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Problem 1: You cannot compute $55^{1000} \bmod 1001$ directly by Fermat's little Theorem because $1001 = 7 \cdot 11 \cdot 13$ is not prime. Instead,
(a) Use Fermat's little Theorem to compute 55^{1000} modulo 7, 11 and 13.
(b) Use the Chinese Remainder Theorem to find $55^{1000} \bmod 1001$.

Solution: **(a)** FlT says that $a^{p-1} \bmod p = 1$ if a and p are relatively prime, so with $p = 7$ and $a = 55 \equiv 6 \pmod 7$ we have

$$55^{1000} \equiv 6^{1000 \bmod 6} \equiv 6^4 \equiv \boxed{1} \pmod 7$$

Since $11|55$,

$$55^{1000} \equiv \boxed{0} \pmod{11}.$$

By FlT with $p = 13$,

$$55^{1000} \equiv 3^{1000 \bmod 12} \equiv 3^4 \equiv \boxed{3} \pmod{13}$$

(b) Seeking $x = x_1 \cdot 11 \cdot 13 + x_2 \cdot 7 \cdot 13 + x_3 \cdot 7 \cdot 11$ gives three independent congruences

$$\begin{aligned} 3x_1 &\equiv 1 \pmod 7 \\ 3x_2 &\equiv 0 \pmod{11} \\ -x_2 &\equiv 3 \pmod{13} \end{aligned}$$

with solutions $x_1 = 5$, $x_2 = 0$ and $x_3 = 10$. Hence

$$55^{1000} \bmod 1001 = 5 \cdot 11 \cdot 13 + 10 \cdot 7 \cdot 11 = \boxed{484}$$

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Problem 2: In the fall season, leaves drop from trees at an accelerating rate. Suppose that a leaf which has survived to day $k = 0, 1, 2, \ldots$ of fall has (conditional) probability $p_k = k/(1+k)$ of dropping on day k.

(a) Show that a random leaf survives through days $0, 1, \ldots, k-1$ and then drops on day k with probability

$$p(f = k) = \frac{k}{(k+1)!}.$$

Here $f(x)$ is the random variable which reports the day when leaf x drops.

(b) Suppose one of the $5041 = 7! + 1$ leaves in Tilden Park is plastic and will never drop. At the end of day $k = 6$ you find a leaf which has not dropped. What is the probability that it is the plastic one?

Solution: Let F_k be the event that the leaf falls on day k, and S_k be the event that the leaf survives through day k.

(a) Since $p(F_k|S_{k-1}) = k/(k+1)$, conditioning on repeated survival gives

$$\begin{aligned} p(F_k) &= p(F_k|S_{k-1})p(S_{k-1}|S_{k-2})\cdots p(S_1|S_0)p(S_0) \\ &= \frac{k}{1+k}\left(1-\frac{k-1}{k}\right)\cdots\left(1-\frac{1}{2}\right)(1) = \frac{k}{1+k}\frac{1}{k}\frac{1}{k-1}\cdots\frac{1}{2} = \frac{k}{(1+k)!}. \end{aligned}$$

(b) Let P be the event that the leaf is plastic. Then $p(P) = \frac{1}{5041}$ and $p(\overline{P}) = \frac{5040}{5041}$. If the leaf is not plastic, its probability of survival through day 6 is

$$p(S_6|\overline{P}) = 1 - \sum_{k=1}^{6} p(F_k) = 1 - \sum_{k=1}^{6} \frac{k}{(1+k)!} = 1 - \sum_{k=1}^{6} \frac{1}{k!} - \frac{1}{(1+k)!} = 1 - 1 + \frac{1}{7!} = \frac{1}{7!}$$

Thus by Bayes' theorem,

$$p(P|S_6) = \frac{p(S_6|P)p(P)}{p(S_6|P)p(P) + p(S_6|\overline{P})p(\overline{P})} = \frac{(1)(\frac{1}{5041})}{(1)\frac{1}{5041} + \frac{1}{7!}\frac{5040}{5041}} = \frac{1}{1+1} = \frac{1}{2}.$$

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Problem 3: The Fibonacci numbers f_n satisfy the recurrence relation

$$f_{n+2} = f_{n+1} + f_n$$

for $n \geq 0$. Use strong induction on $k \geq 1$ to prove that

$$f_{n+2k} = \sum_{j=0}^{k} \binom{k}{j} f_{n+j}$$

for all $n \geq 0$. Note that the base case $k = 1$ is the recurrence relation.

Solution: Since $\binom{1}{0} = \binom{1}{1} = 1$, the recurrence relation constitutes the base case $k = 1$. (The case $k = 0$ is a tautology.) Fix $k \geq 2$. Then by the inductive hypothesis,

$$f_{n+2(k-1)} = f_{n+2k-2} = \sum_{j=0}^{k-1} \binom{k-1}{j} f_{n+j}$$

for $n \geq 0$. Shifting n up by 2 gives

$$f_{n+2k} = \sum_{j=0}^{k-1} \binom{k-1}{j} f_{n+2+j}$$

and shifting the case $k = 1$ (or the recurrence relation) up by j gives

$$f_{n+2k} = \sum_{j=0}^{k-1} \binom{k-1}{j} \left(f_{n+j+1} + f_{n+j}\right).$$

Splitting the sum into two sums and shifting an index of summation gives

$$f_{n+2k} = \sum_{j=1}^{k} \binom{k-1}{j-1} f_{n+j} + \sum_{j=0}^{k-1} \binom{k-1}{j} f_{n+j}$$

and Pascal's identity

$$\binom{k-1}{j-1} + \binom{k-1}{j} = \binom{k}{j}$$

gives the desired result.