Solutions to Homework 6, Math 55

Section 5.1

6. There are 4 aces, 13 hearts, and one ace of hearts; thus, by inclusion-exclusion, $4 + 13 - 1 = 16$ cards are aces or hearts. Therefore, the probability a random card is an ace or a heart is $\frac{16}{52} = \frac{4}{13}$.

10. There are $\binom{52}{5}$ total combinations of cards, out of which $\binom{50}{3}$ contain the two of diamonds and the three of spades. Therefore, the probability a hand has these two cards is $\binom{50}{3}/\binom{52}{5} = \frac{19600}{2598960} = \frac{5}{663}$.

16. For each suit, there are $\binom{13}{5}$ flushes in that suit, so there are $4 \cdot \binom{13}{5}$ hands which are flushes. Thus, the probability of a flush is $4 \cdot \binom{13}{5}/\binom{52}{5} = \frac{5148}{2598960} = \frac{13}{16660}$.

22. Out of the 100 positive integers not exceeding 100, $\lfloor \frac{100}{3} \rfloor = 33$ are divisible by 3; thus, the probability such an integer is divisible by 3 is $\frac{33}{100}$.

28. There are $\binom{80}{7}$ possible tickets; out of these, $\binom{11}{7}$ are winners. Therefore, the probability of winning is $\binom{11}{7}/\binom{80}{7} = 3/28879240$.

    Alternately, if we treat the ticket as being fixed, then there are $\binom{80}{11}$ ways for the lottery commissioners to choose the 11 winning numbers. Out of these, there are $\binom{73}{4}$ ways for them to choose your 7 numbers and 4 others. Therefore, the probability of winning is $\binom{73}{4}/\binom{80}{11} = 3/28879240$.

36. When two dice are rolled, there are 5 ways to get a total of 8; thus, the probability of getting 8 is $\frac{5}{36}$. When three dice are rolled, the number of ways to get a total of 8 is the same as the number of solutions to $x_1 + x_2 + x_3 = 5$, where x_i is one less than the number on the ith die. This is $\binom{5+3-1}{5} = \binom{7}{2} = 21$, so the probability of getting a total of 8 is $\frac{21}{6^3} = \frac{7}{72}$. (Note that since the sum was 5, we didn't have to worry about the fact that each $x_i \leq 5$.) Since $\frac{5}{36} > \frac{7}{72}$, getting a total of 8 is more likely with two dice than with three.

38. (a) We have $p(E_1) = \frac{4}{8} = \frac{1}{2}$; $p(E_2) = \frac{1}{2}$; and $p(E_1 \cap E_2) = \frac{2}{8} = \frac{1}{4}$. Therefore, since $\frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2}$, E_1 and E_2 are independent.

    (b) We have $p(E_1) = \frac{1}{2}$; $p(E_2) = 2/2^3 = \frac{1}{4}$; and $p(E_1 \cap E_2) = \frac{1}{8}$. Since $\frac{1}{8} = \frac{1}{2} \cdot \frac{1}{4}$, E_1 and E_2 are independent.

    (c) We have $p(E_1) = \frac{1}{2}$; $p(E_2) = \frac{2}{8} = \frac{1}{4}$; and $p(E_1 \cap E_2) = 0$. Since $0 \neq \frac{1}{2} \cdot \frac{1}{4}$, E_1 and E_2 are not independent.

40. If you do not change, there is a probability of $\frac{1}{4}$ that you chose the correct door.

    If you do change, then you win by first choosing an incorrect door, then switching to the correct door out of the two left. The probability of choosing an incorrect door first is $\frac{3}{4}$, and then the probability of switching to the correct door is $\frac{1}{2}$. Therefore, with this strategy the probability of winning is $\frac{3}{8}$.

Section 5.2

2. Let p be the probability of getting a 1, so that the probability of getting each number other than 3 is p, and the probability of getting a 3 is $2p$. Then $p + p + 2p + p + p + p = 7p = 1$, so $p = \frac{1}{7}$. Thus, the probability of each number other than 3 is $\frac{1}{7}$, and the probability of 3 is $\frac{2}{7}$.

12. Since $E \subseteq E \cup F$, $p(E \cup F) \geq p(E) = 0.8$. On the other hand, by rewriting the inclusion-exclusion rule, we get $p(E \cap F) = p(E) + p(F) - p(E \cup F)$. Since $p(E \cup F) \leq 1$, this means $p(E \cap F) \geq 0.8 + 0.6 - 1 = 0.4$.

14. First, for $n = 1$, the statement just says $p(E_1) \geq p(E_1)$, which is obvious. For the case $n = 2$, as in the previous problem, we have $p(E_1 \cap E_2) = p(E_1) + p(E_2) - p(E_1 \cup E_2) \geq p(E_1) + p(E_2) - 1$.

Now, for the inductive step, assume $p(E_1 \cap E_2 \cap \cdots \cap E_n) \geq p(E_1) + p(E_2) + \cdots + p(E_n) - (n-1)$. Then

$$\begin{aligned} p(E_1 \cap \cdots \cap E_n \cap E_{n+1}) &\geq p(E_1 \cap \cdots \cap E_n) + p(E_{n+1}) - 1 \\ &\geq (p(E_1) + \cdots + p(E_n) - (n-1)) + p(E_{n+1}) - 1 \\ &= p(E_1) + \cdots + p(E_n) + p(E_{n+1}) - n. \end{aligned}$$

(Here, for the first step, we used the result of the previous paragraph, and for the second step, we used the inductive hypothesis.) This completes the induction.

