Math 55: Practice Final Exam Problems, Fall 1996

Problem 1: Use the procedure of the Chinese Remainder Theorem to find an integer x with $0 \leq x \leq 1000$ such that

$$x \bmod 7 = 1$$

$$x \bmod 11 = 3$$

$$x \bmod 13 = 1.$$

Show all steps in the calculation.

Problem 2: A high school has n students and n lockers. At 8 am all the lockers are closed. Student 1 opens them all, student 2 closes the even numbered ones. Student three changes the state of lockers 3, 6, 9, ...: he opens them if they are closed and closes them if they are open. Student j changes the state of lockers j, $2j$, etc, for $j = 4, 5, ..., n$. How many lockers are left closed after all n students have had their way?

Problem 3: A computer network consists of n computers, each one directly connected to zero or more of the others. Prove or disprove: There are at least two computers in the network that are directly connected to the same number of other computers.

Problem 4: For each of the following, either give an example or prove that there are none:
(a) A simple graph with 6 vertices, whose degrees are 2, 2, 2, 3, 4, 4.
(b) A simple graph with 8 vertices, whose degrees are 0, 1, 2, 3, 4, 5, 6, 7.
(c) A simple graph with 7 vertices that has a Hamilton circuit but no Euler circuit.
(d) A simple graph with 6 vertices that has a Euler circuit but no Hamilton circuit.
(e) A simple graph with 6 vertices and 16 edges.

Problem 5: Find the coefficient a_{-2} of x^{-2} in the binomial expansion of

$$(x + 2/x)^{12} = \sum_{k=-12}^{12} .$$

Problem 6: Find the number of triples of integers (x, y, z) satisfying

$$x + y + z = 32$$

(a) where x, y and z are nonnegative
(b) where $x \geq 7$, $y \geq 15$ and $z \geq 0$.

Problem 7: Let S be the set of 5-digit decimal integers like 03214 or 30789. Pick an element x of S at random.
(a) What is the probability that x begins with 7575?
(b) What is the probability x contains no 4s?

Problem 8: Pick a string of 5 bits at random. Let E_1 be the event that the string ends in 1, E_2 that it begins with 1, E_3 the event that it contains exactly three 1s.
(a) Find $p(E_1|E_3)$.
(b) Find $p(E_3|E_2)$.
(c) Find $p(E_2|E_3)$.
(d) Find $p(E_3|E_1 \cap E_2)$.
(e) Are E_1 and E_2 independent?
(f) Are E_2 and E_3 independent?

Problem 9: Show that $12! \bmod 13 = -1$.

Problem 10: Define a sequence a_n recursively by $a_1 = 5$, $a_2 = 10$, and

$$a_n = a_{n-1} + a_{n-2} + 125$$

for $n \geq 3$. Prove that 5 divides a_n for all $n \geq 1$.

Problem 11: Prove or disprove:
(a) $\forall a, b \in \mathbf{Z} : a \bmod 5 = b \bmod 5 \quad \rightarrow \quad a^2 \bmod 5 = b^2 \bmod 5$
(b) $\forall a, b \in \mathbf{Z} : a^2 \bmod 5 = b^2 \bmod 5 \quad \rightarrow \quad a \bmod 5 = b \bmod 5$

Problem 12: How many different strings can be made by reordering the letters of the words
(a) ALGORITHM
(b) DISCRETE
(c) NONSENSE

Problem 13: Suppose $k \geq 1$ is an odd integer. Prove $k^4 \bmod 16 = 1$.

Problem 14: Does there exist a connected simple graph with an even number of vertices and an odd number of edges which has an Euler circuit? Exhibit one or prove that no such graph can exist.

Problem 15:
(a) Determine the largest integer n such that 13^n divides $100!$.
(b) Determine the largest integer n such that 10^n divides $100!$.

Problem 16: Consider the following pseudocode:

```
function F (n in Z, A = (a1, ..., an), m in Z,  B = (b1, ..., bm), f: A --> B)
do i := 1, m
      s := 0
      do j := 1, n
            if(f(aj) = bi) s := s + 1
      end do
      if(s > 1) return F := False
end do
return F := True
```

(a) What does it compute?
(b) What is its worst-case complexity in terms of `m` and `n` in big-O notation?
(c) Modify the pseudocode to work correctly even if `m` is infinite and `n` is finite.

