Math 55: Midterm #2, 3 March 2008

Write your name, your student ID number, your section time and number, and a three-problem grading grid, on the cover of your blue book. Hand in your exam book at 11:00 am. Books, notes, calculators, scratch paper and/or collaboration are not allowed. Justify your computations: correct answers with inadequate explanation may receive partial credit.

1: **(a)** This is the number of onto functions from an 8-element set to a 7-element set, divided by the total number of functions. It can be directly evaluated by choosing a squirrel to receive two pecans, choosing two pecans for that squirrel to get, and permuting the other 6 pecans/squirrels:

$$p = \frac{7\binom{8}{2}6!}{7^8}.$$

A common wrong answer is

$$\binom{7}{1} / \binom{8+7-1}{8},$$

which is wrong because the final configurations don't have equal probabilities.
(b) Let f_i be the indicator of the event F_i that the ith squirrel gets a pecan, so we are asked for the expectation of $f = f_1 + \cdots + f_7$. By symmetry, this is $7E(f_1) = 7p(F_1) = 7(1 - p(\bar{F}_1))$. Since there are 6^8 ways for the first squirrel not to get a pecan,

$$E(f) = 7E(f_1) = 7(1 - (6/7)^8).$$

(c) This is the expected number of successes in $n = 8$ Bernoulli trials with success probability $p = 1/7$, which is $np = 8/7$.
(d) This is the variance of the number of successes in $n = 8$ Bernoulli trials with success probability $p = 1/7$, which is $npq = 48/49$.

2: **(a)** A non-Martian student gets a perfect score with probability $(2/3)^4$.

(b) Let P be the event of a perfect score, M the event that the student is Martian, N the event that the student is not Martian. Since N and M partition the sample space into a disjoint union,

$$p(P) = p(P \cap N) + p(P \cap M) = p(P|N)p(N) + p(P|M)p(M)$$

by definition of conditional probability. Since the conditionals are given, we have

$$p(P) = (2/3)^4 \cdot (81/82) + 1 \cdot (1/82) = 17/82.$$

(c) By (b) and the definition of conditional probability,

$$p(M|P) = \frac{p(M \cap P)}{p(P)} = \frac{p(P|M)p(M)}{p(P)} = \frac{1 \cdot 1/82}{17/82} = \frac{1}{17}.$$

4:

(a) The inclusion-exclusion formula reads

$$|P \cup Q \cup R| = |P| + |Q| + |R| - |P \cap Q| - |P \cap R| - |Q \cap R| + |P \cap Q \cap R|.$$

(b) Define

$$P = \{1 \leq n \leq pqr \mid p|n\}$$

and similarly for Q and R. Then $|P| = \lfloor pqr/p \rfloor = qr$, while $|Q| = pr$ and $|R| = pq$. Similarly $|P \cap Q| = \lfloor pqr/pq \rfloor = r$ since p, q and r are all distinct primes, and $|P \cap R| = q$, $|Q \cap R| = p$. Finally $|P \cap Q \cap R| = 1$ so we can apply inclusion-exclusion:

$$\begin{aligned} |\{1 \leq n \leq pqr \mid \gcd(n, pqr) = 1\}| &= pqr - |\{1 \leq n \leq pqr \mid p|n \vee q|n \vee r|n\}| \\ &= pqr - |P \cup Q \cup R| \\ &= pqr - qr - pr - pq + r + q + p - 1. \end{aligned}$$

(c) Putting $p = 2$, $q = 3$, $r = 5$ gives 8 numbers 1, 7, 11, 13, 17, 19, 23, 29 between 1 and $pqr = 30$ inclusive and relatively prime to 30. Our formula gives

$$pqr - qr - pr - pq + r + q + p - 1 = 30 - 15 - 10 - 6 + 5 + 3 + 2 - 1 = 8.$$

5:

(a) Since F is all of S except for 11, 33, 55, 77, 99, it has cardinality 45 and $p(F) = 45/50 = 9/10 = 0.9$.

(b) Since expectation is linear, $E(f) = E(d_2) + E(d_1)$. Since the digits are equally probable, $E(d_2) = (0 + 1 + 2 + \cdots + 9)/10 = 4.5$ and $E(d_1) = (1 + 3 + 5 + 7 + 9)/5 = 5$. Summing up, $E(f) = 9.5$.

