbf{S}_{\theta'}(\mathcal{R}) \otimes \mathcal{L}^k \otimes \mathbf{S}_\eta(\mathcal{Q}^*)$ *has nonzero cohomology which is isomorphic, as* $\mathbf{GL}(V)$*-representations, to the cohomology of* $\mathbf{S}_\theta(\mathcal{R}) \otimes \mathcal{L}^k \otimes \mathbf{S}_\nu(\mathcal{Q}^*)$.

Conversely, if η *is a partition with* $0 \le \eta_1 - k - 1 \le n-1$ *and* $\mathbf{S}_{\theta'}(\mathcal{R}) \otimes \mathcal{L}^k \otimes \mathbf{S}_\eta(\mathcal{Q}^*)$ *has nonzero cohomology, and* $\theta' = [c_1, \dots, c_e]$*, then* $\eta_1 - k - 1 \in \{c_1, \dots, c_e\}$.

Proof. The border strip assumption means there exists i so that $\eta_1 - k - 1 = -(\nu_i - k - i)$. In particular, $k-(\eta_1-k-1)+1 = \nu_i - i + 1$, so $\eta_1 - k - 1 \notin \{b_1, \dots, b_e\}$ (see proof of Lemma 2.2.2). Set $\theta' = [b_1, \dots, b_e, \eta_1 - k - 1]$ (then $\ell(\theta') \le n$ since $\ell(\theta) \le n$ and $\eta_1 - k - 1 \le n-1$). We claim that the two sequences

$$(k-\theta_n+n, \dots, k-\theta_1+1, \nu_1, \nu_2-1, \dots) \quad (k-\theta'_n+n, \dots, k-\theta'_1+1, \eta_1, \eta_2-1, \dots)$$

are permutations of each other. Suppose that when we sort $\{b_1, \dots, b_e, \eta_1-k-1\}$, η_1-k-1 becomes the jth largest number. Then $\theta'_j = j-1+(\eta_1-k-1)$, so $k-\theta'_j+j = 2k+2-\eta_1$. Swap this entry of the second sequence with η_1. Then $(2k+2-\eta_1, \eta_2-1, \dots)$ can be sorted to become $(\nu_1, \nu_2-1, \dots)$ (see [SSW, Proof of Proposition 3.5]).

Similarly, $(k-\theta'_n+n, \dots, \eta_1, \dots, k-\theta'_1+1)$ can be sorted to become $(k-\theta_n+n, \dots, k-\theta_1+1)$. To see this, subtract $k+1$ from both sequences and negate and reverse them to get $(\theta_1, \theta_2-1, \dots, \theta_n-n+1)$ and $(\theta'_1, \dots, (k+1)-\eta_1, \dots, \theta'_n-n+1)$. It is enough to assume that $j=1$: the first $j-1$ entries of both are the same, so we can ignore them and subtract $j-1$ from the result to reduce to this case. Adapting the argument from [SSW, Proof of Proposition 3.5], removing the border strip of length $2\eta_1-2k-2$ from θ' amounts to negating the first entry $(k+1)-\eta_1$ and then sorting. So the claim is proven and implies that $\mathbf{S}_{\theta'}(\mathcal{R}) \otimes \mathcal{L}^k \otimes \mathbf{S}_\eta(\mathcal{Q}^*)$ has nonzero cohomology.

On the other hand, assume that $\mathbf{S}_{\theta'}(\mathcal{R}) \otimes \mathcal{L}^k \otimes \mathbf{S}_\eta(\mathcal{Q}^*)$ has nonzero cohomology and that $0 \le \eta_1-k-1 \le n-1$. Set $\theta' = [c_1, \dots, c_e]$. By Lemma 1.1.4, there exists i such that $k+1-\eta_1 = \pm(-\theta'_i+i-1)$. If $k+1-\eta_1 = -\theta'_i+i-1$, then $k-\theta'_i+i = 2k+2-\eta_1$, so we can reverse the steps above to get $\eta_1-k-1 \in \{c_1, \dots, c_e\}$. Otherwise, we have $k+1-\eta_1 = \theta'_i-i+1$, which shows that the sequence $(k-\theta'_n+n, \dots, k-\theta'_1+1, \eta_1, \eta_2-1, \dots)$ has a repeat, which contradicts our assumption. ∎

