Math 55: Midterm #2, 2 April 1998

Problem 1: (20 points)
(a) How many different strings can be made using all the ten letters in the word GOOGOLPLEX?
(b) How many start with G?

Solution: **(a)** There are ten letters, so there are 10! ways to arrange them. However, interchanging the two G's or the three O's or the two L's does not change the resulting string, so we must divide 10! by the permutations of identical letters: thus the answer is

$$\frac{10!}{3!2!2!} = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 151200.$$

(b) If we make the first letter G, then we have nine letters left to rearrange, and there are fewer duplicate letters: thus the answer is

$$\frac{9!}{3!2!} = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 2 = 30240.$$

Problem 2: (20 points) Consider the sequence a_n defined by the initial conditions

$$a_0 = 0, \qquad a_1 = 1$$

and the recurrence relation

$$a_n = 3a_{n-1} - 2a_{n-2}, \qquad n \geq 2.$$

(a) Write down the characteristic equation.
(b) Find its roots p and q.
(c) Find the coefficients t and s of the linear combination

$$a_n = tp^n + sq^n$$

which satisfies both the recurrence relation and the initial conditions.

Solution: **(a)** The characteristic equation is found by seeking solutions of the form $a_n = r^n$. It is

$$r^2 - 3r + 2 = 0.$$

(b) The roots are given by the quadratic formula:

$$p, q = \frac{3 \pm \sqrt{9-8}}{2} = 2, 1.$$

(c) Since p and q are roots of the characteristic equation, we know the recurrence relation is automatically satisfied (if we did the arithmetic right). The initial conditions give us two equations in two unknowns:

$$\begin{aligned} t + s &= 0 \\ 2t + s &= 1. \end{aligned}$$

Solving, we get $t = 1$ and $s = -1$. Hence $a_n = 2^n - 1$.

Problem 3: (20 points) Roll a six-sided die with faces numbered 1 through 6 and a twelve-sided die with faces numbered 1 through 12. Assume each die is fair. Find the expected sum of the numbers rolled. Justify your answer.

Solution: Since expectation is linear, we can simply add the expectation of each die separately. For the six-sided die, the expected value is

$$\frac{1}{6}(1+2+3+4+5+6) = 3.5$$

while for the twelve-side die the expected value is

$$\frac{1}{12}(1+2+3+4+5+6+7+8+9+10+11+12) = 6.5.$$

Adding, we get 10 for the expected sum.

Problem 4: (20 points) Choose a nonnegative integer solution (x_1, x_2, x_3) of the equation

$$x_1 + x_2 + x_3 = 42$$

at random, with equal probabilities. What is the probability that at least one of the x_i's is exactly equal to 7?

Solution: By stars and bars, the set S of nonnegative integer solutions to this equation has cardinality $|S| = \binom{44}{42} = \binom{44}{2}$. Each solution

(x_1, x_2, x_3) has equal probability $p = 1/|S|$ of being chosen. Let $A_i = \{x_i = 7\} \cap S$ be the set of solutions with x_i equal to 7, for $i = 1$ through 3. The probability that at least one of the x_i's is exactly 7 is

$$\frac{|A_1 \cup A_2 \cup A_3|}{|S|},$$

so we should use inclusion-exclusion. The cardinality of A_1 (and of A_2 and A_3, by symmetry) is the number of nonnegative integer solutions to

$$x_2 + x_3 = 35,$$

which is

$$|A_1| = \binom{36}{1}$$

by stars and bars. The cardinality of $A_1 \cap A_2$ (etc.) is the number of nonnegative integer solutions to

$$x_3 = 28,$$

which is 1. The cardinality of $A_1 \cap A_2 \cap A_3$ is 0. Thus by inclusion-exclusion,

$$\begin{aligned} |A_1 \cup A_2 \cup A_3| &= \sum_{1 \le i \le 3} |A_i| - \sum_{1 \le i < j \le 3} |A_i \cap A_j| + \sum_{1 \le i < j < k \le 3} |A_i \cap A_j \cap A_k| \\ &= \sum_{1 \le i \le 3} \binom{36}{1} - \sum_{1 \le i < j \le 3} 1 \\ &= \binom{3}{1} \cdot \binom{36}{1} - \binom{3}{1} \cdot 1 \end{aligned}$$

Thus the probability that some x_i is exactly equal to 7 is

$$\frac{\binom{3}{1} \cdot \binom{36}{1} - \binom{3}{1} \cdot 1}{\binom{44}{2}} = \frac{105}{946}.$$

Problem 5: (20 points) Consider the first 100 Fibonacci numbers

$$f_0, f_1, \ldots, f_{99} = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots, 218922995834555169026.$$

They satisfy the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

for $n \geq 2$.

(a) Show that there are at least 34 of them which have the same remainder **mod** 3.

(b) For which n is f_n even?

(c) Use strong induction to prove that your answer to **(b)** is correct.

Solution: **(a)** There are only three possible remainders **mod** 3, and they are 0, 1, and 2. Thus by the generalized pigeonhole principle, one of them must get $\lceil 100/3 \rceil = 34$ of the 100 Fibonacci numbers.

(b) From the first few shown above, we guess that f_n is even if and only if n is divisible by 3.

(c) Theorem: f_n is even if and only if $3|n$.

Base: The theorem holds for $n \leq 3$ by inspection.

Induction: Assume $n > 3$ and the theorem is true for all $k < n$. If $3|n$ then $f_n = f_{n-1} + f_{n-2}$ is the sum of two odd numbers, by the inductive hypothesis, hence even. Otherwise, f_n is the sum of an even number and an odd number, by the inductive hypothesis, hence odd.