(c) If $f(n) = 10$ then n is one of the 5 numbers 19, 37, 55, 73 and 91. Only 55 is not in F, so

$$p(G|F) = \frac{p(G \cap F)}{p(F)} = \frac{4/50}{45/50} = \frac{4}{45}.$$

6:

(a) For $n = 1$, there are three 1-letter strings, and none of them have two consecutive As. Thus $a_1 = 3$. For $n = 2$, there are $3^2 = 9$ 2-letter strings, so excluding AA we find $a_2 = 8$. For $n = 3$, there are $3^3 = 27$ 3-letter strings, of which we must exclude 3 strings of the form AAx and three of the form xAA where x is A, B or C. Howver, this excludes AAA twice, so adding it back we get $a_3 = 27 - 3 - 3 + 1 = 22$.

(b) A string in Y_n must begin with either AB, AC, B or C and continue without consecutive As, so

$$\begin{aligned} a_n &= |\{x_1 x_2 x_3 \ldots x_n\}| \\ &= |\{ABx_3x_4 \ldots x_n\}| + |\{ACx_3x_4 \ldots x_n\}| + |\{Bx_2x_3 \ldots x_n\}| + |\{Cx_2x_3 \ldots x_n\}| \end{aligned}$$

and $a_n = 2(a_{n-1} + a_{n-2})$.

(c) Our relation gives $a_2 = 2(a_1 + a_0) = 2(3 + 1) = 8$ and $a_3 = 2(a_2 + a_1) = 2(8 + 3) = 22$, which checks.

(d) The characteristic equation is $r^2 - 2r - 2 = 0$ and its roots are

$$r = \frac{2 \pm \sqrt{4+8}}{2} = \frac{1 \pm \sqrt{3}}{2}$$

so $r_1 = 1 + \sqrt{3}$ and $r_2 = 1 - \sqrt{3}$. Thus a_n is a linear combination $sr_1^n + tr_2^n$ determined by the initial conditions

$$s + t = 1, \qquad sr_1 + tr_2 = 3.$$

Solving these equations gives

$$s = \frac{1}{2} + \frac{1}{\sqrt{3}}, \qquad t = \frac{1}{2} - \frac{1}{\sqrt{3}},$$

and $a_n = \left(\frac{1}{2} + \frac{1}{\sqrt{3}}\right)\left(1+\sqrt{3}\right)^n + \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)\left(1-\sqrt{3}\right)^n$. Checking, we get

$$\begin{aligned} a_2 &= \left(\frac{1}{2} + \frac{1}{\sqrt{3}}\right)\left(1+\sqrt{3}\right)^2 + \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)\left(1-\sqrt{3}\right)^2 \\ &= \left(\frac{1}{2} + \frac{1}{\sqrt{3}}\right)\left(4+2\sqrt{3}\right)^2 + \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)\left(4-2\sqrt{3}\right)^2 = 8. \end{aligned}$$

7:

(a) This procedure computes the number s_1 of nonnegative integer solutions of

$$x_1 + x_2 + x_3 + x_4 + x_5 = 832,$$

which is the number of ways to choose 832 objects of 5 kinds. By stars and bars, the number is

$$\binom{832+5-1}{5-1} = \binom{836}{4}.$$

(b) Now we are imposing the constraints

$$x_1 \geq 1, \quad x_2 \geq 2, \quad x_3 \geq 3, \quad x_4 \geq 4$$

and $0 \leq x_5 \leq 3$. The first four constraints can be handled by selecting j objects of kind j for $1 \leq j \leq 4$, leaving 822 objects of 5 kinds to select subject to the constraint on x_5. There are two ways to count these possibilities, and they give

$$\binom{825}{3} + \binom{824}{3} + \binom{823}{3} + \binom{822}{3} = \binom{826}{4} - \binom{821}{4} = 462846965.$$

8: **(a)** Number the doors from (say) 0 to 2 and use two Boolean variables to indicate whether you are offered the choice to switch and whether you accept.

