Solutions to Homework 5, Math 55

Section 4.1

6. By the product rule, there are $4 \cdot 6 = 24$ such routes.

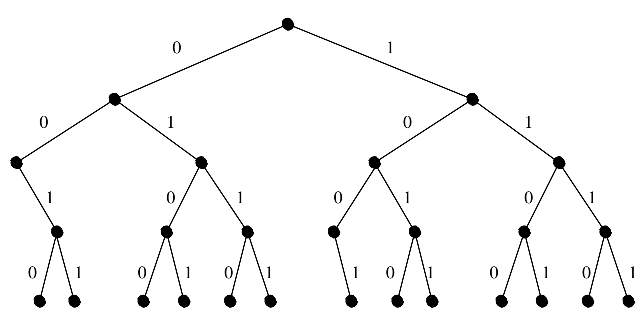
18. (a) The positive integers less than 1000 which are divisible by 7 are $7 \cdot 1, 7 \cdot 2, \ldots, 7 \cdot \lfloor \frac{999}{7} \rfloor$. Therefore, $\lfloor \frac{999}{7} \rfloor = 142$ numbers less than 1000 are divisible by 7.
    (b) The numbers less than 1000 which are divisible by both 7 and 11 are the ones divisible by 77; therefore, there are $\lfloor \frac{999}{77} \rfloor = 12$ of these. Subtracting, $142 - 12 = 130$ of the numbers divisible by 7 are not divisible by 11.
    (c) By the previous part, 12 are.
    (d) There are $\lfloor \frac{999}{11} \rfloor = 90$ multiples of 11 less than 1000. Therefore, by inclusion-exclusion, $142 + 90 - 12 = 220$ are divisible by either 7 or 11.
    (e) By part (b), there are 130 multiples of 7 not divisible by 11. Similarly, there are $90 - 12 = 78$ multiples of 11 not divisible by 7. Therefore, $130 + 78 = 208$ are divisible by exactly one of 7 and 11.
    (f) Since 220 are divisible by either 7 or 11 by part (d), $999 - 220 = 779$ are divisible by neither 7 nor 11.
    (g) There are 9 such 1-digit numbers. For a 2-digit number, we can choose the first digit to be 1 through 9, then we have 9 choices for the second digit; thus, there are $9 \cdot 9 = 81$ such 2-digit numbers. Similarly, there are $9 \cdot 9 \cdot 8 = 648$ such 3-digit numbers. In total, there are $9 + 81 + 648 = 738$ such numbers less than 1000.
    (h) There are 4 such 1-digit numbers. For a 2-digit number, if we choose one of the 4 possible even digits for the first digit, there are 4 possibilities for the second digit; if we choose one of the 5 possible odd digits for the first digit, there are 5 possibilities for the second digit. Therefore, there are $4 \cdot 4 + 5 \cdot 5 = 41$ such 2-digit numbers. Similarly, for a three-digit number, if the first digit is even (4 choices), then there are 4 choices for the third digit, then 8 choices for the second digit. If the first digit is odd (5 choices), then there are 5 choices for the third digit, then 8 choices for the second digit. Therefore, there are $4 \cdot 4 \cdot 8 + 5 \cdot 5 \cdot 8 = 328$ such 3-digit numbers. In total, there are $4 + 41 + 328 = 373$ such numbers less than 1000.

22. (a) Since there are 10 ways to choose the first digit, then 9 ways to choose the second, and so on, there are $10 \cdot 9 \cdot 8 \cdot 7 = 5040$ strings of four decimal digits which do not contain the same digit twice.
    (b) There are 10 ways to choose each of the first three digits, and 5 ways to choose the last digit as even. Therefore, there are $10^3 \cdot 5 = 5000$ such strings.
    (c) There are 4 ways to choose which digit is *not* a 9, then 9 ways to choose that digit. Therefore, there are $4 \cdot 9 = 36$ such strings.

40. Out of these bit strings, $2^5 = 32$ start with two 0's; $2^4 = 16$ end with three 1's; and $2^2 = 4$ start with two 0's and end with three 1's. Therefore, by inclusion-exclusion, $2^5 + 2^4 - 2^2 = 44$ start with two 0's or end with three 1's.

44. By inclusion-exclusion, there are $38 + 23 - 7 = 54$ total students.

48. We get the following tree:



Counting the nodes on the fourth level, we get 13 bit strings of length 4 with no three consecutive 0s.

58. (a) The largest possible value of a 16-bit string is $2^{16} - 1 = 65535$.

    (b) The largest possible value of a 4-bit string is $2^4 - 1 = 15$. Since each 32-bit block is 4 octets, the maximum total header length is $15 \cdot 4 = 60$.

    (c) Since the header is at least 20 octets long, and the total length of the datagram is at most 65535, the maximum possible length of the data area is $65535 - 20 = 65515$.

    (d) If the datagram is as long as possible, then there are $65515 \cdot 8 = 524120$ bits in the data area, so there are 2^{524120} possible contents for the data area.

