## Math 55: Final Exam, 15 May 1998, 12:30-3:30 pm

**Problem 1:** Find the remainder of $2^{326}$ **(a)** mod 2, **(b)** mod 5, **(c)** mod 47. **(d)** State the Chinese Remainder Theorem and use it to compute $2^{326}$ mod 470.

**Solution:** **(a)** Since $2 \equiv 0 \pmod 2$, so is $2^{326}$.
**(b)** Since $2^4 = 16 \equiv 1 \pmod 5$, we can drop all multiples of 4 from the exponent, leaving just 2. Thus $2^{326} \equiv 2^2 \equiv 4 \pmod 5$.
**(c)** By Fermat's Little Theorem, $2^{46} \equiv 1 \pmod{47}$. Since $326 = 7 \cdot 46 + 4$, we find $2^{326} \equiv 2^4 \equiv 16 \pmod{47}$.
**(d)** By the Chinese Remainder Theorem, there is a unique solution $x$ of the congruences

$$\begin{aligned} x &\equiv 0 \pmod 2 \\ x &\equiv 4 \pmod 5 \\ x &\equiv 16 \pmod{47} \end{aligned}$$

in the range $0 \leq x < 470$. From parts (a-c) above, we also know that $x = 2^{326} \bmod 470$ satisfies these congruences. The solution is $x = 204$ by inspection, so $2^{326} \equiv 204 \pmod{470}$.

**Problem 2:** Evaluate the sum

$$\sum_{k=0}^{17} \binom{17}{k} \left(\frac{2}{3}\right)^k 3^{17}$$

mod 17.

**Solution:** By the binomial theorem, we have

$$\sum_{k=0}^{17} \binom{17}{k} 2^k 3^{17-k} = (2+3)^{17} = 5^{17},$$

and by Fermat's Little Theorem, we have

$$5^{17} \equiv 5 \pmod{17}.$$

**Problem 3:** **(a)** How many triples $(x_1, x_2, x_3)$ of nonnegative integers satisfy the equation $x_1 + x_2 + x_3 = 23$?
**(b)** How many triples $(x_1, x_2, x_3)$ of nonnegative integers satisfy the equation $x_1 + x_2 + x_3 = 23$ plus the constraints $x_1 < 10, x_2 < 10, x_3 < 10$?

**Solution:** **(a)** By stars and bars, this is the number of ways to separate 23 stars into three groups with two bars, which is $\binom{25}{2} = 300$.
**(b)** Define sets $A$ and $A_i$ by

$$\begin{aligned} A &= \{(x_1, x_2, x_3) | x_1 + x_2 + x_3 = 23\} \\ A_i &= A \cap \{(x_1, x_2, x_3) | x_i \geq 10\} \end{aligned}$$

for $i = 1, 2, 3$. Then the number of solution is given by the inclusion-exclusion formula

$$\begin{aligned} \left| A - \bigcup_{i=1}^{3} A_i \right| &= |A| - \left| \bigcup_{i=1}^{3} A_i \right| \\ &= |A| - \sum_{i=1}^{3} |A_i| + \sum_{1 \leq i < j \leq 3} |A_i \cap A_j| \\ &= \binom{25}{2} - \sum_{i=1}^{3} \binom{15}{2} + \sum_{1 \leq i < j \leq 3} \binom{5}{2} \\ &= \binom{25}{2} - \binom{3}{1} \cdot \binom{15}{2} + \binom{3}{2} \cdot \binom{5}{2} \\ &= 15. \end{aligned}$$

**Problem 4:** **(a)** Suppose $g_n$ is a given arbitrary sequence of integers and $W_n$, $E_n$ are two sequences satisfying $W_0 = E_0$ and

$$W_{n+1} \leq 2W_n + g_n, \qquad E_{n+1} = 2E_n + g_n \qquad \text{for } n \geq 0.$$

Prove that $W_n \leq E_n$ for $n \geq 0$.
**(b)** The worst-case complexity of merge-sorting $n$ records satisfies

$$W_n \leq W_{\lceil n/2 \rceil} + W_{\lfloor n/2 \rfloor} + n.$$

Prove that $W_n = O(n \log n)$ for $n$ a power of 2 and find explicit constants (from the definition of big-$O$).

