Solutions to Homework 8, Math 55

Section 6.1

4. (a) Since $0 = -3 \cdot 0 + 4 \cdot 0$, $a_n = 0$ is obviously a solution.
   (b) Similarly, since $1 = -3 \cdot 1 + 4 \cdot 1$, $a_n = 1$ is a solution.
   (c) Plugging in, we need to prove $(-4)^n = -3(-4)^{n-1} + (-4)^{n-2}$. Factoring, this is equivalent to $16(-4)^{n-2} = 12(-4)^{n-2} + 4(-4)^{n-2}$, which is obvious.
   (d) Since $a_n = 1$ and $a_n = (-4)^n$ are solutions by the previous two parts, we calculate

$$2(-4)^n + 3 = 2[-3(-4)^{n-1} + 4(-4)^{n-2}] + 3[-3 \cdot 1 + 4 \cdot 1] = -3(2(-4)^{n-1} + 3) + 4(2(-4)^{n-2} + 3).$$

10. (a) In the nth year, the account will pay $0.09a_{n-1}$ in interest, which added to the starting balance of a_{n-1} gives $a_n = a_{n-1} + 0.09a_{n-1} = 1.09a_{n-1}$. We also have that $a_0 = \$1000$.
    (b) We easily see that $a_n = (1.09)^n \cdot \$1000$.
    (c) From the previous part, $a_{100} = (1.09)^{100} \cdot \$1000 \approx \$5,529,041$.

14. (a) We have $a_n = a_{n-1} + 1000 + (0.05)a_{n-1} = (1.05)a_{n-1} + 1000$.
    (c) We find that $a_n = -20000$ is a particular solution to this recurrence relation, and the corresponding homogeneous equation has solution $b_n = C(1.05)^n$. Thus, $a_n = C(1.05)^n - 20000$ for some C, and since $a_0 = 50000$, we see $C = 70000$, and $a_n = 70000(1.05)^n - 20000$..
    (b) Plugging in $n = 8$ to the solution from part (c), we get $a_8 \approx \$83,422$.

24. (a) The number of such bit strings ending with a 1 is a_{n-1}; the number of such bit strings ending with "10" is a_{n-2}; the number ending with "100" is a_{n-3}; and the number ending with "000" is 2^{n-3}. Therefore, $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$. (Another possible recurrence relation, gotten by splitting according to whether or not the last three digits are the first occurrence of a triple 0, is $a_n = 2a_{n-1} + (2^{n-4} - a_{n-4})$.)
    (b) We can easily see $a_1 = a_2 = 0$, and $a_3 = 1$.
    (c) Using the recurrence relation. we calculate $a_4 = a_3 + a_2 + a_1 + 2^1 = 3$; similarly, $a_5 = 8$; $a_6 = 20$; and $a_7 = 47$.

40. The number of such bit strings ending with a 1 is equal to a_{n-1}, and the number of such bit strings ending with a 0 is equal to the number of bit strings of length $n-1$ with an odd number of zeros, which is $2^{n-1} - a_{n-1}$. Thus, $a_n = a_{n-1} + (2^{n-1} - a_{n-1}) = 2^{n-1}$.

44. We calculate directly that

$$\begin{aligned} f_n &= f_{n-1} + f_{n-2} = (f_{n-2} + f_{n-3}) + f_{n-2} = 2f_{n-2} + f_{n-3} = 2(f_{n-3} + f_{n-4}) + f_{n-3} \\ &= 3f_{n-3} + 2f_{n-4} = 3(f_{n-4} + f_{n-5}) + 2f_{n-4} = 5f_{n-4} + 3f_{n-5}. \end{aligned}$$

    The given initial conditions are obvious.

    We now prove by induction that f_{5n} is divisible by 5, for $n = 1, 2, 3, \ldots$. For $n = 1$, this just says $f_5 = 5$ is divisible by 5, which is true. Now, assuming $5 \mid f_{5n}$, we have $f_{5n+5} = 5f_{5n+1} + 3f_{5n}$; since $5 \mid 5f_{5n+1}$, and $5 \mid 3f_{5n}$ using the inductive hypothesis, this implies $5 \mid f_{5n+5}$.

