HW Set 9 of Math 55, Spring 2001 (Prof. Strain)
5.4: 2,8,10,30,32,44,58
5.5: 2,6,10,12,14,24
5.6: 4,6,8
Solutions prepared by Nick Meyer
5.4:
2. $1+4x+16x^2+64x^3+256x^4$, which is a geometric series equaling $\frac{1-(4x)^5}{1-4x}$

8. In this problem, we assume the given function equals $\sum_{k=0}^{\infty} a_k x^k$, and try to find the a_k.
a. Well, $(1+x)^3 = \sum_{k=0}^{3} C(3,k)x^k$ (by the Binomial Theorem, which is equation 1 in Table 1), so $(1+x^2)^3 = \sum_{k=0}^{3} C(3,k)x^{2k}$. (Or you can jump to this step by using equation 3 in Table 1). Thus $(1+x^2)^3 = C(3,0)+C(3,1)x^2+C(3,2)x^4+C(3,3)x^6 = 1+3x^2+3x^4+x^6$, so $a_0=1, a_2=3, a_4=3, a_6=1$, and all other a_k are 0.
b. This is similar to part a. $(3x-1)^3 = -(1-3x)^3 = -(\sum_{k=0}^{3} C(3,k)(-3x)^k) = \sum_{k=0}^{3} -C(3,k)(-3)^k x^k$, so $a_k = -C(3,k)(-3)^k$ for $0 \le k \le 3$, and 0 otherwise; i.e., $a_0=-1, a_1=9, a_2=-27, a_3=27$.
c. By replacing x with $2x^2$ in equation 5 of Table 1, we obtain $\frac{1}{1-2x^2} = \sum_{k=0}^{\infty}(2x^2)^k = \sum_{k=0}^{\infty} 2^k x^{2k}$; thus for all $k \ge 0$, $a_{2k} = 2^k$, and $a_{2k+1} = 0$ (i.e., there are no terms with odd powers of x).
d. By multiplying equation 9 of Table 1 by x^2, and setting $n=3$, we obtain $\frac{x^2}{(1-x)^3} = \sum_{k=0}^{\infty} C(3+k-1,k)x^{k+2} = \sum_{k=2}^{\infty} C(k,k-2)x^k = \sum_{k=0}^{\infty} C(k,2)x^k$. Hence $a_0=a_1=0$, and $a_k = C(k,2)$ for $k \ge 2$.
e. By setting $a=3$ in equation 6 of Table 1, we obtain that $x-1+\frac{1}{1-3x} = x-1+\sum_{k=0}^{\infty} 3^k x^k = 4x+\sum_{k=2}^{\infty} 3^k x^k$, so $a_0=0, a_1=4$, and $a_k=3^k$ for $k \ge 2$.
f. We write $\frac{1+x^3}{(1+x)^3}$ as $\frac{1}{(1+x)^3}+\frac{x^3}{(1+x)^3}$, and solve each part separately. By using equation 10 of Table 1 (with $n=3$), we obtain $\frac{1}{(1+x)^3} = \sum_{k=0}^{\infty} C(k+2,k)(-1)^k x^k = \sum_{k=0}^{\infty} C(k+2,2)(-1)^k x^k$; hence $\frac{x^3}{(1+x)^3} = \sum_{k=0}^{\infty} C(k+2,2)(-1)^k x^{k+3} = \sum_{k=3}^{\infty} C(k-1,2)(-1)^{k-3}x^k$. Thus, for $\frac{1+x^3}{(1+x)^3}$ we have $a_k = C(k+2,2)(-1)^k$ for $k<3$ and $a_k = C(k+2,2)(-1)^k + C(k-1,2)(-1)^{k-3}$ for $k \ge 3$. Ah, but $C(k+2,2)(-1)^k + C(k-1,2)(-1)^{k-3} = (-1)^k(C(k+2,2)-C(k-1,2))$ (since $(-1)^k$ and $(-1)^{k-3}$ have different signs), and this is equal to $(-1)^k(\frac{(k+2)(k+1)}{2} - \frac{(k-1)(k-2)}{2}) = (-1)^k 3k$, so in fact $a_k = (-1)^k 3k$ for $k \ge 3$.
g. The clever solution: multiply top and bottom by $(1-x)$, thereby obtaining $\frac{x-x^2}{1-x^3} = \frac{x}{1-x^3} - \frac{x^2}{1-x^3} = (x+x^4+x^7+x^{10}+\ldots)-(x^2+x^5+x^8-\ldots)$. So for all $k \ge 0$, $a_{3k}=0, a_{3k+1}=1, a_{3k+2}=-1$.