Proposition 2.2.4. *Suppose that* $\tau^{\mathrm{C}}_{2k}(\beta) = \alpha$ *and* $\beta_1 \le n+2d$*. Set* $s = (|\beta|-|\alpha|)/2$*. Then*

$$\mathrm{H}^i(\mathbf{Gr}(n,V); \textstyle\bigwedge^j(\bigwedge^2\mathcal{R}) \otimes \mathcal{L}^k \otimes \mathbf{S}_{\alpha^\dagger}\mathcal{Q}^*) \cong \mathrm{H}^{i+\iota^{\mathrm{C}}_{2k}(\beta)+s}(\mathbf{Gr}(n,V); \bigwedge^{j+s}(\bigwedge^2\mathcal{R}) \otimes \mathcal{L}^k \otimes \mathbf{S}_{\beta^\dagger}\mathcal{Q}^*)$$

as representations of $\mathbf{GL}(V)$ *for all* i, j.

Proof. Let $\{b_1, \dots, b_e\}$ be the set in Lemma 2.2.2 for the partition α. Let $2B_1, \dots, 2B_f$ be the lengths of the border strips removed from β to get α. By Lemma 2.2.3, $\{B_1, \dots, B_f\} \cap \{b_1, \dots, b_e\} = \varnothing$; set $\theta = [b_{i_1}, \dots, b_{i_r}]$ and $\theta' = [B_1, \dots, B_f, b_{i_1}, \dots, b_{i_r}]$ for some subset $\{i_1, \dots, i_r\} \subseteq \{1, \dots, e\}$. Also by Lemma 2.2.3, $\mathbf{S}_{\theta'}(\mathcal{R}) \otimes \mathcal{L}^k \otimes \mathbf{S}_{\beta^\dagger}(\mathcal{Q}^*)$ and $\mathbf{S}_\theta(\mathcal{R}) \otimes \mathcal{L}^k \otimes \mathbf{S}_{\alpha^\dagger}(\mathcal{Q}^*)$ have isomorphic cohomology (ignoring homological degree), and we just need to determine where it is. If the degree for the non-vanishing cohomology of the second vector bundle is N, then we claim that the degree for the first one is $N + \iota^{\mathrm{C}}_{2k}(\beta) + (|\beta|-|\alpha|)/2$.

Let γ be the partition obtained from β by removing a border strip of length $2\beta^\dagger_1 - 2k - 2$ and let θ'' be obtained by removing a hook partition of length $2\beta^\dagger_1 - 2k - 2$ from θ'. Let c be the number of columns that this border strip occupies. At the end of the proof of Lemma 2.2.3, it is shown that there is a unique index i such that $2k+2-\beta^\dagger_1 = k-\theta'_i+i$. Let $t_{a,b}$ be the transposition that swaps a and b. Define the permutation

$$v = t_{n+c-1,n+c} \cdots t_{n+1,n+2} t_{i-(\beta^\dagger_1-k-2), i-(\beta^\dagger_1-k-2)+1} \cdots t_{i-1,i} t_{i,n+1}.$$

Then

$$v(k-\theta'_n+n,\dots,k-\theta'_1+1,\beta^\dagger_1,\beta^\dagger_2-1,\dots)=(k-\theta''_n+n,\dots,k-\theta''_1+1,\gamma^\dagger_1,\gamma^\dagger_2-1,\dots).$$

This is a product of $c+(|\beta|-|\gamma|)/2$ transpositions and they are all simple transpositions except for $t_{i,n+1}$. Let u be the permutation that takes the above sequence to a strictly decreasing sequence δ. Each of the simple transpositions in the expression for v above creates a new inversion so each adds 1 to the length of u. We have

$$t_{i,n+1}v^{-1}u^{-1}\delta=t_{i,n+1}(k-\theta'_n+n,\dots,k-\theta'_1+1,\beta^\dagger_1,\beta^\dagger_2-1,\dots)=\varepsilon.$$