Section 6.2

4. (a) The characteristic equation here is $r^2 - r - 6 = 0$, which has roots $r = 3, r = -2$. Thus, $a_n = \alpha_1 3^n + \alpha_2(-2)^n$. The initial conditions give $\alpha_1 + \alpha_2 = 3$, $3\alpha_1 - 2\alpha_2 = 6$; this has solutions $\alpha_1 = \frac{12}{5}$, $\alpha_2 = \frac{3}{5}$. Thus, $a_n = \frac{1}{5}(12 \cdot 3^n + 3(-2)^n)$.
   (b) The characteristic equation is $r^2 - 7r + 10 = 0$, with roots $r = 5$, $r = 2$. Thus, $a_n = \alpha_1 5^n + \alpha_2 2^n$. The initial conditions give $\alpha_1 + \alpha_2 = 2$, $5\alpha_1 + 2\alpha_2 = 1$, which has solution $\alpha_1 = -1$, $\alpha_2 = 3$. Therefore, $a_n = 3 \cdot 2^n - 5^n$.

(c) The characteristic equation is $r^2 - 6r + 8 = 0$, with roots $r = 4, r = 2$. Thus, $a_n = \alpha_1 4^n + \alpha_2 2^n$. The initial conditions give $\alpha_1 + \alpha_2 = 4$, $4\alpha_1 + 2\alpha_2 = 10$, which has solution $\alpha_1 = 1$, $\alpha_2 = 3$. So $a_n = 4^n + 3 \cdot 2^n$.

(d) The characteristic equation is $r^2 - 2r + 1 = 0$, which has a double root $r = 1$. Thus, $a_n = \alpha_1 + \alpha_2 n$. The initial conditions give $\alpha_1 = 4$, $\alpha_1 + \alpha_2 = 1$, so $\alpha_2 = -3$. Thus, $a_n = 4 - 3n$.

(e) The characteristic equation is $r^2 - 1 = 0$, which has roots $r = \pm 1$. Thus, $a_n = \alpha_1 + \alpha_2(-1)^n$. The initial conditions give $\alpha_1 + \alpha_2 = 5$, $\alpha_1 - \alpha_2 = -1$, with solution $\alpha_1 = 2$, $\alpha_2 = 3$. Therefore, $a_n = 2 + 3(-1)^n$. Alternately, it is easy to see directly that

$$a_n = \begin{cases} 5, & \text{if } n \text{ is even;} \\ -1, & \text{if } n \text{ is odd.} \end{cases}$$

(f) The characteristic equation is $r^2 + 6r + 9 = 0$, which has a double root $r = -3$. This means $a_n = \alpha_1(-3)^n + \alpha_2 n(-3)^n$. The initial conditions give $\alpha_1 = 3$, $-3\alpha_1 - 3\alpha_2 = -3$, so $\alpha_2 = -2$. Thus, $a_n = (3 - 2n)(-3)^n$.

(g) The characteristic equation is $r^2 + 4r - 5 = 0$, which has roots $r = 1, r = -5$. Thus, $a_n = \alpha_1 + \alpha_2(-5)^n$. The initial conditions give $\alpha_1 + \alpha_2 = 2$, $\alpha_1 - 5\alpha_2 = 8$, which has solution $\alpha_1 = 3$, $\alpha_2 = -1$. Thus, $a_n = 3 - (-5)^n$.

22. The general form would be $a_n = \alpha_1(-1)^n + \alpha_2 n(-1)^n + \alpha_3 n^2(-1)^n + \beta_1 2^n + \beta_2 n 2^n + \gamma_1 5^n + \gamma_2 n 5^n + \delta_1 7^n$.

24. (a) We need to check that $n2^n = 2(n-1)2^{n-1} + 2^n$. Dividing both sides by 2^{n-1}, this is equivalent to proving $2n = 2(n-1) + 2$, which is clear.

    (b) The general solution to the associated homogeneous equation $b_n = 2b_{n-1}$ is $b_n = \alpha 2^n$. Thus, the general solution to the original equation is $a_n = n2^n + \alpha 2^n$.

    (c) Since $a_0 = \alpha$, we get $\alpha = 2$, so $a_n = (n+2)2^n$.