Alternate solution: factor the denominator, and use partial fractions. This gets kind of ugly, but is doable.
h. Replacing x with $3x^2$ in the penultimate equation of Table 1, we obtain $e^{3x^2} = \sum_{k=0}^{\infty}\frac{(3x^2)^k}{k!} = \sum_{k=0}^{\infty}\frac{3^k x^{2k}}{k!}$, so $e^{3x^2}-1 = \sum_{k=1}^{\infty}\frac{3^k x^{2k}}{k!}$. Hence $a_0=0$, and $a_k = \frac{3^k x^{2k}}{k!}$ for $k \ge 1$.
10. Note: there are multiple ways to solve these, of which brute force is probably the easiest. However, for pedagogical purposes, we'll try to present methods that will work on larger-scale problems.
a. The given power series equals $(\frac{1}{1-x^3})^3$, which equals $\sum_{k=0}^{\infty} C(2+k,k)x^{3k}$ by equation 9 of Table 1 (setting $n=3$ and replacing x with x^3). The x^9 coefficient of this is $C(5,3)=10$.
b. The given power series equals $x^6(1+x+x^2+x^3+\ldots)^3 = \frac{x^6}{(1-x)^3} = x^6\sum_{k=0}^{\infty} C(2+k,k)x^k$. The x^9 coefficient is $C(5,3)=10$ again.
c. We can factor the given expression to obtain $x^7(1+x^2+x^3)(1+x)(1+x+x^2+\ldots) = x^7(1+x+x^2+2x^3+x^4)(1+x+x^2+\ldots$. We need the x^2 coefficient in $(1+x+x^2+2x^3+x^4)(1+x+x^2+\ldots$; this is easily seen to be 3.
d. Brute-force is the most practical way to proceed here. Only two terms yield an x^9; $x \cdot x^8$ and $x^7 \cdot x^2$. So the answer is 2.
e. Note that the given expression is of degree 6, so there is no x^9 term. Answer: 0.

30. We know $G(x) = a_0 + a_1x + a_2x^2 + \ldots = \sum_{k=0}^{\infty} a_k x^k$, so:
a. $2a_0 + 2a_1x + \ldots = 2G(x)$.
b. $0 + a_0x + a_1x^2 + a_2x^3 + \ldots = xG(x)$.
c. $a_2x^4 + a_3x^5 + a_4x^6 + \ldots = x^2(G(x) - a_0 - a_1x)$.
d. $a_2 + a_3x + a_4x^2 + \ldots = (G(x) - a_0 - a_1x)/x^2$.
e. $a_1 + 2a_2x + 3a_3x^3 + \ldots = \sum_{k=1}^{\infty} ka_kx^{k-1}$, which is the derivative of $\sum_{k=0}^{\infty} a_kx^k$, so the answer is $G'(x)$.
f. Here, it's important to note that the subscripts of each summand in the x^n term add to n; this indicates (following Theorem 1) that the given power series is a product of two simpler ones. Once you see this, it's easy to guess and check the correct solution, which is $(G(x))^2$.

32. Say our solution is $G(x) = \sum_{k=0}^{\infty} a_kx^k$. We know $a_k = 7a_{k-1}$ for $k \geq 1$, so $G(x)$ also equals $5 + \sum_{k=1}^{\infty} 7a_{k-1}x^k = 5 + 7\sum_{k=0}^{\infty} a_kx^{k+1} = 5 + 7x\sum_{k=0}^{\infty} a_kx^k = 5 + 7xG(x)$. So $G(x) = 5 + 7xG(x)$, hence $G(x) = \frac{5}{1-7x} = 5\sum_{k=0}^{\infty}(7x)^k = \sum_{k=0}^{\infty} 5(7^k)x^k$, which makes $a_k = 5(7^k)$.