**Solution:** **(a)** Subtraction of the equality from the inequality gives

$$W_{n+1} - E_{n+1} \leq 2(W_n - E_n) \qquad \text{for } n \geq 0,$$

which suggests a proof by induction.
**Base:** Clearly $W_0 - E_0 = 0 \leq 0$, so $W_0 \leq E_0$.
**Induction Step:** Suppose $W_k \leq E_k$ for $0 \leq k \leq n$. Then $W_{n+1} - E_{n+1} \leq 2(W_n - E_n) \leq 0$, so $W_{n+1} \leq E_{n+1}$.
**(b)** For $n$ a power of 2, the inequality becomes

$$W_n \leq 2W_{n/2} + n$$

so iterating gives

$$\begin{aligned} W_n &\leq 2W_{n/2} + n \\ &\leq 2(2W_{n/4} + n/2) + n = 4W_{n/4} + n + n \\ &\leq 4(2W_{n/8} + n/4) + 2n = 8W_{n/8} + 3n \\ &\leq 2^k W_{n/2^k} + kn = 2^k W_1 + kn \end{aligned}$$

where $n = 2^k$, $k = \log_2 n$. Thus $W_n = O(n \log n)$. To find the constants, we need

$$W_1 + \log n \leq C \log n$$

for $n \geq k$. This will hold, for example, if $C = W_1 + 1$ and $k = 3$.

**Problem 5:** Let $S$ be the set of bit strings of length less than or equal to 3:

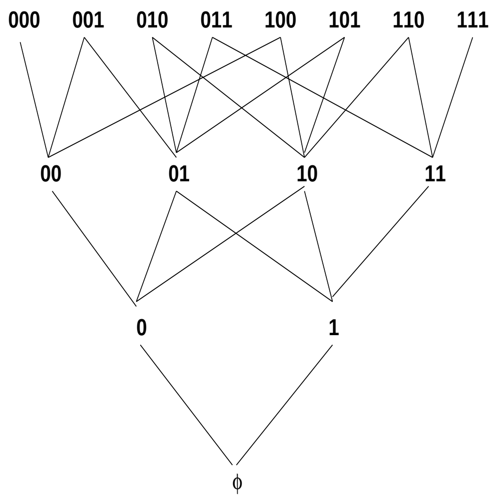
$$S = \{\phi, 0, 1, 00, 01, 10, 11, 000, 001, 010, 100, 011, 101, 110, 111\}.$$

**(a)** Define a relation $R$ on $S$ by letting $aRb$ whenever $a$ and $b$ have the same number of 1's. Prove from the definition of an equivalence relation that $R$ is an equivalence relation.
**(b)** Write down the partition of $S$ which corresponds to the equivalence relation $R$.
**(c)** Define another relation $\preceq$ on $S$ by letting $a \preceq b$ whenever $a$ is a substring of $b$. (Precise definition: $a$ is a substring of $b$ iff $b$ is obtained from $a$ by prefixing and postfixing $a$ with some nonnegative number of 0 and 1 bits.) Prove from the definition of a partial order that $\preceq$ is a partial order on $S$.
**(d)** Draw the Hasse diagram for $\preceq$ on $S$ and use it to list $S$ in a topologically sorted order.

**Solution:** **(a)**
**Reflexive:** Clearly $a$ and $a$ have the same number of 1's.
**Symmetric:** The definition is clearly symmetric.
**Transitive:** Suppose $aRb$ and $bRc$. Then $a$ has the same number $k$ of 1's as $b$, and $c$ has the same number of 1's as $b$, so $c$ has $k$ 1's as well. Hence $aRc$.
**(b)** The relation partitions $S$ as follows:

$$\{\{\phi, 0, 00, 000\}, \{1, 01, 10, 001, 010, 100\}, \{11, 011, 101, 110\}, \{111\}\}$$