Note that $\varepsilon_{n+1}=k-\theta'_i+i=2k+2-\beta^\dagger_1$ and $\varepsilon_i=\beta^\dagger_1$, so we have $\varepsilon_{n+1}<\varepsilon_i$, which is not an inversion, but becomes one upon applying $t_{i,n+1}$. Furthermore, for each $i+1\le j\le n$, we have $\varepsilon_{n+1}>\varepsilon_j$ and hence $\varepsilon_i>\varepsilon_j$, so none of these can create new inversions. We conclude that $\ell(uv)=\ell(u)+c+(|\beta|-|\gamma|)/2$. By induction, $\ell(u)=N+\iota^{\mathrm{C}}_{2k}(\gamma)+(|\gamma|-|\alpha|)/2$. Since $c+\iota^{\mathrm{C}}_{2k}(\gamma)=\iota^{\mathrm{C}}_{2k}(\beta)$ by definition of ι^{C}_{2k}, we get $\ell(uv)=N+\iota^{\mathrm{C}}_{2k}(\beta)+(|\beta|-|\alpha|)/2$, and the claim is proven.

By Lemma 2.2.3, we have exhausted all possible θ' with nonzero cohomology, so this finishes the proof. □

2.3. **Lifting argument.** Set $\xi=\bigwedge^2(\mathcal{R})$ and define η by the sequence of vector bundles over $\mathbf{Gr}(n,V^*)$:

$$0\to\bigwedge^2(\mathcal{R})\to\bigwedge^2(V^*)\to\eta\to 0$$

where V^* is the trivial bundle $V^*\times\mathbf{Gr}(n,V^*)$. For a partition ν, set $\mathcal{V}_\nu=\mathbf{S}_\nu(\mathcal{Q}^*)\otimes\mathcal{L}^k\otimes(\det V^*)^k$ and let $\mathcal{K}_\nu=\bigwedge^\bullet(\bigwedge^2(\mathcal{R}))\otimes\mathcal{V}_\lambda$ be the twisted Koszul complex over $\bigwedge^2(V^*)\times\mathbf{Gr}(n,V^*)$. The factor of $(\det V^*)^k$ is just for normalization purposes; the final claims are about $\mathbf{Sp}(V)$ (and $\det V$ is trivial when restricted to $\mathbf{Sp}(V)$). Let $\pi\colon\bigwedge^2(V^*)\times\mathbf{Gr}(n,V^*)\to\bigwedge^2(V^*)$ be the projection map. Apply Theorem 1.5.1 to get a minimal complex $\mathbf{F}^\nu_\bullet$ of free $\mathrm{Sym}(\bigwedge^2(V))$-modules with $\mathbf{F}^\nu_\bullet\simeq\mathrm{R}\pi_*(\mathcal{K}_\nu)$. The symplectic form gives a map $\bigwedge^2(V)\to\mathbf{k}$ and we get an algebra homomorphism $\mathrm{Sym}(\bigwedge^2(V))\to\mathbf{k}$. Let $\mathbf{k}_\omega$ denote $\mathbf{k}$ with this $\mathrm{Sym}(\bigwedge^2(V))$-module structure.

Proposition 2.3.1. *For $i\ge 0$ and any partition ν, we have a $\mathbf{Sp}(V)$-equivariant isomorphism $\mathrm{H}_{-i}(\mathbf{F}^\nu_\bullet\otimes_{\mathrm{Sym}(\bigwedge^2(V))}\mathbf{k}_\omega)=\mathrm{H}^i(\mathbf{IGr}(n,V);\mathcal{E}_\nu)$.*

Proof. The derived projection formula [H, Proposition II.5.6] gives a quasi-isomorphism

$$\mathrm{R}\pi_*(\mathcal{K}_\nu)\otimes^{\mathrm{L}}_{\mathrm{Sym}(\bigwedge^2(V))}\mathbf{k}_\omega\simeq\mathrm{R}\pi_*(\mathcal{K}_\nu\otimes_{\mathcal{O}_{\bigwedge^2(V^*)\times\mathbf{Gr}(n,V^*)}}\mathrm{L}\pi^*\mathbf{k}_\omega).$$

Theorem 1.5.1 implies that $\mathrm{R}\pi_*(\mathcal{K}_\nu)\simeq\mathbf{F}^\nu_\bullet$. Since $\mathbf{F}^\nu_\bullet$ is a complex of free $\mathrm{Sym}(\bigwedge^2 V)$-modules, we do not need to derive the tensor product in the left hand side, and it simplifies to $\mathbf{F}^\nu_\bullet\otimes_{\mathrm{Sym}(\bigwedge^2(V))}\mathbf{k}_\omega$. Since π is flat, we can replace $\mathrm{L}\pi^*$ with π^*. The right hand side is then $\mathrm{R}\pi_*$ applied to the complex (2.1.2) twisted by $\mathcal{V}_\nu$. Using the spectral sequence (2.1.3), the homology of the resulting complex is $\mathrm{H}^\bullet(\mathbf{IGr}(n,V);\mathcal{E}_\nu)$. □