Problem 17: Prove or disprove that the following pairs of graphs are isomorphic. Find a Hamilton circuit and an Euler circuit if they exist, or explain why they cannot.
(a)

(b)

Problem 18: Let A be uncountable and $A \subseteq B$. Prove B is uncountable.

Problem 19:
(a) List all possible partitions of the set $\{a, b, c\}$.
(b) Define a relation between partitions P_1 and P_2 of $\{a, b, c\}$ by $P_1 R P_2$ iff P_1 is a refinement of P_2, in other words every set in P_1 is a subset of some set in P_2. Show that R is a partial order on the set of partitions of $\{a, b, c\}$.
(c) Draw the Hasse diagram for R.

Problem 20: Prove that 4 divides $9^n - 5^n$ for every integer $n \geq 0$.

Problem 21:
(a) Let $A \subseteq B$. Show $P(A) \subseteq P(B)$.
(b) Suppose $P(A) \subseteq P(B)$. Must it be true that $A \subseteq B$?

Problem 22: Let A and B be subsets of a finite set U, and suppose that all the elements of the following set are distinct:

$$X = \{A, B, A \cup B, A \cap B, U, \emptyset, A - B, B - A, A \oplus B\}.$$

Construct the Hasse diagram for the poset $P = (X, \subseteq)$ and sort P topologically. Include only relations which must be true for every such A and B, for example $\phi \subseteq A$.

Problem 23: Prove or disprove: Whenever A and B are sets then $A \cap (A \cup B) = A$.

Problem 24: Suppose f and $f \circ g$ are one to one. Prove or disprove: g must be one to one. Repeat with "onto" in place of "one to one".

Problem 25: For each of the following relations, give an example or prove there are none:
(a) On the set $\{1, 2, 3\}$, a reflexive transitive relation which is not antisymmetric.
(a) On the set $\{1, 2\}$, a symmetric transitive relation which is not reflexive.
(a) On the set $\{1, 2, 3\}$, a reflexive transitive relation which is not symmetric.

Problem 26: Find functions $f : \mathbf{Z} \quad \rightarrow \quad \mathbf{Z}$ which are
(a) one to one but not onto,
(b) onto but not one to one,
(c) one to one and onto but not the identity function,
(d) neither one to one nor onto.

Problem 27:
(a) Define what it means for a set A to be countable.
(b) Use your definition to show $\mathbf{N} \times \mathbf{N}$ is countable.
(c) Show that there are only countably many correct computer programs.
(d) Are the real numbers countable? Why or why not?
(e) Is the set of functions $f : \mathbf{N} \quad \rightarrow \quad \mathbf{B}$ countable, where $B = \{0, 1\}$?

Problem 28: Prove that $(p \wedge q) \quad \rightarrow \quad p$ is a tautology.

Problem 29: Suppose A and B are subsets of $\mathbf{R}$ and $f : \quad \rightarrow \quad B$ is strictly increasing, in other words $x > y \quad \rightarrow \quad f(x) > f(y)$.
(a) Prove that f is one to one.
(b) Suppose another function $g : B \quad \rightarrow \quad C \subseteq \mathbf{R}$ is also strictly increasing. Prove that $g \circ f$ is one to one.
(c) Must **(a)** be true if A and B are arbitrary partially ordered sets?

Problem 30: Determine whether the given relation R on the given set A is reflexive, symmetric, antisymmetric, transitive:
(a) $A = \mathbf{N}$, $aRb \quad \leftrightarrow \quad a|b$.
(b) $A = \mathbf{Z}$, $aRb \quad \leftrightarrow \quad |a - b| \leq 1$.
(c) $A = \mathbf{R}$, $aRb \quad \leftrightarrow \quad a^2 = b^2$.
(d) $A = \mathbf{R}$, $aRb \quad \leftrightarrow \quad a \neq b$.
(e) $A = \mathbf{Z}$, $aRb \quad \leftrightarrow \quad a \bmod 10 = b \bmod 10$.
(f) $A = P(\{1, 2, 3, 4\})$, $SRT \quad \leftrightarrow \quad S \subseteq T$.
(g) $A = \mathbf{R}$, $aRb \quad \leftrightarrow \quad a - b \in \mathbf{Z}$.

