

Solutions to Homework 7, Math 55

1. The probability that Beatrix starts with the crown and throws it to Andrew is $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$; similarly, the probability that Charles starts with the crown and throws it to Andrew is $\frac{1}{6}$. Therefore, the probability that Andrew has the crown after one turn is $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$. Similarly, the probability that Charles has the crown after one turn is $\frac{1}{3}$ from Andrew, and $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$ from Beatrix, so the total probability is $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$. On the other hand, if Beatrix gets the crown, it must be from Charles; therefore, the probability that Beatrix has the crown after one turn is $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$.

2. We have $p(A) = \frac{4}{52} = \frac{1}{13}$; $p(B) = \frac{13}{52} = \frac{1}{4}$; and $p(C) = \frac{26}{52} = \frac{1}{2}$. Now $p(A \cap B) = \frac{1}{52} = p(A)p(B)$, and $p(A \cap C) = \frac{2}{52} = p(A)p(C)$, but $p(B \cap C) = \frac{13}{52} \neq p(B)p(C)$. Thus, A and B are independent, and so are A and C , but B and C are not independent.

If we add a joker to the deck, then $p(A) = \frac{4}{53}$; $p(B) = \frac{13}{53}$; and $p(C) = \frac{26}{53}$. However, $p(A \cap B) = \frac{1}{53} \neq p(A)p(B)$; $p(A \cap C) = \frac{2}{53} \neq p(A)p(C)$; and $p(B \cap C) = \frac{13}{53} \neq p(B)p(C)$. Thus, in this case, no pair of events from A, B, C is independent.

3. We have $p(A' \cap B') = p(S - (A \cup B)) = 1 - p(A \cup B) = 1 - (p(A) + p(B) - p(A \cap B)) = 1 - p(A) - p(B) + p(A \cap B) = (1 - p(A))(1 - p(B)) = p(A')p(B')$. Therefore, A' and B' are independent.

Similarly, $p(A \cap B') = p(A - (A \cap B)) = p(A) - p(A \cap B) = p(A) - p(A)p(B) = p(A)(1 - p(B)) = p(A)p(B')$, so A and B' are independent. (Note that in the step $p(A - (A \cap B)) = p(A) - p(A \cap B)$, we need to use the fact that $A \cap B \subseteq A$.) However, if A and A' are independent, then $p(A)p(A') = p(A \cap A') = p(\emptyset) = 0$, which implies $p(A) = 0$ or $p(A') = 0$, and in the latter case, $p(A) = 1 - p(A') = 1$. Thus, A and A' are not independent unless $p(A) = 0$ or $p(A) = 1$.

4. Let E be the event that the number is divisible by 3, and F be the event that the number is divisible by 5. Then $p(E) = \frac{300}{900} = \frac{1}{3}$; and $p(F) = \frac{180}{900} = \frac{1}{5}$. Also, $E \cap F$ is the event that the number is divisible by 15, which has probability $\frac{60}{900} = \frac{1}{15} = p(E)p(F)$. Therefore, the two given events are independent.

By the previous problem, this implies that E' and F' are independent, so the probability of the event that the number has no factor in common with 15 (which is just $E' \cap F'$) is $p(E')p(F') = \frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}$.

5. The probability that the value is \$5000 is $\frac{1}{10000}$; the probability that the value is \$700 is $\frac{1}{10000}$; the probability that the value is \$100 is $\frac{3}{10000}$; and the probability that the value is \$0 is $\frac{9995}{10000}$. Therefore, the expected value of the ticket is

$$\frac{1}{10000} \cdot \$5000 + \frac{1}{10000} \cdot \$700 + \frac{3}{10000} \cdot \$100 + \frac{9995}{10000} \cdot \$0 = \$0.60.$$

(Thus, if you include the purchase price in the value, then the expected value is $-\$0.40$.)

6. (a) Let R_i be the expected number of red balls we get if we draw from the i th box, and similarly let W_i be the expected number of white balls we get from the i th box. Then we see that $R_1 = W_3$, $R_2 = W_1$, and $R_3 = W_2$; also, by symmetry, $R_4 = W_4$. Since the expected number of red balls overall is $\frac{1}{4}(R_1 + R_2 + R_3 + R_4) = \frac{1}{4}(W_3 + W_1 + W_2 + W_4)$, this is the same as the expected number of white balls overall.