44. a. Say our solution is $G(x) = \sum_{n=0}^{\infty} a_nx^n$, where $a_n = 1^2 + 2^2 + \ldots + n^2$. We know a_n satisfies the recurrence relation $a_n = a_{n-1} + n^2$ for $n \geq 1$, so $G(x) = 0 + \sum_{n=1}^{\infty}(a_{n-1} + n^2)x^n = \sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} n^2x^n = \sum_{n=0}^{\infty} a_nx^{n+1} + \sum_{n=1}^{\infty} n^2x^n = x\sum_{n=0}^{\infty} a_nx^n + \sum_{n=1}^{\infty} n^2x^n = xG(x) + \sum_{n=1}^{\infty} n^2x^n$.

So what's $B(x) = \sum_{n=1}^{\infty} n^2x^n$? Well, $b_n = n^2$ satisfies the recurrence relation $b_n = b_{n-1} + 2n - 1$ for $n \geq 1$, so $B(x) = 0 + \sum_{n=1}^{\infty}(b_{n-1} + 2n - 1)x^n = xB(x) + \sum_{n=1}^{\infty} 2nx^n - \sum_{n=1}^{\infty} x^n = xB(x) - (\frac{1}{1-x} - 1) + \sum_{n=1}^{\infty} 2nx^n$.

So what's $C(x) = \sum_{n=1}^{\infty} 2nx^n$? Well, $c_n = 2n$ satisfies the recurrence relation $c_n = c_{n-1} + 2$ for $n \geq 1$, so $C(x) = 0 + \sum_{n=1}^{\infty}(c_{n-1} + 2)x^n = xC(x) + \sum_{n=1}^{\infty} 2x^n = xC(x) + 2(\frac{1}{1-x} - 1)$, so $C(x)(1-x) = 2(\frac{1}{1-x} - 1)$, so $C(x) = 2\frac{1}{(1-x)^2} - 2\frac{1}{1-x}$.

Hence $B(x)(1-x) = 2\frac{1}{(1-x)^2} - 2\frac{1}{1-x} - (\frac{1}{1-x} - 1)$, so $B(x) = 2\frac{1}{(1-x)^3} - 3\frac{1}{(1-x)^2} + \frac{1}{1-x}$.

Hence $G(x)(1-x) = 2\frac{1}{(1-x)^3} - 3\frac{1}{(1-x)^2} + \frac{1}{1-x}$, so $G(x) = 2\frac{1}{(1-x)^4} - 3\frac{1}{(1-x)^3} + \frac{1}{(1-x)^2} = \frac{2-3(1-x)+(1-x)^2}{(1-x)^4} = \frac{x^2+x}{(1-x)^4}$, as desired.

Alternate solution: repeated clever applications of equation 9 of Table 1.

b. Using equation 9 of Table 1, we obtain that $\frac{x^2+x}{(1-x)^4} = (x^2 + x)\sum_{k=0}^{\infty} C(3+k, k)x^k = \sum_{k=0}^{\infty} C(3+k, k)x^{k+2} + \sum_{k=0}^{\infty} C(3+k, k)x^{k+1} = \sum_{k=2}^{\infty} C(1+k, k-2)x^k + \sum_{k=1}^{\infty} C(2+k, k-1)x^k = x + \sum_{k=2}^{\infty}(C(1+k, k-2) + C(2+k, k-1))x^k$, so $a_1 = 1$ and $a_k = C(1+k, k-2) + C(2+k, k-1) = C(1+k, 3) + C(2+k, 3) = (k+1)(k)(k-1)/3! + (k+2)(k+1)(k)/3! = k(k+1)(2k+1)/6$ for $k \geq 2$. (Note that this formula also works for $k = 1$).