Then the sample space is $S = Z_3 \times B \times B \times Z_3$, where the four entries of each point $x = (x_1, x_2, x_3, x_4) = (p, c, a, w) \in S$ are $x_1 = p \in Z_3$ for the door you pick, $x_2 = c \in B$ for the offering of the choice, $x_3 = a \in B$ for accepting the choice, and $x_4 = w \in Z_3$ for the number of the winning door. Since some possible combinations never occur, they have zero probability, so the points do not all have equal probabilities. Smaller sample spaces can be used if they describe all the details of the experiments, but they will complicate the calculations.

(b) Let C be the event that you are offered the choice to switch, W be the event that your original choice is the winner, and L be the complementary event that your original choice loses. Then we are given conditional probabilities $P(C|W) = 1$, $P(C|L) = p$, and equal a priori probabilities $P(W) = 1/3$, $P(L) = 2/3$. Since W and L form a disjoint partition of the sample space, i.e. are mutually exclusive events covering every possibility,

$$P(C) = P(C \cap W) + P(C \cap L).$$

By definition of conditional probability,

$$P(C) = P(C|W)P(W) + P(C|L)P(L) = 1 \cdot 1/3 + p \cdot 2/3 = (1 + 2p)/3.$$

(c) Switching wins iff your original choice lost, so we are computing

$$P(L|C) = \frac{P(L \cap C)}{P(C)} = \frac{P(C \cap L)}{P(C)} = \frac{P(C|L)P(L)}{P(C)} = \frac{2p/3}{(1+2p)/3} = \frac{2p}{1+2p}.$$

(d) In the usual Monty Hall problem you are always offered the chance to switch, so $p = 1$ and switching wins with probability $2/3$.

9: (a) By stars and bars, the set S of nonnegative integer solutions to this equation has cardinality $|S| = \binom{24+3-1}{3-1} = \binom{26}{2} = 325$. Each solution (x_1, x_2, x_3) has equal probability $p = 1/|S| = 1/325$ of being chosen.

(b) Since the three x_i's sum to 24, by symmetry we must have $E(x_1) = 8$.

(c) Let $A_i = \{x_i = 8\} \cap S$ be the set of solutions with x_i equal to 8, for $i = 1$ through 3. The probability that at least one of the x_i's is exactly 8 is

$$\frac{|A_1 \cup A_2 \cup A_3|}{|S|},$$

so we should use inclusion-exclusion. The cardinality of A_1 (and of A_2 and A_3, by symmetry) is the number of nonnegative integer solutions to

$$x_2 + x_3 = 16,$$

which is

$$|A_1| = \binom{17}{1}$$

by stars and bars. The cardinality of $A_1 \cap A_2$ (etc.) is the number of nonnegative integer solutions to

$$x_3 = 8,$$

which is 1. The cardinality of $A_1 \cap A_2 \cap A_3$ is 1. Thus by inclusion-exclusion,

$$\begin{aligned}
|A_1 \cup A_2 \cup A_3| &= \sum_{1 \le i \le 3} |A_i| - \sum_{1 \le i < j \le 3} |A_i \cap A_j| + \sum_{1 \le i < j < k \le 3} |A_i \cap A_j \cap A_k| \\
&= \sum_{1 \le i \le 3} \binom{17}{1} - \sum_{1 \le i < j \le 3} 1 + 1 \\
&= \binom{3}{1} \cdot \binom{17}{1} - \binom{3}{2} \cdot 1 + 1
\end{aligned}$$

Thus the probability that some x_i is exactly equal to 8 is

$$\frac{\binom{3}{1} \cdot \binom{17}{1} - \binom{3}{2} \cdot 1 + 1}{\binom{26}{2}} = \frac{49}{325}.$$

(d) Brute force computation gives

$$E(x_1^2) = \frac{1}{|S|} \sum_{x_1 + x_2 + x_3 = 24} x_1^2 = \frac{1}{|S|} \sum_{x_1} x_1^2 \sum_{x_2 + x_3 = 24 - x_1} 1 = \frac{1}{|S|} \sum_{x_1 = 0}^{2} 4x_1^2 \binom{24 - x_1 + 2 - 1}{2 - 1}$$

which works out to

$$E(x_1^2) = \frac{1}{|S|} \sum_{x_1} 25x_1^2 - x_1^3 = \frac{1}{325} \left(25 \frac{25 \cdot 26}{2} - \frac{25^2 \cdot 26^2}{4} \right) = 100,$$

so $V(x_1) = 36$ and $\sigma = 6$.