- (b) Let R, W, B be the random variables counting the number of red, white, and blue balls drawn, respectively. The previous part shows that $E(R) = E(W)$; similarly, $E(R) = E(B)$, so all three have the same expected value. However, we always have $R + W + B = 12$, so $E(R) + E(W) + E(B) = 12$. Therefore, $3E(B) = 12$, so $E(B) = 4$.

7. (a) Let $a \in \{0, 1, 2\}$ and $b \in \{0, 1, 2, 3\}$ be any two possible values for f and g , respectively. Then $p(f = a) = \frac{1}{3}$ since exactly 300 numbers in S are congruent to $a \pmod{3}$, and similarly, $p(g = b) = \frac{1}{4}$. On the other hand, by the Chinese Remainder Theorem, for any values of a and b there is a unique solution c up to congruence $\pmod{12}$ to the system $c \equiv a \pmod{3}$ and $c \equiv b \pmod{4}$. Therefore, the event $f = a \wedge g = b$ is exactly the same as the event that the number is congruent to $c \pmod{12}$. Since exactly $\frac{900}{12} = 75$ of the numbers in S satisfy this condition, $p(f = a \wedge g = b) = \frac{1}{12} = p(f = a)p(g = b)$.

(b) We have $p(f = 0) = p(f = 1) = p(f = 2) = \frac{1}{3}$, so $E(f) = \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 2 = 1$. Similarly, $E(g) = \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 3 = \frac{3}{2}$. Therefore, $E(f + g) = E(f) + E(g) = \frac{5}{2}$. Also, since f and g are independent, $E(fg) = E(f)E(g) = \frac{3}{2}$.
 Also, $E(f^2) = \frac{1}{3} \cdot 0^2 + \frac{1}{3} \cdot 1^2 + \frac{1}{3} \cdot 2^2 = \frac{5}{3}$, so $V(f) = E(f^2) - E(f)^2 = \frac{2}{3}$. Similarly, $E(g^2) = \frac{1}{4} \cdot 0^2 + \frac{1}{4} \cdot 1^2 + \frac{1}{4} \cdot 2^2 + \frac{1}{4} \cdot 3^2 = \frac{7}{2}$, so $V(g) = E(g^2) - E(g)^2 = \frac{5}{4}$. Thus, since f and g are independent, $V(f + g) = V(f) + V(g) = \frac{2}{3} + \frac{5}{4} = \frac{23}{12}$.

8. This problem can be reduced to a series of 9 Bernoulli trials, with the probability of success (having an emale) on each trial being $p = \frac{1}{3}$. Therefore, the expected number of emales is $np = 9 \cdot \frac{1}{3} = 3$, and the variance is $npq = 9 \cdot \frac{1}{3} \cdot \frac{2}{3} = 2$.

9. Let X denote the number of complete novels the person can read. There are $\binom{10}{5} = 252$ ways to choose the books. The only way to get no complete novels is to choose one volume from each novel; there are $2^5 = 32$ ways to do this, so $p(X = 0) = \frac{32}{252} = \frac{8}{63}$.

If $X = 1$, then there are 5 ways to choose which novel is complete, then $\binom{4}{3}$ ways to choose which novels are incomplete, and 2^3 ways to choose one volume from each of these. Thus, $p(X = 1) = 5 \cdot \binom{4}{3} \cdot 2^3 / 252 = \frac{160}{252} = \frac{40}{63}$. Finally, if $X = 2$, then there are $\binom{5}{2}$ ways to choose which two novels are complete, then 6 ways to choose the remaining volume. Thus, $p(X = 2) = \binom{5}{2} \cdot 6 / 252 = \frac{60}{252} = \frac{5}{21}$.