**(c)**
**Reflexive:** Obvious since $a$ is a substring of $a$.
**Antisymmetric:** Suppose $a \preceq b$ and $b \preceq a$. Then by definition each is obtained from the other by prefixing and postfixing, so they must be equal.
**Transitive:** Suppose $a \preceq b$ and $b \preceq c$. Then by definition $a$ is a substring of $b$ which is a substring of $c$, so $a$ is a substring of $c$. Hence $a \preceq c$.
**(d)** The Hasse diagram for $\preceq$ on $S$ is:




A topologically sorted order on $S$ is found by pulling off a minimal element at each step; for example, one such order is

$$\phi, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111.$$

**Problem 6:** Let $S$ be the set of permutations of 10 objects.
**(a)** How many elements of $S$ leave the third object fixed?
**(b)** Consider the experiment of choosing an element of $S$ randomly with equal probabilities. Let $f$ be the random variable defined by $f(\sigma) = |\{j : \sigma(j) = j\}|$ , so $f(\sigma)$ is the number of fixed points of $\sigma$. Compute the expectation of $f$.

**Solution:**
**(a)** Permutations which fix the third object correspond precisely to permutations of the $n - 1 = 9$ objects numbered $1, 2, 4, 5, \ldots, 10$. Thus there are $9!$ of them.
**(b)** Let $n = 10$. For $j = 1$ to $n$ define a random variable $f_j$ by $f_j(\sigma) = 1$ if $\sigma$ fixes $j$ and $f_j(\sigma) = 0$ otherwise. Then the number of fixed points is $f = f_1 + f_2 + \ldots + f_n$. Since expectation is linear,

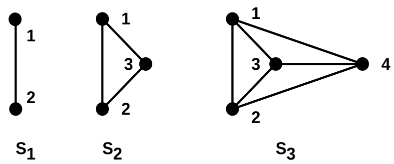
$$E(f) = E(f_1) + \ldots + E(f_n).$$

But precisely $(n-1)!$ of the permutations of $n$ objects fix the first object, as in (a), so

$$E(f_1) = \frac{(n-1)!}{n!} = \frac{1}{n}.$$

Since there was nothing special about the first object, $E(f_j) = 1/n$ for each $j$. Hence $E(f) = n \cdot 1/n = 1$.

**Problem 7:** The *simplex graph* $S_n = (V_n, E_n)$ in $n$ dimensions is defined recursively as follows. $S_1$ has two vertices and a single edge between them. Given $S_n = (V_n, E_n)$, we form $V_{n+1}$ by adding one more vertex $v$. The edge set $E_{n+1}$ is $E_n$ plus edges connecting $v$ to every vertex in $V_n$. The first few look like this:




**(a)** Find the number $v_n$ of vertices in $S_n$.
**(b)** Find the degree $d$ of each vertex in $S_n$.
**(c)** Use the Handshaking Theorem to find the number $e_n$ of edges in $S_n$.
**(d)** Derive a first-order inhomogeneous recurrence relation for the number $e_n$ of edges in $S_n$. Solve your relation by iteration and check against (c).

**Solution:** **(a)** Since $v_1 = 2$ and $v_{n+1} = v_n + 1$, we have $v_n = n + 1$.
**(b)** Since the new vertex $v$ in $S_{n+1}$ connects to $v_n = n + 1$ old vertices, it has degree $d = n + 1$. Since each old vertex had degree $n$ (by induction) and added one edge, each old vertex has degree $n + 1$ as well.
**(c)** Since we know the degree of each vertex and the number of vertices in $S_n$, the Handshaking Theorem

