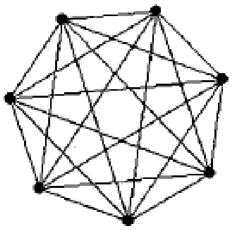
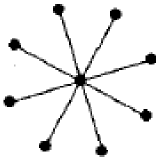
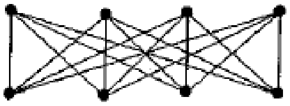
20. a) This graph has 7 vertices, with an edge joining each pair of distinct vertices.



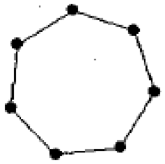
b) This graph is the complete bipartite graph on parts of size 1 and 8; we have put the part of size 1 in the middle.



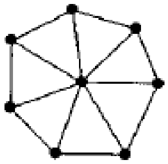
c) This is the complete bipartite graph with 4 vertices in each part.



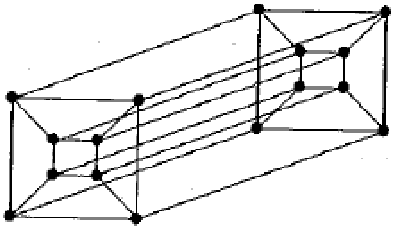
d) This is the 7-cycle.



e) The 7-wheel is the 7-cycle with an extra vertex joined to the other 7 vertices.



f) We take two copies of Q_3 and join corresponding vertices.



36. **a)** Since the number of odd-degree vertices has to be even, no graph exists with these degrees. Another reason no such graph exists is that the vertex of degree 0 would have to be isolated but the vertex of degree 5 would have to be adjacent to every other vertex, and these two statements are contradictory.

b) Since the number of odd-degree vertices has to be even, no graph exists with these degrees. Another reason no such graph exists is that the degree of a vertex in a simple graph is at most 1 less than the number of vertices.

c) A 6-cycle is such a graph. (See picture below.)

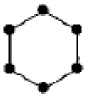
d) Since the number of odd-degree vertices has to be even, no graph exists with these degrees.

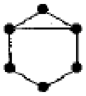
e) A 6-cycle with one of its diagonals added is such a graph. (See picture below.)

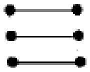
f) A graph consisting of three edges with no common vertices is such a graph. (See picture below.)

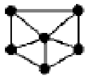
g) The 5-wheel is such a graph. (See picture below.)

h) Each of the vertices of degree 5 is adjacent to all the other vertices. Thus there can be no vertex of degree 1. So no such graph exists.

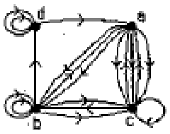

(c)


(e)


(f)


(g)

24. This is the adjacency matrix of a directed multigraph, because the matrix is not symmetric and it contains entries greater than 1.



28. For an undirected graph, the sum of the entries in the i^{th} row is the same as the corresponding column sum, namely the number of edges incident to the vertex i, which is the same as the degree of i minus the number of loops at i. (See the solution to Exercise 29.)In a directed graph, the answer is dual to the answer for Exercise 29. The sum of the entries in the i^{th} row is the number of edges that have i as their initial vertex, i.e., the out-degree of i.

44. The easiest way to show that these graphs are not isomorphic is to look at their complements. The complement of the graph on the left consists of two 4-cycles. The complement of the graph on the right is an 8-cycle. Since the complements are not isomorphic, the graphs are also not isomorphic.

54. **a)** There are just two graphs with 2 vertices—the one with no edges, and the one with one edge.

b) A graph with three vertices can contain 0, 1, 2, or 3 edges. There is only one graph for each number of edges, up to isomorphism. Therefore the answer is 4.

c) Here we look at graphs with 4 vertices. There is 1 graph with no edges, and 1 (up to isomorphism) with a single edge. If there are two edges, then these edges may or may not be adjacent, giving us 2 possibilities. If there are three edges, then the edges may form a triangle, a star, or a path, giving us 3 possibilities. Since graphs with four, five, or six edges are just complements of graphs with two, one, or no edges (respectively), the number of isomorphism classes must be the same as for these earlier cases. Thus our answer is $1+1+2+3+2+1+1=11$.