36. We have

$$a_n = \sum_{k=1}^{n} k^2 = \left(\sum_{k=1}^{n-1} k^2\right) + n^2 = a_{n-1} + n^2$$

and

$$a_1 = \sum_{k=1}^{1} k^2 = 1^2 = 1.$$

To solve this, we note that the associated homogeneous equation is $b_n = b_{n-1}$, where the characteristic equation is $r - 1 = 0$. Thus, the solutions to this are $b_n = C$ for some constant C. Now, for a particular solution, since the equation is equivalent to $a_n - a_{n-1} = n^2 \cdot 1^n$, and $r = 1$ is a single root of the characteristic equation, we look for a solution of the form $c_n = \alpha n^3 + \beta n^2 + \gamma n$. Plugging in, we need

$$\begin{aligned} \alpha n^3 + \beta n^2 + \gamma n &= \alpha(n-1)^3 + \beta(n-1)^2 + \gamma(n-1) + n^2 \\ &= \alpha n^3 + (-3\alpha + \beta + 1)n^2 + (3\alpha - 2\beta + \gamma)n + (-\alpha + \beta - \gamma). \end{aligned}$$

Equating coefficients gives $\alpha = \alpha$, $\beta = -3\alpha + \beta + 1$, $\gamma = 3\alpha - 2\beta + \gamma$, and $-\alpha + \beta - \gamma = 0$, which has solution $\alpha = \frac{1}{3}$, $\beta = \frac{1}{2}$, and $\gamma = \frac{1}{6}$.

Therefore, in general, $a_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n + C$. The initial condition $a_1 = 1$ implies $C = 0$, so $\sum_{k=1}^{n} k^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n + C = \frac{2n^3+3n^2+n}{6} = \frac{n(n+1)(2n+1)}{6}$.

40. By substituting $n-1$ for n in the first equation, we get $a_{n-1} = 3a_{n-2} + 2b_{n-2}$. Also, we have

$$a_n = 3a_{n-1} + 2b_{n-1} = 3(3a_{n-2} + 2b_{n-2}) + 2(a_{n-2} + 2b_{n-2}) = 11a_{n-2} + 10b_{n-2}.$$

Now, to eliminate b_{n-2}, we calculate

$$a_n - 5a_{n-1} = (11a_{n-2} + 10b_{n-2}) - 5(3a_{n-2} + 2b_{n-2}) = -4a_{n-2}.$$

Thus, $a_n - 5a_{n-1} + 4a_{n-2} = 0$. Also, we have initial conditions $a_0 = 1$ and $a_1 = 3a_0 + 2b_0 = 7$.

We now solve this in the usual way: the characteristic equation is $r^2 - 5r + 4 = 0$, which has roots $r = 1$ and $r = 4$, so $a_n = \alpha_1 + \alpha_2 4^n$. The initial conditions give $\alpha_1 + \alpha_2 = 1$ and $\alpha_1 + 4\alpha_2 = 7$, so $\alpha_1 = -1$ and $\alpha_2 = 2$. Thus, $a_n = 2 \cdot 4^n - 1$. Finally, we have $2b_{n-1} = a_n - 3a_{n-1} = (8 \cdot 4^{n-1} - 1) - 3(2 \cdot 4^{n-1} - 1) = 2 \cdot 4^{n-1} + 2$, so in other words, $b_{n-1} = 4^{n-1} + 1$, which implies $b_n = 4^n + 1$.

Section 6.3

8. (a) We have $f(2) = 2f(1) + 3 = 13$.
   (b) Since $f(4) = 2f(2) + 3 = 29$, then $f(8) = 2f(4) + 3 = 61$.
   (c) Similarly, $f(16) = 2f(8) + 3 = 125$; $f(32) = 2f(16) + 3 = 253$; and $f(64) = 2f(32) + 3 = 509$.
   (d) We continue in this way to get $f(128) = 1021$, $f(256) = 2045$, $f(512) = 4093$, and $f(1024) = 8189$.

   (Alternately, we could use the method from the following problems to find $f(n) = 8n - 3$ for $n = 2^k$.)

12. Since $f(3^k) = 2f(3^{k-1}) + 4$, letting $a_k = f(3^k)$ gives $a_k = 2a_{k-1} + 4$, with initial condition $a_0 = f(3^0) = 1$. Solving this recurrence using the methods of the previous section gives $a_k = 5 \cdot 2^k - 4$. Thus, for $n = 3^k$, $f(n) = 5 \cdot 2^k - 4 = 5 \cdot 2^{\log_3 n} - 4 = 5 \cdot n^{\log_3 2} - 4$.

14. Since a tournament for n players consists of one round followed by a tournament for $n/2$ players, we get $f(n) = f(n/2) + 1$. Also, $f(1) = 0$, since no matches are necessary when we start with only 1 player.

16. Letting $a_k = f(2^k)$, we get $a_k = a_{k-1} + 1$ with $a_0 = 0$. This easily has solution $a_k = k$, so for $n = 2^k$, $f(n) = k = \log_2 n$.