Math 55: Final Exam Answers, Fall 1998

Problem 1: Since $2250 - 1 = 2 \cdot 3^2 \cdot 5^3 - 1$, it is not divisible by 2, 3 or 5. Since $\gcd(2250, 7) = 1$, Fermat's Little Theorem shows that $2250^6 \equiv 1 \pmod 7$, so $2250^{666} - 1 \equiv 0 \pmod 7$ and $\boxed{\textbf{(c)}}$ is the answer.

Problem 2: By stars and bars, the number of nonnegative integer solutions to

$$x_1 + \cdots + x_p = m$$

is $\binom{m+p-1}{p-1}$, and we want to sum this from $m = 0$ to $M = 9$ for $p = 6$. Looking at the $p-1$st column of Pascal's triangle shows us that

$$\sum_{m=0}^{M} \binom{m+p-1}{p-1} = \binom{M+p}{p},$$

so the answer is $\binom{15}{6} = \binom{15}{9}$ which is $\boxed{\textbf{(d)}}$.

Problem 3: There are $\binom{52}{4}$ hands with $\binom{39}{4}$ having no clubs. Hence a correct answer is $\boxed{\textbf{(e)}}$. Answer $\boxed{\textbf{(a)}}$ is also correct. Answer **(b)** overcounts.

Problem 4: Let E be the event that some box contains exactly three balls and F be the event that the first two balls are in different boxes, so $|F| = 4 \cdot 3 \cdot 4 \cdot 4$. If some box contains three balls then another box has precisely one, so an element of $E \cap F$ is determined by choosing one of four boxes to put three balls in, one of the remaining three to put one ball in, and which of the two chosen boxes contains the first ball. Thus $|E \cap F| = 4 \cdot 3 \cdot 2$ and

$$P(E|F) = \frac{|E \cap F|}{|F|} = \frac{4 \cdot 3 \cdot 2}{4 \cdot 3 \cdot 4 \cdot 4} = \frac{1}{8}$$

which is $\boxed{\textbf{(b)}}$.

Problem 5: There are $n!$ permutations of $1, 2, 3, \ldots, n$ and $(n-2)!$ permutations of $123, 4, 5, 6, \ldots, n$, so the answer is $\boxed{\textbf{(b)}}$.

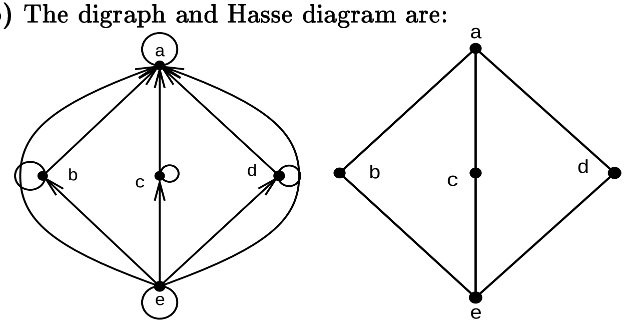
Problem 6: (a) The easy way to list equivalence relations is to list partitions into equivalence classes:

$$a = \{\{1,2,3\}\}, b = \{\{1,2\},\{3\}\}, c = \{\{1,3\},\{2\}\}, d = \{\{2,3\},\{1\}\}, e = \{\{1\},\{2\},\{3\}\}.$$

The corresponding relations are

$$\begin{aligned}
&\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\},\\
&\{(1,1),(1,2),(2,1),(2,2),(3,3)\},\\
&\{(1,1),(1,3),(2,2),(3,1),(3,3)\},\\
&\{(1,1),(2,2),(2,3),(3,2),(3,3)\},\\
&\{(1,1),(2,2),(3,3)\}.
\end{aligned}$$

(b) The digraph and Hasse diagram are:



(c) A topologically sorted order for E is e, b, c, d, a.