Let V' be a symplectic vector space of dimension $2d$ and let $\widetilde{V}=V\oplus V'$. Let $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{Q}}$ be the tautological subbundle and quotient bundle on $\mathbf{Gr}(n+2d,\widetilde{V}^*)$. Define $\widetilde{\xi}=\bigwedge^2(\widetilde{\mathcal{R}})$ and $\widetilde{\eta}$ by

$$0\to\bigwedge^2(\widetilde{\mathcal{R}})\to\bigwedge^2(\widetilde{V}^*)\to\widetilde{\eta}\to 0.$$

Set $\widetilde{\mathcal{V}}_\nu = \mathbf{S}_\nu(\widetilde{\mathcal{Q}}^*) \otimes (\det \widetilde{\mathcal{R}}^*)^k \otimes (\det \widetilde{V}^*)^k$. Let $\widetilde{\pi}\colon \bigwedge^2(\widetilde{V}^*) \times \mathbf{Gr}(n+2d, \widetilde{V}^*) \to \bigwedge^2(\widetilde{V}^*)$ be the projection map. Theorem 1.5.1 gives us a minimal complex $\widetilde{\mathbf{F}}^\nu_\bullet$ of free $\mathrm{Sym}(\bigwedge^2(\widetilde{V}))$-modules with $\widetilde{\mathbf{F}}^\nu_\bullet \simeq \mathrm{R}\widetilde{\pi}_* \widetilde{\mathcal{K}}_\nu$ where $\widetilde{\mathcal{K}}_\nu = \bigwedge^\bullet(\bigwedge^2(\widetilde{\mathcal{R}})) \otimes \widetilde{\mathcal{V}}_\mu$ is the twisted Koszul complex which lives on $\bigwedge^2(\widetilde{V}^*) \times \mathbf{Gr}(n+2d, \widetilde{V}^*)$. Also let $\mathbf{k}_{\widetilde{\omega}}$ be the $\mathrm{Sym}(\bigwedge^2(\widetilde{V}))$-module structure on $\mathbf{k}$ coming from the symplectic form on $\widetilde{V}$.

Order the basis of $\widetilde{V}$ as $x_1, \dots, x_{2n+4d}$ so that the symplectic form is defined by $\widetilde{\omega}(x_i, x_{2n+4d+1-j}) = \delta_{i,j}$ and $\widetilde{\omega}(x_i, x_j) = \widetilde{\omega}(x_{2n+4d+1-i}, x_{2n+4d+1-j}) = 0$ for $i,j = 1, \dots, n+2d$. With respect to this basis, the set of upper triangular matrices in $\mathbf{Sp}(\widetilde{V})$ is a Borel subgroup.

Pick a partition α with $\ell(\alpha) \le 2n+4d$ and set $\alpha' = (\alpha_1 - \alpha_{2n+4d}, \alpha_2 - \alpha_{2n+4d-1}, \dots, \alpha_{n+2d} - \alpha_{n+2d+1})$. With the above choice of Borel subgroup, the highest weight vector for $\mathbf{GL}(\widetilde{V})$ of weight α in $\mathbf{S}_\alpha(\widetilde{V})$ is a highest weight vector for $\mathbf{Sp}(\widetilde{V})$ of weight α'.

Lemma 2.3.2. *If $\mathbf{S}_{[\gamma]}(\widetilde{V}) \subseteq \mathbf{S}_\alpha(\widetilde{V})$ and $\gamma \supseteq \alpha'$, then $\gamma = \alpha'$.*

Proof. Set $N = 2n+4d$. First, as a representation of the diagonal matrices in $\mathbf{GL}(\widetilde{V})$, the weights $(\nu_1, \dots, \nu_N)$ of $\mathbf{S}_\alpha(\widetilde{V})$ correspond to semistandard Young tableaux of shape α and weight ν (see, for example, [W, Proposition 2.1.15(b)]), i.e., a filling of the Young diagram of α with the numbers $1, \dots, N$ where i appears with multiplicity ν_i, the entries in each row are weakly increasing from left to right, and the entries in each column are strictly increasing from top to bottom. Upon restriction to $\mathbf{Sp}(\widetilde{V})$, this weight becomes $\nu' = (\nu_1 - \nu_N, \dots, \nu_{n+2d} - \nu_{n+2d+1})$. So in our situation, we have some ν so that $\nu' = \gamma$ and so that there exists a semistandard Young tableau of shape α and weight ν.