58. Note: this is the third time we've done this problem during this course.
a. We know from experience that $p(X(s) = k) = q^{k-1}p$, where $q = 1 - p$ and $k \geq 1$. Hence $G_X(x) = \sum_{k=1}^{\infty} q^{k-1}px^k = px\sum_{k=0}^{\infty} q^kx^k = \frac{px}{1-qx}$.
b. By Exercise 57b, we get that $E(X) = (\frac{px}{1-qx})'(1)$. This is $\frac{p(1-qx)-px(-q)}{(1-qx)^2}$ evaluated at $x = 1$, which is $\frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$.

By Exercise 57c, we get that $V(X) = (\frac{px}{1-qx})''(1) + E(X) - (E(X))^2$. The double derivative yields $\frac{2pq}{(1-qx)^3}$; evaluated at $x = 1$, we get $\frac{2pq}{(1-q)^3} = \frac{2pq}{p^3} = \frac{2q}{p^2}$. Hence $V(X) = \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{q}{p^2}$.

Section 5.5

2. By a very straightforward application of Inclusion-Exclusion, we obtain $|C \cup D| = |C| + |D| - |C \cap D| = 345 + 212 - 188 = 369$.

6. a. Here $A_1 \cup A_2 \cup A_3 = A_3$, so the answer is 10000.
b. Here all intersections are empty , so $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| = 11100$.
c. Here, $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3 = 100 + 1000 + 10000 - 2 - 2 - 2 + 1 = 11095$.

10. Here it's easier to look at the complement of the set in question: the set of integers not exceeding 100 that ARE divisible by either 5 or 7 has cardinality $\lfloor 100/5 \rfloor + \lfloor 100/7 \rfloor - \lfloor 100/35 \rfloor = 20 + 14 - 2 = 32$, so the set we want has size $100 - 32 = 68$.

12. Here, we want $\lfloor \sqrt{1000} \rfloor + \lfloor \sqrt[3]{1000} \rfloor - \lfloor \sqrt[6]{1000} \rfloor = 31 + 10 - 3 = 38$.

14. Again, it's easier to look at the complement; how many permutations of the alphabet contain either fish, rat, or bird? Well, $|F \cup R \cup B| = |F| + |R| + |B| - |F \cap R| - |F \cap B| - |R \cap B| + |F \cap R \cap B| = 23! + 24! + 23! - 21! - 0 - 0 + 0$. (Note 1: to find F, for example, glue the letters F,I,S,H together; now we have 23 total "letters", which can be arranged in 23! ways. Note 2: no permutation can contain both fish and bird, since they share an i, and no permutation can contain both bird and rat, since they share an r). So the answer we want is 26! - 23! - 24! - 23! + 21!, which is around $4 \cdot 10^{26}$.

24. Let E_1, E_2, E_3 be the three given events. $p(E_1) = C(5,3)/32 = 10/32, p(E_2) = 1/4 = 8/32, p(E_3) = 1/4 = 8/32, p(E_1 \cap E_2) = C(3,1)/32 = 3/32, p(E_1 \cap E_3) = 1/32, p(E_2 \cap E_3) = 2/32, p(E_1 \cap E_2 \cap E_3) = 1/32$, so by inclusion-exclusion, $p(E_1 \cup E_2 \cup E_3) = 21/32$.

Section 5.6

4. Following Example 1 slavishly, we find the answer to be $C(4+17-1,17) - C(4+13-1,13) - C(4+12-1,12) - C(4+11-1,11) - C(4+8-1,8) + C(4+8-1,8) + C(4+7-1,7) + C(4+4-1,4) + C(4+6-1,6) + C(4+3-1,3) + C(4+2-1,2) - C(4+2-1,2) - 0 - 0 - 0 + 0 = 20$, for those among you who are patient.

6. Here our properties are "divisible by 4", "divisible by 9", "divisible by 25" and "divisible by 49" (note we only have to consider squares of primes). N, of course, is 99. Thus our answer is $99 - \lfloor 99/4 \rfloor - \lfloor 99/9 \rfloor - \lfloor 99/25 \rfloor - \lfloor 99/49 \rfloor + \lfloor 99/36 \rfloor + 0 + 0 + 0 + 0 - 0 = 61$.

8. Using Theorem 1, we immediately obtain the answer to be $5^7 - C(5,1)4^7 + C(5,2)3^7 - C(5,3)2^7 + C(5,4)1^7 = 16800$.