Solutions to Homework 9, Math 55

Section 6.4

2. The generating function is $1 + 4x + 16x^2 + 64x^3 + 256x^4 = \sum_{n=0}^{4}(4x)^n = \frac{1-(4x)^5}{1-4x} = \frac{1-1024x^5}{1-4x}$.

8. (a) Since $(x^2+1)^3 = (1+x^2)^3 = \sum_{i=0}^{3}\binom{3}{i}x^{2i}$, we have $a_n = \binom{3}{n/2}$ if $0 \le n \le 6$ and n is even; otherwise, $a_n = 0$.

   (b) Similarly, $(3x-1)^3 = \sum_{n=0}^{3}\binom{3}{n}(3x)^n(-1)^{3-n}$; thus, if $0 \le n \le 3$, then $a_n = (-1)^{3-n}3^n\binom{3}{n}$, and otherwise $a_n = 0$.

   (c) We have $\frac{1}{1-2x^2} = \sum_{k=0}^{\infty}(2x^2)^k = \sum_{k=0}^{\infty}2^k x^{2k}$. Thus, if n is even, then $a_n = 2^{n/2}$, while if n is odd, then $a_n = 0$.

   (d) The given function is equal to $x^2(1-x)^{-3}$. By the binomial theorem, this is equal to $x^2\sum_{k=0}^{\infty}\binom{-3}{k}(-1)^k x^k = \sum_{k=0}^{\infty}\binom{-3}{k}(-1)^k x^{k+2}$. Thus, to get a term x^n we set $k = n-2$; if $n < 2$, this gives $a_n = 0$, while if $n \ge 2$, this gives $a_n = \binom{-3}{n-2}(-1)^n$. By the result of example 8 on page 438, we can write this as $a_n = \binom{n}{n-2} = \binom{n}{2}$.

   (e) The given function is equal to $x - 1 + \sum_{n=0}^{\infty}3^n x^n$. Thus, $a_0 = 3^0 - 1 = 0$; $a_1 = 3^1 + 1 = 4$; and if $n \ge 2$, then $a_n = 3^n$.

   (f) We rewrite $\frac{1+x^3}{(1+x)^3} = 1 - \frac{3x+3x^2}{(1+x)^3} = 1 - \frac{3x}{(1+x)^2}$. This is equal to $1 - 3x\sum_{k=0}^{\infty}\binom{-2}{k}x^k = 1 - \sum_{k=0}^{\infty}3\binom{-2}{k}x^{k+1}$. Thus, we have $a_0 = 1$, while for $n \ge 1$, we set $k = n-1$ to get the x^n term, so $a_n = -3\binom{-2}{n-1}$. Again using example 8, we can rewrite this as $a_n = 3n(-1)^n$.

   (g) If we multiply the top and bottom by $1-x$, then this is equal to $\frac{x-x^2}{1-x^3} = (x-x^2)\sum_{k=0}^{\infty}x^{3k} = \sum_{k=0}^{\infty}(x^{3k+1} - x^{3k+2})$. Thus, we get

   $$a_n = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 3; \\ 1, & \text{if } n \equiv 1 \pmod 3; \\ -1, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

   Alternately, let $\omega = \frac{-1+i\sqrt{3}}{2}$; then we can factor $1 + x + x^2 = (1-\omega x)(1-\overline{\omega}x)$, where $\overline{\omega} = \frac{-1-i\sqrt{3}}{2}$ is the complex conjugate. We can now use partial fractions to write

   $$\frac{x}{1+x+x^2} = \frac{1}{i\sqrt{3}}\left(\frac{1}{1-\omega x} - \frac{1}{1-\overline{\omega}x}\right) = \frac{1}{i\sqrt{3}}\sum_{n=0}^{\infty}(\omega^n - \overline{\omega}^n)x^n.$$

   Thus, $a_n = \frac{1}{i\sqrt{3}}(\omega^n - \overline{\omega}^n)$. Since $\omega = e^{2\pi i/3}$ and $\overline{\omega} = e^{-2\pi i/3}$, this can be rewritten in terms of real numbers as $a_n = \frac{2}{\sqrt{3}}\sin(\frac{2\pi n}{3})$.

   (h) The given function expands to $-1 + \sum_{k=0}^{\infty}(3x^2)^k/k! = -1 + \sum_{k=0}^{\infty}(3^k/k!)x^{2k}$. Thus, if n is odd, then $a_n = 0$; $a_0 = -1 + 1 = 0$; and if $n \ge 2$ is even, then $a_n = 3^{n/2}/(n/2)!$.