We will prove by induction on i that $\nu_i = \alpha_i$ and $\nu_{N+1-i} = \alpha_{N+1-i}$. First note that by definition of semistandard Young tableaux, one gets the dominance inequalities $\alpha_1 + \cdots + \alpha_m \ge \nu_1 + \cdots + \nu_m$ for all $1 \le m \le N$. Using that $|\alpha| = |\nu|$, we also get the inequalities $-(\alpha_N + \cdots + \alpha_{N-m}) \ge -(\nu_N + \cdots + \nu_{N-m})$. Since $\gamma \supseteq \alpha'$, we have $\nu_1 - \nu_N = \gamma_1 \ge \alpha_1 - \alpha_N$. Combining this with $\alpha_1 \ge \nu_1$, we get $-\nu_N \ge -\alpha_N$, but we also know that the reverse inequality holds, so $\nu_N = \alpha_N$ and hence $\nu_1 \ge \alpha_1$ (which again implies $\nu_1 = \alpha_1$).

Now suppose we have shown that $\nu_i = \alpha_i$ and $\nu_{N+1-i} = \alpha_{N+1-i}$. Consider the dominance inequality $\alpha_1 + \cdots + \alpha_{i+1} \ge \nu_1 + \cdots + \nu_{i+1}$. Then $\alpha_{i+1} \ge \nu_{i+1}$ by our induction hypothesis. Also, we have $\nu_{i+1} - \nu_{N-i} = \gamma_{i+1} \ge \alpha_{i+1} - \alpha_{N-i}$, so $-\nu_{N-i} \ge -\alpha_{N-i}$. Now consider the dominance inequality $-(\alpha_N + \cdots + \alpha_{N-i}) \ge -(\nu_N + \cdots + \nu_{N-i})$ to get the reverse inequality. As before, we conclude that $\nu_{N-i} = \alpha_{N-i}$ and $\nu_{i+1} = \alpha_{i+1}$. ☐

Lemma 2.3.3. *Let $\mu \subseteq (n+2d+2k+1) \times (n+2d)$ be a partition such that $\tau^{\mathrm{C}}_{2k}(\mu)$ is defined. For all $i > -\iota^{\mathrm{C}}_{2k}(\mu)$, the minimal generators of $\widetilde{\mathbf{F}}^{\mu^\dagger}_i$ map injectively into $\widetilde{\mathbf{F}}^{\mu^\dagger}_{i-1}$.*

Proof. Let $\mathbf{S}_\alpha(\widetilde{V})$ be a subset of the minimal generators of $\widetilde{\mathbf{F}}^{\mu^\dagger}_i$ and that $i > -\iota^{\mathrm{C}}_{2k}(\mu)$. Suppose that its image in $\widetilde{\mathbf{F}}^{\mu^\dagger}_{i-1}$ is zero. Set $\mathbf{G}_\bullet = \widetilde{\mathbf{F}}^{\mu^\dagger}_\bullet \otimes_{\mathrm{Sym}(\bigwedge^2(\widetilde{V}))} \mathbf{k}_{\widetilde{\omega}}$. By Proposition 2.3.1 and Lemma 1.6.3, $\mathrm{H}_i(\mathbf{G}_\bullet) = 0$. Note that $\mathbf{G}_\bullet$ is equivariant for $\mathbf{Sp}(\widetilde{V})$ and that $\mathbf{S}_{[\alpha']}(\widetilde{V}) \subseteq \mathbf{S}_\alpha(\widetilde{V}) \subseteq \mathbf{G}_i$ where $\alpha' = (\alpha_1 - \alpha_{2n+4d}, \dots, \alpha_{n+2d} - \alpha_{n+2d+1})$. Hence there must be minimal generators $\mathbf{S}_\beta(\widetilde{V}) \subseteq \widetilde{\mathbf{F}}^{\mu^\dagger}_{i+1}$ which map to $\mathbf{S}_\alpha(\widetilde{V})$ such that we have $\mathbf{S}_{[\alpha']}(\widetilde{V}) \subseteq \mathbf{S}_\beta(\widetilde{V}) \subseteq \mathbf{G}_{i+1}$ mapping isomorphically to this copy of $\mathbf{S}_{[\alpha']}(\widetilde{V})$ in $\mathbf{G}_i$. Since $\widetilde{\mathbf{F}}^{\mu^\dagger}_\bullet$ is $\mathbf{GL}(\widetilde{V})$-equivariant, we have $\alpha \subseteq \beta$. Using (2.3.5), we can apply Lemma 1.3.3 (with $n+2d$ in place of k) to conclude that $\beta' \subseteq \alpha'$ where $\beta' = (\beta_1 - \beta_{2n+4d}, \dots, \beta_{n+2d} - \beta_{n+2d+1})$. Lemma 2.3.2 implies

