Math 55: Midterm #2, 3 March 2008

Write your name, your student ID number, your section time and number, and a three-problem grading grid, on the cover of your blue book. Hand in your exam book at 11:00 am. Books, notes, calculators, scratch paper and/or collaboration are not allowed. Justify your computations: correct answers with inadequate explanation may receive partial credit.

1: Five pecans p_1, p_2, p_3, p_4 and p_5 are given to three squirrels s_1, s_2 and s_3, with each pecan given to one squirrel chosen independently with equal probabilities.
(a) Describe the sample space and compute its cardinality.
(b) Compute the probability that the first squirrel s_1 gets all the pecans.
(c) Compute the expected number of pecans the last squirrel s_3 gets.
(d) Compute the variance of the number of pecans the last squirrel gets.
(e) Compute the probability that each squirrel gets at least one pecan.

2: A discrete math midterm consists of 19 independent true–false questions on Martian literature. On each question, any non-Martian student has a 1/3 chance of guessing the correct answer. Two students (in the class of 28 students) are Martian, and therefore will get perfect scores.
(a) What is the probability that a student chosen at random from the non-Martian students gets a perfect score?
(b) What is the probability that a student chosen at random from the whole class gets a perfect score?
(c) Given that a student named Zr̈thjkpq got a perfect score on the midterm, compute the probability that Zr̈thjkpq is a Martian.

3: For any integer $k \geq 0$, let T_k be the set of all k–letter strings of Xs, Ys and Zs that **HAVE** two consecutive Xs. For example, $T_3 = \{XXY, XXZ, YXX, ZXX, XXX\}$. Let $t_k = |T_k|$.
(a) Evaluate t_0, t_1, t_2 and t_3 from the definition of T_k.
(b) Show that t_k satisfies the recurrence relation

$$t_{k+2} = 2t_{k+1} + 2t_k + 3^k$$

for $k \geq 0$.

(c) Find a closed form (like e^x/x or $1/(1-x)$) for the generating function

$$G(x) = \sum_{k=0}^{\infty} t_k x^k.$$

4: Let p, q and r be distinct primes.

(a) Write down the inclusion-exclusion formula for the cardinality of the union $P \cup Q \cup R$ of three sets P, Q and R.

(b) How many integers n with $1 \leq n \leq pqr$ are relatively prime to pqr?

(c) Let $p = 2$, $q = 3$ and $r = 5$. Write down all integers n with $1 \leq n \leq pqr$ and relatively prime to pqr, and check your formula gives the correct number of them.

5: Let the sample space S be the set of two-digit *odd* integers $n = d_2 \cdot 10 + d_1$ with $01 \leq n \leq 99$, chosen at random with equal probabilities.

(a) Find the probability of the event F that an element of S has distinct digits $d_1 \neq d_2$.

(b) Define a random variable f to be the sum of the digits of n: thus $f(n) = d_1 + d_2$. Find the expectation $E(f)$.

(c) Let G be the event that $f(n) = 10$ and F as in (a). Find the conditional probability $p(G|F)$.

6: For any integer $n \geq 1$ let Y_n be the set of all n-letter strings of As, Bs and Cs that do not have two consecutive As. For example, ABCABA is an element of Y_6. Let $a_n = |Y_n|$.

(a) Evaluate a_1, a_2 and a_3 from the definition of Y_n.

(b) Find a linear homogeneous recurrence relation for a_n.

(c) Given that $a_0 = 1$, check that your recurrence relation gives the correct result for a_2 and a_3.

(d) Find an explicit non-recursive formula for a_n valid for all $n \geq 0$.

7: Express the results s_1 and s_2 of the following procedures as binomial coefficients or sums or differences of binomial coefficients.