10. (a) The given function is equal to $(\frac{1}{1-x^3})^3 = (1-x^3)^{-3} = \sum_{n=0}^{\infty}\binom{-3}{n}(-1)^n x^{3n}$. To get x^9 in this sum we set $n = 3$ to get a coefficient of $-\binom{-3}{3} = -\frac{(-3)(-4)(-5)}{3\cdot 2\cdot 1} = 10$.

    (b) The given function is equal to $(\frac{x^2}{1-x})^3 = \frac{x^6}{(1-x)^3} = x^6\sum_{n=0}^{\infty}\binom{-3}{n}(-1)^n x^n = \sum_{n=0}^{\infty}\binom{-3}{n}(-1)^n x^{n+6}$. To get x^9 in this sum we set $n = 3$ to get a coefficient of $-\binom{-3}{3} = 10$.

    (c) To get x^9 in this product, we can take $x^3\cdot x^3\cdot x^3$, $x^3\cdot x^4\cdot x^2$, or $x^5\cdot x^3\cdot x$; therefore, the coefficient of x^9 is 3.

    (d) Here, to get x^9 in the product, we can take $x\cdot x^8$ or $x^7\cdot x^2$; therefore, the coefficient of x^9 is 2.

    (e) Since $(1+x+x^2)^3$ is a polynomial of degree 6, the coefficient of x^9 is 0.

30. (a) The generating function here is $\sum_{n=0}^{\infty} 2a_n x^n = 2\sum_{n=0}^{\infty} a_n x^n = 2G(x)$.
    (b) The resulting generating function is $a_0x + a_1x^2 + a_2x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^{n+1} = x\sum_{n=0}^{\infty} a_n x^n = xG(x)$.
    (c) The result is $a_2x^4 + a_3x^5 + \cdots = \sum_{n=2}^{\infty} a_n x^{n+2} = x^2\sum_{n=2}^{\infty} a_n x^n = x^2(G(x) - a_0 - a_1x) = x^2G(x) - a_0x^2 - a_1x^3$.
    (d) Here we get $a_2 + a_3x^2 + a_4x^3 + \cdots = \sum_{n=2}^{\infty} a_n x^{n-2} = x^{-2}\sum_{n=2}^{\infty} a_n x^n = (G(x) - a_0 - a_1x)/x^2$.
    (e) Since $G(x) = \sum_{n=0}^{\infty} a_n x^n$, $G'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2x + 3a_3x^2 + \cdots$, which is the desired generating function.
    (f) We calculate $(G(x))^2 = (a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)^2 = a_0^2 + 2a_0a_1x + (2a_0a_2 + a_1^2)x^2 + (2a_0a_3 + 2a_1a_2)x^3 + \cdots$; thus, the desired generating function is $(G(x))^2$.

32. Since $a_k = 7a_{k-1}$ for $k \geq 1$, we have $\sum_{k=1}^{\infty} a_k x^k = 7\sum_{k=1}^{\infty} a_{k-1}x^k$. If $G(x) = \sum_{k=0}^{\infty} a_k x^k$, then the left hand side is equal to $G(x) - a_0 = G(x) - 5$. Also, setting $k-1 = n$, we get $\sum_{k=1}^{\infty} a_{k-1}x^k = \sum_{n=0}^{\infty} a_n x^{n+1} = xG(x)$. Thus, the above equation reduces to $G(x) - 5 = 7xG(x)$; this has solution $G(x) = \frac{5}{1-7x} = 5\sum_{k=0}^{\infty} 7^k x^k$. Thus, since a_k is the coefficient of x^k in $G(x)$, we get $a_k = 5 \cdot 7^k$.

44. (a) As in the solution to problem 10 from Lenstra's notes, we find $\sum_{n=0}^{\infty} n^2 x^n = \frac{x^2+x}{(1-x)^3}$. Now if we multiply by $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, the coefficient of x^n in the product is $\sum_{k=0}^{n} k^2 \cdot 1$. Thus, the generating function of a_n is $\frac{x^2+x}{(1-x)^3} \cdot \frac{1}{1-x} = \frac{x^2+x}{(1-x)^4}$.
    (b) Since $x^2+x = (1-x)^2 - 3(1-x) + 2$, we have $\frac{x^2+x}{(1-x)^4} = \frac{1}{(1-x)^2} - \frac{3}{(1-x)^3} + \frac{2}{(1-x)^4}$. By the binomial theorem, the coefficient of x^n in this expression is $a_n = (-1)^n\binom{-2}{n} - 3(-1)^n\binom{-3}{n} + 2(-1)^n\binom{-4}{n}$. Using the result of example 8 on page 438, we can rewrite this as $a_n = \binom{n+1}{n} - 3\binom{n+2}{n} + 2\binom{n+3}{n} = 2\binom{n+3}{3} - 3\binom{n+2}{2} + (n+1)$.