Problem 31: Find the matrix and the digraph for each of the following relations:
(a) $A = \{x, y, z\}$, $R = \{(x, x), (y, z), (z, y)\}$
(b) $A = \{a, b, c, d\}$, $R = \{(a, a), (a, b), (b, a), (b, b), (b, d), (c, c), (d, c), (d, d)\}$
(c) $A = P(\{1, 2, 3\})$, $SRT \quad \leftrightarrow \quad S \subseteq T$.

Problem 32: Show that

$$\sum_{k=1}^{n^2} k^4 = O(n^{10}).$$

Problem 33: How many positive integers less than 10000 are not the second or higher power of an integer?

Problem 34: Suppose $a \equiv b \bmod q$ and $c \equiv d \bmod q$. Show that $ac \equiv bd \bmod q$.

Problem 35:
(a) Use the Euclidean algorithm to compute $\gcd(89, 55)$.
(b) Use your work from **(a)** to find integers s and t such that $\gcd(89, 55) = s \cdot 89 + t \cdot 55$.
(c) Let F_n be the nth Fibonacci number defined by $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Give a big-O formula in terms of n for the number of steps required by the Euclidean algorithm to compute $\gcd(F_{n+1}, F_n)$. Explain why your formula is correct.

Problem 36: Let $A = \{2, 3, 4, , 510, 12, 16\}$. Define a partial order on A by $a \preceq b \quad \leftrightarrow \quad a|b$. Draw and label the Hasse diagram for $(A, \preceq)$. Find all minimal elements. Sort A topologically.

Problem 37: Define a relation on the set of bit strings $b_1b_2b_3b_4$ by declaring two strings to be related if they have the same number of 0s. Find the equivalence class of the string 1010.

Problem 38: Suppose x is an integer with $0 \leq x \leq 1000$ and

$$x \bmod 7 = 3, \qquad x \bmod 11 = 5, \qquad x \bmod 13 = 7.$$

(a) Is x uniquely determined by this information? Why or why not?
(b) Calculate $x^2 \bmod 7$ and $x^3 \bmod 11$.

Problem 39: Use mathematical induction to show that 8 divides $n^4 - 1$ whenever n is an odd positive integer.

Problem 40: Find the smallest equivalence relation R on the set $\{1, 2, 3\}$ which has $1R2$ and $2R3$.

Problem 41: Is $ab \leq 0$ an equivalence relation on $\mathbf{Z}$?

Problem 42: How many positive integers less than 10000 have the sum of their decimal digits equal to 5?

Problem 43: Suppose p and q are prime numbers and $n = pq$. What is the probability that a randomly chosen positive integer less than n is *not* divisible by either p or q?

Problem 44: How many positive integers less than or equal to 1000000 are either odd or the square of an integer?

Problem 45: Consider the experiment of rolling twelve dice once.
(a) What is the sample space S?
(b) Let $f : S \quad \rightarrow \quad \mathbf{Z}$ be the random variable which gives the sum of the twelve dice. Find $E(f)$ and $\sigma(f)$.
(c) State Chebyshev's inequality.
(d) Prove that there is a 75% probability that f is between 30 and 54 inclusive.

Problem 46: Suppose p and q are distinct primes. Use inclusion-exclusion to find the number of integers less than or equal to pq which are relatively prime to pq.

Problem 47:
(a) How many integers between 1 and 1000 inclusive are divisible by either 4 or 29?
(b) How many integers between 1 and 1000 inclusive are divisible by either 9 or 12?

Problem 48: How many permutations of the 26 letters abcdefghijklmnopqrstuvwxyz do not contain any of the strings *fish*, *rat* or *bird*? How about (dog, big, oil), (cart, show, like), or (sword, plant, carts)?

Problem 49: How many ways are there to put 17 objects into 5 boxes if box 1 can hold at most 3, box 2 at most 4, box 3 at most 6, box 4 at most 2 and box 5 at most 1?

Problem 50: Use inclusion-exclusion to find the number of primes less than 120.