(a)

```
procedure atherwood
  s_1 := 0
  for x_1 = 0 to 10000
    for x_2 = 0 to 10000
      for x_3 = 0 to 10000
        for x_4 = 0 to 10000
          for x_5 = 0 to 10000
            if x_1 + x_2 + x_3 + x_4 + x_5 = 832 then s_1 := s_1 + 1
  return s_1
```

(b)

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procedure sarif
  s_2 := 0
  for x_1 = 1 to 10000
    for x_2 = 2 to 10000
      for x_3 = 3 to 10000
        for x_4 = 4 to 10000
          for x_5 = 0 to 3
            if x_1 + x_2 + x_3 + x_4 + x_5 = 832 then s_2 := s_2 + 1
  return s_2
```

8: Consider the following variant of a well-known game. Of three doors, one is secretly a winner (with equal probability). You choose a door to start. If your door is the winner, the host opens one of the others, showing it is empty, and offers you the choice to switch or stay. If your door is not the winner, the host either ends the game (with conditional probability $1-p$) or opens one of the others, showing it is empty, and offers the choice to switch or stay.

(a) Describe the sample space. Do all the points in the sample space have the same probability?

(b) Compute the probability that the host offers you the choice to switch or stay.

(c) Compute the conditional probability that switching wins, given that you are offered the choice.
(d) For what value of p does this game become the usual Monty Hall problem and what does your analysis then suggest?

9: Choose a nonnegative integer solution (x_1, x_2, x_3) of the equation

$$x_1 + x_2 + x_3 = 24$$

at random, where each solution has equal probability.
(a) What is the probability of selecting $(8, 8, 8)$?
(b) Compute the expectation $E(x_1)$. Don't work too hard.
(c) What is the probability that at least one of the x_i's is exactly equal to $E(x_i)$?
(d) Compute the expectation $E(x_1^2)$, the variance $V(x_1)$ and the standard deviation σ. You may find the following identities useful:

$$\sum_{j=0}^{n} j = \frac{n(n+1)}{2}, \qquad \sum_{j=0}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{j=0}^{n} j^3 = \frac{n^2(n+1)^2}{4}.$$

(e) State Chebyshev's inequality.
(f) Use Chebyshev's inequality to show that at least seventy-five percent of the triples (x_1, x_2, x_3) solving this equation have $x_1 \leq 20$.

10: Let $a_0 = 1/4$ and a_k be the solution of the recurrence relation

$$a_{k+1} = \sum_{j=0}^{k} a_j a_{k-j}$$

for $k \geq 0$.
(a) Evaluate a_0, a_1, a_2 and a_3 from the recurrence relation.
(b) Show that the generating function

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

satisfies the quadratic equation

$$xG(x)^2 - G(x) + 1/4 = 0.$$

You may find the following identity useful:

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n} = \sum_{j=0}^{\infty}\sum_{n=j}^{\infty}.$$

(c) Find a closed form (like e^x/x or $1/(1-x)$) for the generating function $G(x)$.
(d) Use the binomial series

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} C(\alpha, k)x^k$$

to express the solution a_k of the recurrence relation in terms of binomial coefficients.

11: (20 points) Consider the first 100 Fibonacci numbers

$f_0, f_1, \ldots, f_{99} = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots, 218922995834555169026.$

They satisfy the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

for $n \geq 2$.
(a) Show that there are at least 34 of them which have the same remainder **mod** 3.
(b) For which n is f_n even?
(c) Use strong induction to prove that your answer to **(b)** is correct.

12: (20 points) Consider the sequence a_n defined by the initial conditions

$$a_0 = 0, \qquad a_1 = 1$$

and the recurrence relation

$$a_n = 3a_{n-1} - 2a_{n-2}, \qquad n \geq 2.$$

(a) Write down the characteristic equation.
(b) Find its roots p and q.

(c) Find the coefficients t and s of the linear combination

$$a_n = tp^n + sq^n$$

which satisfies both the recurrence relation and the initial conditions.

13: (20 points) Roll a six-sided die with faces numbered 1 through 6 and a twelve-sided die with faces numbered 1 through 12. Assume each die is fair. Find the expected sum of the numbers rolled. Justify your answer.

14: (20 points)
(a) How many different strings can be made using all the ten letters in the word GOOGOLPLEX?
(b) How many start with G?