$\beta' = \alpha'$. Since $\alpha \subseteq \beta$, this implies $\beta = \alpha$. But $\mathrm{Sym}(\bigwedge^2(\widetilde{V}))$ does not contain non-constant $\mathbf{GL}(\widetilde{V})$-invariants, so this contradicts that the map $\widetilde{\mathbf{F}}^{\mu^\dagger}_{i+1} \to \widetilde{\mathbf{F}}^{\mu^\dagger}_{i}$ is minimal. □

Proposition 2.3.4. *Suppose $\tau^{\mathrm{C}}_{2k}(\beta) = \alpha$ and $\beta \subseteq (n + 2d + 2k + 1) \times (n + 2d)$. We have an isomorphism of chain complexes $\widetilde{\mathbf{F}}^{\beta^\dagger}_{\bullet}[\iota^{\mathrm{C}}_{2k}(\beta)] \cong \widetilde{\mathbf{F}}^{\alpha^\dagger}_{\bullet}$.*

Proof. Set $I = \iota^{\mathrm{C}}_{2k}(\beta)$. We have $\widetilde{\mathbf{F}}^{\alpha^\dagger}_{i} \cong \widetilde{\mathbf{F}}^{\beta^\dagger}_{i-I}$ for all i by Proposition 2.2.4. Set $\nu = (k - \alpha^\dagger_{n+2d}, \dots, k - \alpha^\dagger_1)$. Then $\widetilde{\mathcal{V}}_{\alpha^\dagger} = \mathbf{S}_\nu(\mathcal{Q})$, so we are in the situation of [SSW, §3.5, Step b]. In particular, $\widetilde{\mathbf{F}}^{\alpha^\dagger}_{\bullet}$ is acyclic, and

$$\widetilde{\mathbf{F}}^{\alpha^\dagger}_{i} = \bigoplus_{\substack{\theta \\ \tau^{\mathrm{C}}_{2n+4d}(\theta)=\alpha^\dagger \\ \iota^{\mathrm{C}}_{2n+4d}(\theta)=i}} \mathbf{S}_\theta(\widetilde{V}) \otimes \mathrm{Sym}(\textstyle\bigwedge^2(\widetilde{V}))(-(|\theta| - |\alpha|)/2). \tag{2.3.5}$$

The $\mathbf{GL}(\widetilde{V})$-equivariant map $\widetilde{\mathbf{F}}^{\alpha^\dagger}_{1} \to \widetilde{\mathbf{F}}^{\alpha^\dagger}_{0}$ (which is nonzero) is unique up to scalar multiple (see [SSW, Proof of Proposition 3.15]). Similarly, the same is true for $\widetilde{\mathbf{F}}^{\beta^\dagger}_{1-I} \to \widetilde{\mathbf{F}}^{\beta^\dagger}_{-I}$ (and it is nonzero by Lemma 2.3.3). So we have an isomorphism $\mathrm{coker}(\widetilde{\mathbf{F}}^{\beta^\dagger}_{1-I} \to \widetilde{\mathbf{F}}^{\beta^\dagger}_{-I}) \cong \mathrm{coker}(\widetilde{\mathbf{F}}^{\alpha^\dagger}_{1} \to \widetilde{\mathbf{F}}^{\alpha^\dagger}_{0})$ which lifts to a $\mathbf{GL}(\widetilde{V})$-equivariant chain map $\psi \colon \widetilde{\mathbf{F}}^{\beta^\dagger}_{\bullet}[I] \to \widetilde{\mathbf{F}}^{\alpha^\dagger}_{\bullet}$ which is unique up to homotopy. We will show by induction on $i \geq 0$ that ψ_i is an isomorphism. We have just explained that this is true for $i = 0$. Suppose it is true for i. By Lemma 2.3.3, the minimal generators of $\widetilde{\mathbf{F}}^{\beta^\dagger}_{i+1-I}$ map injectively into $\widetilde{\mathbf{F}}^{\beta^\dagger}_{i-I}$. So in order for the diagram