$$\sum_{v \in V} \deg(v) = 2|E|$$

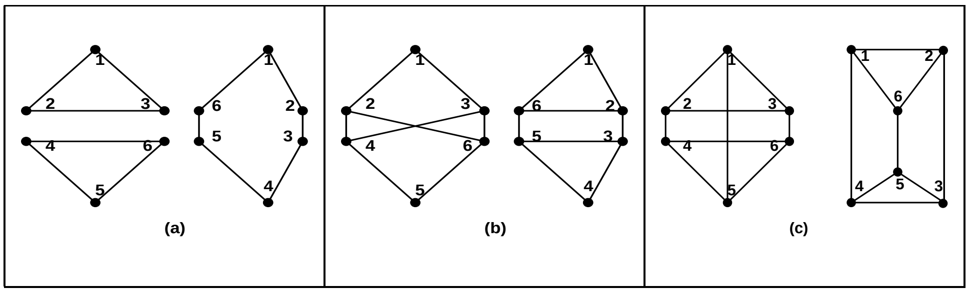
immediately implies

$$v_n \cdot d = (n+1)n = 2e_n$$

and $e_n = n(n+1)/2$. This checks for $n \leq 3$.
**(d)** We obtain $S_{n+1}$ from $S_n$ by adding $v_n$ new edges, so

$$\begin{aligned} e_{n+1} &= e_n + n + 1 \\ &= e_{n-1} + (n+1) + n \\ &= \ldots \\ &= e_0 + (n+1) + n + \ldots + 1 \\ &= n(n+1)/2. \end{aligned}$$

**Problem 8:** For each of the following pairs of graphs, either specify an isomorphism between the left and the right graph, or state why there cannot be one.

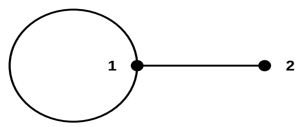



**Solution:** **(a)** These graphs are not isomorphic because the left contains a simple circuit of length 3 and the right does not.

**(b)** These graphs are not isomorphic because the right contains a simple circuit of length 3 and the left does not.

**(c)** These graphs are isomorphic: Take the interior vertices 5 and 6 of the right graph and move them out to the top and bottom. Thus the isomorphism takes $1 \to 6$, $2 \to 1$, $3 \to 2$, $4 \to 4$, $5 \to 5$ and $6 \to 3$.

**Problem 9:** Consider the following undirected pseudograph $G$:




**(a)** Write down the adjacency matrix $A$ of $G$.
**(b)** Find all paths of length 3 from 1 to 2; for example, one is $(1,1,1,2)$, where the path loops twice at 1 then goes from 1 to 2.
**(c)** Use the adjacency matrix $A$ to express the number of paths from 1 to 2 of any specified length $n \geq 1$ in terms of the Fibonacci numbers (defined by $f_0 = 0$, $f_1 = 1$, $f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$).

**Solution:** (a)

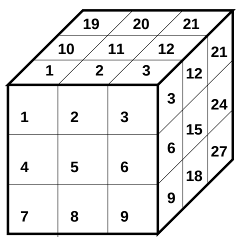
$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

**(b)** There are two paths of length 3 from 1 to 2: one is $(1,1,1,2)$, where the path loops twice at 1 then goes from 1 to 2. The other is $(1,2,1,2)$, where the path goes from 1 to 2, back again, and then from 1 to 2 again..
**(c)** Calculating the first few powers of $A$ gives

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 8 & 5 \\ 5 & 3 \end{bmatrix}, \ldots, A^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}.$$

Hence the number of paths of length $n$ from 1 to 2 is $f_n$.

**Problem 10:** A 3 by 3 by 3 block of cheese is cut into 27 1 by 1 by 1 cubes as shown. Mickey Mouse eats one cube on the first day, starting wherever he likes. On each subsequent day, Mickey eats another cube adjacent to the previous day's cube (adjacent means sharing a face, not a corner or edge). Can Mickey arrange his path so that he eats the center cube on the 27th day? Justify your answer. (Hint: the graph is bipartite.)




**Solution:** Number the cubes starting at a corner, as shown. Consider the graph with 27 vertices, one per cubes, where vertices are connected when they share a face. Then the question asks whether there is a Hamilton path beginning from any point and ending at the center vertex, number 14. Following the hint, we observe that the graph is bipartite. In other words, vertices are numbered odd and even in such a way that two vertices can be adjacent only if one is odd and the other even. Hence any path must contain vertices alternating between odd and even. There are altogether 14 odd and 13 even vertices, so if we end with the even vertex numbered 14, we cannot alternate parity at each step as required. Thus the mouse cannot eat the center cube last.