58. (a) Since $p(X = n) = q^{n-1}p$, the probability generating function is $G_X(x) = \sum_{n=1}^{\infty} q^{n-1}px^n = \frac{p}{q}\sum_{n=1}^{\infty} q^n x^n = \frac{p}{q}(\frac{1}{1-qx} - 1)$. (This is equal to $\frac{px}{1-qx}$, but the form given here is easier to use in taking derivatives.)
    (b) We have $E(X) = G_X'(1)$. Since $G_X'(x) = \frac{p}{q} \cdot \frac{q}{(1-qx)^2} = \frac{p}{(1-qx)^2}$, this gives $E(X) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$. Similarly, we have $G_X''(x) = \frac{2pq}{(1-qx)^3}$, so $G_X''(1) = \frac{2pq}{(1-q)^3} = \frac{2pq}{p^3} = \frac{2q}{p^2}$. Thus, $V(X) = G_X''(1) + G_X'(1) - (G_X'(1))^2 = \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{2q+p-1}{p^2} = \frac{2q-q}{p^2} = \frac{q}{p^2}$.

Section 6.5

2. By inclusion-exclusion, the answer is $345 + 212 - 188 = 369$.

6. (a) If $A_1 \subseteq A_2 \subseteq A_3$, then $A_1 \cup A_2 \cup A_3 = A_3$, so $A_1 \cup A_2 \cup A_3$ has 10000 elements.
    (b) If the sets are pairwise disjoint, then we can just add up the number of elements in A_1, A_2, and A_3 to get $|A_1 \cup A_2 \cup A_3| = 11100$.
    (c) By inclusion-exclusion, we have

$$\begin{aligned}|A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| \\ &= 100 + 1000 + 10000 - 2 - 2 - 2 + 1 = 11095.\end{aligned}$$

10. There are $\lfloor \frac{100}{5} \rfloor = 20$ positive integers not exceeding 100 which are divisible by 5, $\lfloor \frac{100}{7} \rfloor = 14$ which are divisible by 5, and $\lfloor \frac{100}{35} \rfloor = 2$ which are divisible by both 5 and 7. Thus, $20 + 14 - 2 = 32$ of them are divisible by either 5 or 7, which means that $100 - 32 = 68$ are not divisible by either 5 or 7.

12. We have that $\lfloor \sqrt{1000} \rfloor = 31$ of these integers are square numbers, $\lfloor \sqrt[3]{1000} \rfloor = 10$ are cubes, and $\lfloor \sqrt[6]{1000} \rfloor = 3$ are both (i.e. sixth powers). Therefore, the answer is $31 + 10 - 3 = 38$.

14. In order to get a permutation which includes the string *fish*, we first choose one of 23 possible positions for this string, then choose one of the 22! possible permutations of the other letters; thus, 23! of these strings contain the string *fish*. (Alternately, we could think of permuting a set containing the block *fish* and the 22 other letters.) Similarly, 24! contain the string *rat*, and 23! contain the string *bird*. Now no permutations

contain both *fish* and *bird* because of the i, and no permutations contain both *rat* and *bird* because of the r. Thus, the only possible overlap is the strings containing both *fish* and *rat*. The number of permutations containing both is equal to the number of permutations of the block *fish*, the block *rat*, and the 19 other letters, which is 21!.

Therefore, $23!+23!+24!-21!$ permutations contain at least one of these strings, so $26!-23!-23!-24!+21!$ of the permutations contain none of the given strings.