$$\begin{array}{ccc} \widetilde{\mathbf{F}}^{\alpha^\dagger}_{i+1} & \longrightarrow & \widetilde{\mathbf{F}}^{\alpha^\dagger}_{i} \\ {\scriptstyle \psi_{i+1}} \uparrow & & \uparrow {\scriptstyle \psi_i} \\ \widetilde{\mathbf{F}}^{\beta^\dagger}_{i+1-I} & \longrightarrow & \widetilde{\mathbf{F}}^{\beta^\dagger}_{i-I} \end{array}$$

to commute, the minimal generators of $\widetilde{\mathbf{F}}^{\beta^\dagger}_{i+1-I}$ must map injectively under ψ_{i+1}. By Proposition 1.3.2, the partitions indexing the Schur functors in the minimal generators of $\widetilde{\mathbf{F}}^{\beta^\dagger}_{i+1-I} \cong \widetilde{\mathbf{F}}^{\alpha^\dagger}_{i+1}$ are incomparable, so ψ_{i+1} maps minimal generators to minimal generators. So ψ_{i+1} induces an isomorphism between the minimal generators of $\widetilde{\mathbf{F}}^{\beta^\dagger}_{i+1-I}$ and $\widetilde{\mathbf{F}}^{\alpha^\dagger}_{i+1}$, and hence is an isomorphism. □

Remark 2.3.6. The idea for Proposition 2.3.4 comes from the uniqueness of BGG resolutions for irreducible representations of Kac–Moody algebras with integral dominant highest weight (roughly, any complex that looks like the BGG resolution is isomorphic to it; see [K, Theorem 9.2.18] for a precise statement). The resolution $\widetilde{\mathbf{F}}^{\alpha^\dagger}_{\bullet}$ can be given the structure of a BGG resolution, but it resolves an irreducible representation whose highest weight is not integral dominant (see [EH] which builds on [E]). □

Corollary 2.3.7. *Suppose $\tau^{\mathrm{C}}_{2k}(\beta) = \alpha$ and $\beta \subseteq (n + 2d + 2k + 1) \times (n + 2d)$. We have an isomorphism of chain complexes $\mathbf{F}^{\beta^\dagger}_{\bullet}[\iota^{\mathrm{C}}_{2k}(\beta)] \cong \mathbf{F}^{\alpha^\dagger}_{\bullet}$. Furthermore, $\mathbf{F}^{\alpha^\dagger}_{j} = 0$ for $j < 0$.*

Proof. Let $i \colon \mathbf{Gr}(n, V) \to \mathbf{Gr}(n + 2d, \widetilde{V})$ be the map $W \mapsto W \oplus V'$. Then $i^*(\widetilde{\mathcal{R}}) = \mathcal{R} \oplus V'$ and $i^*(\widetilde{\mathcal{Q}}) = \mathcal{Q}$. So for any partition ν with $\ell(\nu) \leq n + d$, $\mathcal{K}_\nu$ is the $\mathbf{GL}(V')$-invariant subcomplex of $i^*(\widetilde{\mathcal{K}}_\nu)$, and hence $\mathbf{F}^{\nu}_{\bullet}$ is the $\mathbf{GL}(V')$-invariant subcomplex of $\widetilde{\mathbf{F}}^{\nu}_{\bullet}$. Now use

Proposition 2.3.4. The last statement follows from the fact that $\widetilde{\mathbf{F}}_j^{\alpha^\dagger} = 0$ for $j < 0$ which was discussed in the proof of Proposition 2.3.4. □

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