18. (a) There are 7^2 possibilities for which days of the week the two people were born, out of which 7 have them born on the same day. Therefore, the probability they were born on the same day of the week is $\frac{7}{7^2} = \frac{1}{7}$.
    (b) There are 7^n possibilities for which days the n people were born; out of these, $P(7, n)$ have them all born on different days. Therefore, $7^n - P(7, n)$ of the possibilities have at least two born on the same day of the week, so the probability this happens is $1 - P(7, n)/7^n$.
    (c) We calculate the probability for $n = 1$ is 0; for $n = 2$ is $\frac{1}{7}$; for $n = 3$ is $\frac{19}{49}$; and for $n = 4$ is $\frac{223}{343}$. So 4 people are needed to make the probability greater than $\frac{1}{2}$.

24. The probability that the first flip is tails is $\frac{1}{2}$. The probabilility that the first flip is tails and that exactly four heads appear is $\frac{1}{32}$. Therefore, the desired conditional probability is $\frac{1}{32}/\frac{1}{2} = \frac{1}{16}$.

28. (a) There are $\binom{5}{3} = 10$ ways to get three boys, and the probability of each is $(0.51)^3(0.49)^2$. Thus, the probability is $10(0.51)^3(0.49)^2 \approx 0.318$.
    (b) The probability of getting 5 girls is $(0.49)^5$; thus, the probability of getting at least one boy is $1 - (0.49)^5 \approx 0.972$.
    (c) The probability of getting 5 boys is $(0.51)^5$; thus, the probability of getting at least one girl is $1 - (0.51)^5 \approx 0.965$.
    (d) The answer is $(0.51)^5 + (0.49)^5 \approx 0.063$.

38. (a) There are 6 ways for the first die to be 6, 6 ways for the second die to be 6, and 1 way for both to be 6. Thus, the probability that at least one die is 6 is $\frac{6+6-1}{36} = \frac{11}{36}$. On the other hand, the probability that the total is 7 and at least one die is 6 is $\frac{2}{36}$. Therefore, the conditional probability is $\frac{2}{36}/\frac{11}{36} = \frac{2}{11}$.
    (b) Again, the probability of getting at least one 5 is $\frac{11}{36}$, and the probability that the total is 7 and at least one die is 5 is $\frac{2}{36}$, so the desired conditional probability is $\frac{2}{11}$.

    (In another interpretation, the question could possibly be saying that the observer tells you that at least one die is 6, and that at least one die is 5. In this case, we know the total is 11, so the conditional probability would be 0.)

Section 5.3

6. The probability of winning is $1/\binom{50}{6} = 1/15890700$. Thus, the probability that the value is 0 is $\frac{15890699}{15890700}$, and the probability that the value is \$10,000,000 is $\frac{1}{15890700}$. We thus calculate that the expected value of the ticket is $0 \cdot \frac{15890699}{15890700} + 10000000 \cdot \frac{1}{15890700} = \frac{10000000}{15890700} \approx \$0.63$. (This means that the expected loss from buying a ticket is about 37 cents.)

16. We have, for example, that $p(X = 0) = \frac{1}{4}$, and $p(Y = 0) = \frac{1}{4}$. However, $p(X = 0 \wedge Y = 0) = 0 \neq \frac{1}{4} \cdot \frac{1}{4}$. Thus, X and Y are not independent.

24. Let $X_i = 1$ if a 6 appears on the ith roll, and $X_i = 0$ otherwise; then we need to calculate $V(X_1 + \cdots + X_{10})$. However, it is easy to see that the X_i are pairwise independent, so the answer is equal to $V(X_1) + V(X_2) + \cdots + V(X_{10})$.

    Now for each i, we have $E(X_i)$ is the probability that a 6 appears on the ith roll, which is $\frac{1}{6}$. On the other hand, $E(X_i^2) = 1^2 \cdot \frac{1}{6} + 0^2 \cdot \frac{1}{6} = \frac{1}{6}$, so $V(X_i) = E(X_i^2) - E(X_i)^2 = \frac{1}{6} - \frac{1}{36} = \frac{5}{36}$. Therefore, the desired answer is $10 \cdot \frac{5}{36} = \frac{25}{18}$.

    (This is just a concrete example of the reasoning in example 18 on page 390. The result of that example directly gives the answer as $10 \cdot \frac{1}{6} \cdot \frac{5}{6}$.)

26. Suppose, for example, that $X = Y$ but X is not a constant random variable, so $V(X) \neq 0$. Then $V(X+Y) = V(2X) = E(4X^2) - E(2X)^2 = 4E(X^2) - 4E(X)^2 = 4V(X) \neq 2V(X) = V(X) + V(Y)$.

32. The total probability that x is in the list is $\sum_{i=1}^{n} \frac{i}{n(n+1)} = \frac{1}{n(n+1)} \cdot \frac{n(n+1)}{2} = \frac{1}{2}$. Thus, the probability that x is not in the list is $\frac{1}{2}$. In this case, according to example 8 on page 384, the linear search uses $2n+2$ comparisons; otherwise, if x is the ith element of the list, then the linear search uses $2i+1$ comparisons.

Thus, the average number of comparisons is

$$\begin{aligned}
\frac{1}{2}(2n+2) + \sum_{i=1}^{n}(2i+1) \cdot \frac{i}{n(n+1)} &= (n+1) + \frac{1}{n(n+1)} \sum_{i=1}^{n}(2i^2+i) \\
&= (n+1) + \frac{1}{n(n+1)} \cdot \left(2 \cdot \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right) \\
&= (n+1) + \frac{2n+1}{3} + \frac{1}{2} = \frac{10n+11}{6}.
\end{aligned}$$