Problem 7: **(a)** Subtracting the number of vertices of C_n from the number of vertices of the complete graph K_n gives $|\bar{E}| = n(n-1)/2 - n = n(n-3)/2$.
(b) Since $n(n-3)/2 \neq n$ for $n \neq 5$, C_n and $\bar{C}_n$ have different numbers of edges and hence cannot be isomorphic. For $n = 5$, they are isomorphic: the map $1 \to 1$, $2 \to 4$, $3 \to 2$, $4 \to 5$, $5 \to 3$ preserves edges.
(c) By symmetry and the handshaking theorem, all vertices of $\bar{C}_n$ have the same degree $\deg(v) = 2 \cdot n(n-3)/2n = (n-3)$, which is even iff n is odd. By Euler's theorem, $\bar{C}_n$ has an Euler circuit iff n is odd.
(d) The degree $n-3$ of each vertex in $\bar{C}_n$ equals or exceeds $n/2$ whenever $n \geq 6$, so Dirac's theorem guarantees that $\bar{C}_n$ has a Hamilton circuit if $n \geq 6$. For $n \leq 4$, $\bar{C}_n$ is disconnected so a hamilton circuit cannot exist, while for $n = 5$ a Hamilton circuit —such as 13524— can be found directly. Hence a Hamilton circuit exists iff $n \geq 5$.

Problem 8: **(a)** A partial order is a relation which is reflexive, antisymmetric and transitive. R is a partial order because for any a, b and $c \in \mathbf{Z}_n$, $a|a$, $a|b \wedge b|a \to a = b$ and $a|b \wedge b|c \to a|c$.
(b) The matrix M has $M_{ij} = 1$ whenever $i|j$, so summing the entries in column j counts exactly $d(j)$ divisors i of $j \in \mathbf{Z}_n$.
(c) A prime p has only two divisors, 1 and p, so $d(p) = 2$. There are $k+1$ powers of two dividing 2^k, namely $1 = 2^0, 2^1, 2^2, \ldots, 2^k$, so $d(2^k) = k+1$.
(d) Plugging **(b)** into the definition of expectation and swapping sums gives

$$E(d) = \frac{1}{n}\sum_{j=1}^{n} d(j) = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{n} M_{ij} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij} = \frac{1}{n}\sum_{i=1}^{n}\lfloor n/i \rfloor \leq \frac{1}{n}\sum_{i=1}^{n} n/i \leq \sum_{i=1}^{n} 1/i.$$

since row i of M contains $\lfloor n/i \rfloor$ 1s.

Problem 9: **(a)** Following through the steps, $S(38, 14)$ returns 2.
(b) S computes $\gcd(a, b)$: First, we observe that S terminates because two passes through the while loop reduce the integer ab by at least a factor of two. Furthermore, $\gcd(ac, bc)$ is a loop invariant. It is clearly preserved by the first three cases of the if-test. The case when a and b are both odd works because

$$\gcd(ac, bc) = \gcd(|a-b|c, \min(a, b)c) = \gcd(ac - bc, bc)$$

if $a \geq b$, and similarly for $a < b$. Since $ac = a$ and $bc = b$ initially, $\gcd(a, b) = \gcd(ac, bc) = ac$ finally.
(c) Since ab is reduced by at least a factor of 2 every two steps, the worst-case complexity of $S(a, b)$ is $O(\log ab) = O(\log a + \log b) = O(\log \max(a, b))$.

Problem 10: **(a)** The adjacency matrix is

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}.$$

(b) There are 10 paths: `1c1c1a2`, `1c1c1b2`, `1a2a1a2`, `1a2a1b2`, `1a2b1a2`, `1a2b1b2`, `1b2a1a2`, `1b2a1b2`, `1b2b1a2`, `1b2b1b2`.
(c) Since powers of the adjacency matrix count paths,

$$A^n = \begin{bmatrix} a_n & b_n \\ b_n & c_n \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} a_{n-1} & b_{n-1} \\ b_{n-1} & c_{n-1} \end{bmatrix} = \begin{bmatrix} a_{n-1} + 2b_{n-1} & b_{n-1} + 2c_{n-1} \\ 2a_{n-1} & 2b_{n-1} \end{bmatrix}$$

gives

$$b_n = b_{n-1} + 2c_{n-1} = b_{n-1} + 4b_{n-2}.$$

(d) $b_3 = b_2 + 2b_1 = 2 + 4 \cdot 2 = 10$ checks.
(e) The characteristic equation and its roots are

$$r^2 - r - 4 = 0, \qquad r_\pm = \frac{1 \pm \sqrt{17}}{2}.$$

The solution has the form $b_n = sr_+^n + tr_-^n$ with initial conditions $b_1 = b_2 = 2$ giving

$$b_n = \frac{2}{\sqrt{17}}\left(\frac{1+\sqrt{17}}{2}\right)^n - \frac{2}{\sqrt{17}}\left(\frac{1-\sqrt{17}}{2}\right)^n$$

which checks.