Section 4.2

6. Since there are only d possible remainders r with $0 \le r < d$, some two of the $d+1$ integers must give the same remainder by the pigeonhole principle.

8. There are $|S|$ elements of S which get mapped to $|T|$ elements of T. Therefore, by the pigeonhole principle, if $|S| > |T|$ then some two elements of S get mapped by f to the same element of T.

12. Since there are 25 possible values for $(a \,\mathbf{mod}\, 5, b \,\mathbf{mod}\, 5)$, we need 26 pairs (a, b) to ensure that some value of $(a \,\mathbf{mod}\, 5, b \,\mathbf{mod}\, 5)$ gets hit twice.

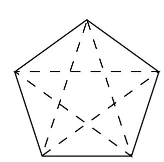
20. As in the text, denote by i_k the length of the longest increasing subsequence starting at a_k, and by d_k the length of the longest decreasing subsequence starting at a_k. Then, starting from the end and working backwards, we can calculate i_k recursively by setting $i_k = 1 + \max\{i_j : j > k \wedge a_j \ge a_k\}$, or $i_k = 1$ if $a_j < a_k$ whenever $j > k$. We have a similar construction for d_k. We can thus construct the following table of values for i_k and d_k:

i_k	2	5	4	4	1	3	2	1	2	1
a_k	22	5	7	2	23	10	15	21	3	17
d_k	3	2	2	1	3	2	2	2	1	1

Therefore, the longest increasing subsequence is 5 long, and the longest decreasing subsequence is 3 long. An example of an increasing subsequence of length 5 is $5, 7, 10, 15, 21$, and an example of a decreasing subsequence of length 3 is $22, 5, 2$.

24. Suppose 5 people are sitting around a circular table; and each person is friends with the two people next to them, but enemies of the other two people. Then each person has only two friends, who are enemies of each other, so there are no three mutual friends. Similarly, each person has only two enemies, who are sitting next to each other and are thus friends; so there are no three mutual enemies.

    The diagram below shows this arrangement. Here, the vertices of the pentagon represent the 5 people; solid lines represent a pair of friends; and dashed lines represent a pair of enemies.



(In fact, this is essentially the only example. By the reasoning in example 13 on page 318, if any person has either three friends or three enemies, then there is a group of 3 mutual friends or a group of 3 mutual enemies. So for any example to work, each person must have 2 friends and 2 enemies; also, each person's two friends must be enemies, and each person's two enemies must be friends. Given this, it's not hard to see that given any example, there's a way to seat the 5 people around a table so that each person is friends with the two people next to them.)

Section 4.3

12. (a) There are $\binom{12}{3} = 220$ ways to choose which 3 bits are 1.

    (b) There are $\binom{12}{0}$ bit strings with no 1s, $\binom{12}{1}$ bit strings with exactly 1 1, $\binom{12}{2}$ bit strings with exactly 2 1s, and $\binom{12}{3}$ bit strings with exactly 3 1s. Therefore, there are $\binom{12}{0}+\binom{12}{1}+\binom{12}{2}+\binom{12}{3} = 1+12+66+220 = 299$ bit strings with at most three 1s.

    (c) Similarly to the previous part, there are $\binom{12}{0} + \binom{12}{1} + \binom{12}{2} = 79$ bit strings with at most two 1s; and there are $2^{12} = 4096$ total bit strings. Therefore, there are $2^{12} - 79 = 4017$ bit strings with at least three 1s.

    (d) There are $\binom{12}{6} = 924$ ways to choose which 6 bits are 1, then the rest must be 0.

28. There are $\binom{40}{17}$ ways to choose which 17 of the answers are true.

32. (a) There are $26^6 = 308915776$ total strings of length 6, of which $25^6 = 244140625$ have no a. Therefore, $26^6 - 25^6 = 64775151$ strings have at least one a.

    (b) Similarly to the previous part, 25^6 strings of length 6 have no b. Also, 24^6 strings of length 6 have no a and no b. Therefore, by inclusion-exclusion, $25^6 + 25^6 - 24^6 = 297178274$ strings of length 6 either have no a or have no b. Thus, $26^6 - 2 \cdot 25^6 + 24^6 = 11737502$ strings of length 6 have both a and b.

    (c) There are 5 possible positions for the ab; then there are $P(24, 4) = 24 \cdot 23 \cdot 22 \cdot 21$ ways to choose the remaining 4 letters. Thus, the answer is $5 \cdot 24 \cdot 23 \cdot 22 \cdot 21 = 1275120$.

    (d) There are $\binom{6}{2} = 15$ ways to choose where the a and b go; then, as before, there are $24 \cdot 23 \cdot 22 \cdot 21$ ways to choose the rest of the letters. Thus, the answer is $15 \cdot 24 \cdot 23 \cdot 22 \cdot 21 = 3825360$.