24. Let E_1 be the event that tails comes up exactly three times, E_2 be the event that the first and last flips come up tails, and E_3 be the event that the second and fourth flips come up heads. Then $p(E_1) = \binom{5}{3}/2^5 = \frac{5}{16}$; $p(E_2) = \frac{1}{4}$; $p(E_3) = \frac{1}{4}$; $p(E_1 \cap E_2) = \frac{3}{32}$; $p(E_1 \cap E_3) = \frac{1}{32}$; $p(E_2 \cap E_3) = \frac{1}{16}$; and $p(E_1 \cap E_2 \cap E_3) = \frac{1}{32}$. Therefore,

$$\begin{aligned} p(E_1 \cup E_2 \cup E_3) &= p(E_1) + p(E_2) + p(E_3) - p(E_1 \cap E_2) - p(E_1 \cap E_3) - p(E_2 \cap E_3) + p(E_1 \cap E_2 \cap E_3) \\ &= \frac{5}{16} + \frac{1}{4} + \frac{1}{4} - \frac{3}{32} - \frac{1}{32} - \frac{1}{16} + \frac{1}{32} = \frac{21}{32}. \end{aligned}$$

Section 6.6

4. Following the method of example 1 on page 458, the desired answer is

$$\binom{17+3}{3} - \binom{20-4}{3} - \binom{20-5}{3} - \binom{20-6}{3} - \binom{20-9}{3} + \binom{20-4-5}{3} + \binom{20-4-6}{3} + \\ \binom{20-4-9}{3} + \binom{20-5-6}{3} + \binom{20-5-9}{3} + \binom{20-6-9}{3} - \binom{20-4-5-6}{3} = 20$$

(where these are the only nonzero terms).

Alternately, we could use generating functions to express the answer as the coefficient of x^{17} in $(1+x+x^2+x^3)(1+x+x^2+x^3+x^4)(1+x+\cdots+x^5)(1+x+\cdots+x^8)$. If we write $1+x+x^2+x^3 = \frac{1-x^4}{1-x}$ and so on, expand the numerator in the resulting product, then write $(1-x)^{-4} = \sum_{n=0}^{\infty} \binom{n+3}{3} x^n$, we will get exactly the same expression as above. Alternately, we could just calculate $(1+x+x^2+x^3)(1+x+x^2+x^3+x^4) = 1+2x+3x^2+4x^3+4x^4+3x^5+2x^6+x^7$; multiplying this by $1+x+\cdots+x^5$ gives $1+3x+6x^2+10x^3+14x^4+17x^5+18x^6+17x^7+14x^8+10x^9+6x^{10}+3x^{11}+x^{12}$; and multiplying this by $1+x+\cdots+x^8$ gives the coefficient of x^{17} as $10+6+3+1=20$.

This calculation suggests yet another solution: if we set $y_1 = 3-x_1, y_2 = 4-x_2, y_3 = 5-x_3, y_4 = 8-x_4$, then solving $x_1+x_2+x_3+x_4 = 17$ with the given constraints is equivalent to solving $y_1+y_2+y_3+y_4 = 3$ with the constraints $y_1 \le 3, y_2 \le 4, y_3 \le 5, y_4 \le 8$. However, since the sum is 3, all these constraints are trivial, so the answer is $\binom{3+4-1}{3} = 20$.

6. Any number divisible by a square number greater than 1 is divisible by the square of a prime number, so we first count the number of integers less than 100 which are divisible by 4, 9, 25, or 49. The only possible overlap between these cases is the numbers divisible by 4 and 9, since any other product of two or more squares of primes is 100 or greater. Thus, using inclusion-exclusion,

$$\left\lfloor \frac{99}{4} \right\rfloor + \left\lfloor \frac{99}{9} \right\rfloor + \left\lfloor \frac{99}{25} \right\rfloor + \left\lfloor \frac{99}{49} \right\rfloor - \left\lfloor \frac{99}{4 \cdot 9} \right\rfloor = 24+11+3+2-2 = 38$$

numbers between 1 and 99 are divisible by 4, 9, 25, or 49. This means that $99-38 = 61$ integers less than 100 are squarefree.

8. By Theorem 1 on page 461, the answer (setting $m = 7$ and $n = 5$) is

$$5^7 - \binom{5}{1}4^7 + \binom{5}{2}3^7 - \binom{5}{3}2^7 + \binom{5}{4}1^7 = 16800.$$

Alternately, in this simple case, we can split into two cases: if some three elements of the domain get mapped to the same image, we can choose these three elements in $\binom{7}{3}$ ways, then choose the function in 5! ways.

Otherwise, there are two pairs of elements of the domain which each get mapped to the same image (but these images are different between the two pairs). There are $\binom{7}{2}\binom{5}{2}/2$ ways to choose these two pairs, then again 5! ways to choose the function. Thus, the answer is $5!\left[\binom{7}{3}+\binom{7}{2}\binom{5}{2}/2\right]=16800$.