22. **a)** Adjacent vertices are in different parts, so every path between them must have odd length. Therefore there are no paths of length 2.

b) A path of length 3 is specified by choosing a vertex in one part for the second vertex in the path and a vertex in the other part for the third vertex in the path (the first and fourth vertices are the given adjacent vertices). Therefore there are $3 \cdot 3 = 9$ paths.

c) As in part (a), the answer is 0.

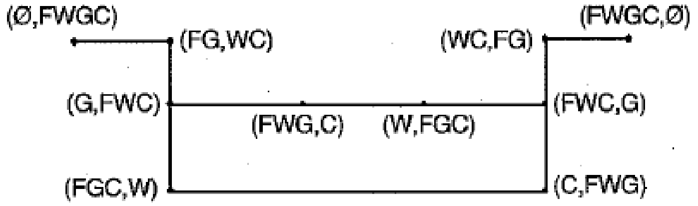
d) This is similar to part **(b)**; therefore the answer is $3^4 = 81$.

42. First we obtain the inequality given in the hint. We claim that the maximum value of $\sum n_i^2$, subject to the constraint that $\sum n_i = n$, is obtained when one of the n_i's is as large as possible, namely $n-k+1$, and the remaining n_i's (there are $k-1$ of them) are all equal to 1. To justify this claim, suppose instead that two of the n_i's were a and b, with $a \geq b \geq 2$. If we replace a by $a+1$ and b by $b-1$, then the constraint is still satisfied, and the sum of the squares has changed by $(a+1)^2+(b-1)^2-a^2-b^2 = 2(a-b)+2 \geq 2$. Therefore the maximum cannot be attained unless the n_i's are as we claimed. Since there are only a finite number of possibilities for the distribution of the n_i's, the arrangement we give must in fact yield the maximum. Therefore $\sum n_i^2 \leq (n-k+1)^2 + (k-1)\cdot 1^2 = n^2-(k-1)(2n-k)$, as desired.

Now by Exercise 41, the number of edges of the given graph does not exceed $\sum C(n_i,2) = \sum (n_i^2+n_i)/2 = ((\sum n_i^2)+n)/2$. Applying the inequality obtained above, we see that this does not exceed $(n^2-(k-1)(2n-k)+n)/2$, which after a little algebra is seen to equal $(n-k)(n-k+1)/2$. The upshot of all this is that the most edges are obtained if there is one component as large as possible, with all the other components consisting of isolated vertices.

54. a) To proceed systematically, we list the states in order of decreasing population on the left shore. The allowable states are then $(FWGC,\emptyset)$, (FWG,C), (FWC,G), (FGC,W), (FG,WC), (WC,FG) (C,FWG), (G,FWC), (W,FGC), and $(\emptyset,FWGC)$. Notice that, for example, (GC,FW) and (WGC,F) are not allowed by the rules.

b) The graph is as shown here. Notice that the boat can carry only the farmer and one other object, so the transitions are rather restricted.



c) The path in the graph corresponds to the moves in the solution.

d) There are two simple paths from $(FWGC,\emptyset)$ to $(\emptyset,FWGC)$ that can be easily seen in the graph. One is $(FWGC,\emptyset)$, (WC,FG), (FWC,G), (W,FGC), (FWG,C), (G,FWC), (FG,WC), $(\emptyset,FWGC)$. The other is $(FWGC,\emptyset)$, (WC,FG), (FWC,G), (C,FWG), (FGC,W), (G,FWC), (FG,WC), $(\emptyset,FWGC)$.

e) Both solutions cost $4.

56. If we use the ordered pair (a,b) to indicate that the three-gallon jug has a gallons in it and the five-gallon jug has b gallons in it, then we start with $(0,0)$ and can do the following things: fill a jug that is empty or partially empty (so that, for example, we can go from $(0,3)$ to $(3,3)$); empty a jug; or transfer some or all of the contents of a jug to the other jug , as long as we either completely empty the donor jug or completely fill the receiving jug. A simple solution to the puzzle uses this directed path: $(0,0) \to (3,0) \to (0,3) \to (3,3) \to (1,5)$.