(e)

$$P(|X - E(X)| \geq r\sigma) \leq \frac{1}{r^2}.$$

(f) For this case,

$$P(|x_1 - 8| \geq 12) \leq \frac{1}{4}$$

implies that seventy-five percent of solutions have $x_1 < 20$.

10: (a) By inspection,

$$a_1 = 1/16, \qquad a_2 = 1/32, \qquad a_3 = 5/256.$$

(b) Multiply the recurrence relation by x^{k+1} and sum from $k = 0$ to ∞ to get

$$G(x) = a_0 + \sum_{k=0}^{\infty} a_{k+1} x^{k+1} = a_0 + \sum_{k=0}^{\infty} \sum_{j=0}^{k} a_{k-j} a_j x^{k+1}$$

after using the recurrence relation. Reverse the order of summation and pull out an x to get

$$G(x) = a_0 + x \sum_{j=0}^{\infty} a_j x^j \sum_{j=k}^{\infty} a_{k-j} x^{k-j} = a_0 + x \sum_{j=0}^{\infty} a_j x^j \sum_{k=0}^{\infty} a_k x^k = a_0 + xG(x)^2,$$

proving the desired identity. Note that we worked forward from the definition to discover the identity, rather than plugging the generating function into the identity and checking that the two sides matched without knowing why.
(c) Solving the quadratic equation gives

$$G(x) = \frac{1 - \sqrt{1 - x}}{2x}$$

where only the negative sign gives $G(0) = a_0$.
(d)

$$G(x) = \frac{1 - 1 - \sum_{k=0}^{\infty} C(1/2, k+1)(-1)^{k+1} x^{k+1}}{2x}.$$

so $a_k = \frac{1}{2}(-1)^k C(1/2, k+1)$.

11: (a) There are only three possible remainders **mod** 3, and they are 0, 1, and 2. Thus by the generalized pigeonhole principle, one of them must get $\lceil 100/3 \rceil = 34$ of the 100 Fibonacci numbers.

(b) From the first few shown above, we guess that f_n is even if and only if n is divisible by 3.
(c) Theorem: f_n is even if and only if $3|n$.
Base: The theorem holds for $n \leq 3$ by inspection.
Induction: Assume $n > 3$ and the theorem is true for all $k < n$. If $3|n$ then $f_n = f_{n-1} + f_{n-2}$ is the sum of two odd numbers, by the inductive hypothesis, hence even. Otherwise, f_n is the sum of an even number and an odd number, by the inductive hypothesis, hence odd.

12: **(a)** The characteristic equation is found by seeking solutions of the form $a_n = r^n$. It is

$$r^2 - 3r + 2 = 0.$$

(b) The roots are given by the quadratic formula:

$$p, q = \frac{3 \pm \sqrt{9-8}}{2} = 2, 1.$$

(c) Since p and q are roots of the characteristic equation, we know the recurrence relation is automatically satisfied (if we did the arithmetic right). The initial conditions give us two equations in two unknowns:

$$\begin{aligned} t + s &= 0 \\ 2t + s &= 1. \end{aligned}$$

Solving, we get $t = 1$ and $s = -1$. Hence $a_n = 2^n - 1$.

13: Since expectation is linear, we can simply add the expectation of each die separately. For the six-sided die, the expected value is

$$\frac{1}{6}(1+2+3+4+5+6) = 3.5$$

while for the twelve-side die the expected value is

$$\frac{1}{12}(1+2+3+4+5+6+7+8+9+10+11+12) = 6.5.$$

Adding, we get 10 for the expected sum.

14: **(a)** There are ten letters, so there are 10! ways to arrange them. However, interchanging the two G's or the three O's or the two L's does not change the resulting string, so we must divide 10! by the permutations of identical letters: thus the answer is

$$\frac{10!}{3!2!2!} = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 151200.$$

(b) If we make the first letter G, then we have nine letters left to rearrange, and there are fewer duplicate letters: thus the answer is

$$\frac{9!}{3!2!} = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 2 = 30240.$$