Math 55: Final Exam, 14 December 2009

1: Define a relation R on simple graphs $G = (V, E)$ by the requirement that $G_1 R G_2$ iff $|V_1| = |V_2|$, $|E_1| = |E_2|$, and there is a bijection $f : V_1 \to V_2$ such that for every $v \in V_1$, $\deg(f(v)) = \deg(v)$.
(a) State the definition of an equivalence relation and prove that R is an equivalence relation.
(b) Draw one representative of each equivalence class C of R, for simple graphs with $n = 4$ vertices and $e = 2$ to 6 edges.
(c) Define isomorphism of simple graphs and prove that if G_1 is isomorphic to G_2 then $G_1 R G_2$.
(d) Find two nonisomorphic simple graphs G_1 and G_2 such that $G_1 R G_2$. Prove that G_1 is not isomorphic to G_2.

2: Define the divisibility relation R on $A_n = \{1, 2, 3, 4, \ldots, n\}$ by $aRb \leftrightarrow a|b$.
(a) Define a partial order and prove or disprove that R is one.
(b) Let M be the matrix of R. Show that the number $d(j)$ of divisors of any integer $j \in A_n$ is given by the sum of the entries in column j of M:

$$d(j) = \sum_{i=1}^{n} M_{ij}.$$

(c) Evaluate $d(pq)$ for any primes p and q such that $pq \in A_n$.
(d) Consider the experiment of selecting an integer j from A_n at random, with equal probabilities. Define the expectation $E(d)$ and show that

$$E(d) \leq \sum_{k=1}^{n} \frac{1}{k}.$$

(e) Prove by strong induction that $E(d) \leq 2 + \log_2 n$.

3: (a) State Fermat's little Theorem.
(b) Use Fermat's little Theorem and modular arithmetic to compute 555^{555} mod 17.
(c) Run the Euclidean algorithm forward and backward to find an inverse of $15 \pmod{17}$.
(d) State the Chinese Remainder Theorem for the case when there are three moduli m_1, m_2 and m_3.
(e) Use the Chinese Remainder Theorem and modular arithmetic to compute 555^{555} mod 255.

4: Define the Lucas numbers by

$$g_0 = 2, \quad g_1 = 1, \quad g_{n+1} = g_n + g_{n-1} \text{ for } n \geq 1.$$

The first few are

$$g_0 = 2, g_1 = 1, g_2 = 3, \; g_3 = 4, \; g_4 = 7, \; g_5 = 11, \; g_6 = 18, \; g_7 = 29, \; \ldots$$

(a) Let

$$g(x) = \sum_{n=0}^{\infty} g_n x^n$$

be the generating function for the Lucas numbers. Use the recurrence relation and initial conditions to find an explicit formula for $g(x)$.

(b) Use (a), partial fractions and the geometric series to show that

$$g_n = \varphi^n + (1-\varphi)^n, \qquad \varphi = \frac{1+\sqrt{5}}{2} = 1.618\ldots.$$

(c) Write each positive integer $n \leq 12$ as a sum

$$n = \sum_{j=0}^{\infty} n_j g_j, \qquad n_j \in \{0, 1\}$$

of one or more *distinct* Lucas numbers. The first few are

$$1 = g_1, \; 2 = g_0, \; 3 = g_2, \; 4 = g_3, \; 5 = g_3 + g_1, \; 6 = g_3 + g_0, \ldots$$

(d) Prove by strong induction that *every* positive integer n can be written as a sum of one or more *distinct* Lucas numbers. (Hint: Choose m so that $g_m \leq n < g_{m+1}$ and let $k = n - g_m$.)