Therefore, $E(X) = 0 \cdot \frac{8}{63} + 1 \cdot \frac{40}{63} + 2 \cdot \frac{5}{63} = \frac{70}{63} = \frac{10}{9}$. Also, $E(X^2) = 0^2 \cdot \frac{8}{63} + 1^2 \cdot \frac{40}{63} + 2^2 \cdot \frac{5}{63} = \frac{100}{63}$; thus, $V(X) = \frac{100}{63} - (\frac{10}{9})^2 = \frac{200}{567}$.

(An alternate way to calculate $E(X)$ is to let $X_i = 1$ if the i th novel is complete, and $X_i = 0$ otherwise. Then $E(X_i)$ is the probability that novel i is complete, which is $\binom{8}{3} / \binom{10}{5} = \frac{2}{9}$, so $E(X) = E(X_1) + \dots + E(X_5) = \frac{10}{9}$. However, since these random variables are not independent, it is more difficult to calculate $V(X)$ using this method.)

10. We calculate $E(f^2) = \sum_{n=0}^{\infty} n^2 p(f = n) = p \sum_{n=0}^{\infty} n^2 q^{n-1}$. However, from the formula $\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$ which was used in Example 8, we see $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$. Differentiating this formula gives

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{d}{dx} \left(\frac{x}{(1-x)^2} \right) = \frac{(1-x)^2 \cdot 1 - x \cdot (-2(1-x))}{(1-x)^4} = \frac{1+x}{(1-x)^3}.$$

Therefore, $E(f^2) = p \cdot \frac{1+q}{(1-q)^3} = \frac{p(1+q)}{p^3} = \frac{1+q}{p^2}$. Thus, $V(f) = E(f^2) - E(f)^2 = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$.

11. Let a be the probability that the value of f is even, and b be the probability that the value of f is odd. Then $a + b = 1$. On the other hand, using the binomial theorem, we see that

$$\begin{aligned} (q-p)^n &= q^n - \binom{n}{1} q^{n-1} p + \binom{n}{2} q^{n-2} p^2 - \dots + (-1)^n p^n \\ &= p(f = 0) - p(f = 1) + p(f = 2) - \dots + (-1)^n p(f = n) = a - b. \end{aligned}$$

Solving $a + b = 1$ and $a - b = (q - p)^n$ simultaneously for a and b , we get $a = \frac{1}{2} + \frac{1}{2}(q - p)^n$, and $b = \frac{1}{2} - \frac{1}{2}(q - p)^n$.

12. Let $X_i = 1$ if the i th child gets a letter which matches his first name. Then for each i , $E(X_i)$ is the probability the i th child gets an appropriate letter, which is $\frac{2}{52} = \frac{1}{26}$. Now by linearity, the expected number of children who get an appropriate letter is $\sum_{i=1}^{52} E(X_i) = 52 \cdot \frac{1}{26} = 2$.

However, without more information, we cannot calculate the variance. For example, if every child's name starts with "A", then there are always exactly two children who get an appropriate letter, so the variance is 0. On the other hand, if for each letter there are two children with that first initial, then it would be possible for all 52 children to get an appropriate letter; in this case, the random variable is not always equal to its expected value of 2, so the variance is nonzero.

13. By Example 12, if f is the random variable counting the number of elements in a random subset of $\{1, \dots, 100\}$, then $E(f) = 50$ and $V(f) = \frac{1}{4} \cdot 100 = 25$. Therefore, by Chebyshev's inequality, $p(|f - 50| \geq 10) \leq V(f)/10^2 = \frac{1}{4}$. Thus, $p(|f - 50| < 10) = 1 - p(|f - 50| \geq 10) \geq \frac{3}{4}$; and the event $|f - 50| < 10$ is exactly the desired event that the number of elements in the subset is between 41 and 59 (inclusive).
16. We have $E(f) = 1 \cdot p(f = 1) + 0 \cdot p(f = 0) = 1 \cdot P(A) + 0 \cdot P(A') = P(A)$. Since f only has values 0 and 1, $f^2 = f$, so $E(f^2) = E(f) = P(A)$. Thus, $V(f) = E(f^2) - E(f)^2 = P(A) - P(A)^2 = P(A)(1 - P(A)) = P(A)P(A')$.
20. This is just a reformulation of the Monty Hall problem; so by the same reasoning, you should switch cups.