Section 4.4

8. In order to get x^8y^9 in the binomial expansion of $(3x+2y)^{17}$, we must take $\binom{17}{8}(3x)^8(2y)^9 = \binom{17}{8} \cdot 3^8 \cdot 2^9 x^8 y^9$. Thus, the coefficient is $\binom{17}{8} \cdot 3^8 \cdot 2^9 = 81662929920$.

10. We have $(x + 1/x)^{100} = \sum_{m=0}^{100} \binom{100}{m} x^m (1/x)^{100-m} = \sum_{m=0}^{100} \binom{100}{m} x^{2m-100}$. Therefore, if k is odd, or if $k < -100$ or $k > 100$, then the coefficient of x^k is 0. Otherwise, to get $2m - 100 = k$ we need $m = \frac{k+100}{2} = \frac{k}{2} + 50$. Plugging this in, we see the coefficient of x^k in this case is $\binom{100}{k/2+50}$.

28. (a) Suppose we want to choose 2 people out of a group of n men and n women. Then since there are $2n$ people total, there are $\binom{2n}{2}$ ways to do this. On the other hand, there are $\binom{n}{2}$ ways to choose two men; $\binom{n}{2}$ ways to choose two women; and $n \cdot n$ ways to choose a man and a woman. Since this covers all possible cases, we see that $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

    (b) We have $\binom{2n}{2} = \frac{(2n)(2n-1)}{2} = n(2n - 1)$; and $2\binom{n}{2} = 2 \cdot \frac{n(n-1)}{2} = n(n - 1)$. Thus, $2\binom{n}{2} + n^2 = n(n - 1) + n^2 = n(n - 1 + n) = n(2n - 1) = \binom{2n}{2}$.

Section 4.5

10. (a) Since there are 6 different types of croissants, the answer is $\binom{12+6-1}{12} = \binom{17}{5} = 6188$.

    (b) Similarly to the previous part, the answer is $\binom{36+6-1}{36} = \binom{41}{5} = 749398$.

    (c) There are $2 \cdot 6 = 12$ fixed croissants, and then 12 excess croissants to distribute among the 6 types. By part (a), the number of ways to do this is 6188.

    (d) There are $\binom{24+6-1}{24} = \binom{29}{5} = 118755$ total ways to have 2 dozen croissants; out of these, the number of combinations which have at least 3 broccoli croissants is $\binom{21+6-1}{21} = \binom{26}{5} = 65780$. Therefore, $118755 - 65780 = 52975$ of these combinations have at most 2 broccoli croissants.

    (e) There are 8 fixed croissants, which leaves 16 croissants to distribute among the 6 types. Thus, the answer is $\binom{16+6-1}{16} = \binom{21}{5} = 20349$.

(f) There are 9 croissants fixed by the first five constraints, which leaves 15 to distribute among the 6 types; there are $\binom{15+6-1}{15} = \binom{20}{5} = 15504$ total ways to do this. Out of these combinations, $\binom{11+6-1}{11} = \binom{16}{5} = 4368$ have at least 4 broccoli croissants. Therefore, $15504 - 4368 = 11136$ of them have at most 3 broccoli croissants.

14. There are $\binom{17+4-1}{17} = \binom{20}{3} = 1140$ solutions.

16. (a) Since each $x_i \geq 2$, 12 of the 29 is already fixed. Thus, to distribute the remaining 17 among the x_i, there are $\binom{17+6-1}{17} = \binom{22}{5} = 26334$ ways to solve this.

    (b) Since $x_5 \geq 6$, we have $x_1 + \cdots + x_6 \geq 22$ already. Since we need to distribute the remaining 7 among the x_i, there are $\binom{7+6-1}{7} = \binom{13}{6} = 1716$ solutions.

    (c) There are $\binom{29+6-1}{29} = \binom{34}{5} = 278256$ total solutions. Out of these, $\binom{23+6-1}{23} = \binom{28}{5} = 98280$ have $x_1 \geq 6$. Therefore, $278256 - 98280 = 179976$ of the solutions have $x_1 \leq 5$.

    (d) The number of solutions with $x_2 \geq 9$ is $\binom{20+6-1}{20} = \binom{25}{5} = 53130$. Out of these, $\binom{12+6-1}{12} = \binom{17}{5} = 6188$ have $x_1 \geq 8$. Therefore, there are $53130 - 6188 = 46942$ solutions with $x_1 < 8$ and $x_2 > 8$.

44. (a) This is the number of solutions to $x_1 + x_2 + x_3 + x_4 = 12$, where x_i is the number of books on shelf i; thus, the answer is $\binom{12+4-1}{12} = \binom{15}{3} = 455$.

    (b) For each placement of the books from the previous part, there are 12! ways to order the books; thus, the answer is $\binom{15}{3} \cdot 12! = 455 \cdot 12! = 217945728000$.

    Alternately, if we follow the hint, then there are 4 ways to place b_1. On the second book, since there are 5 terms in the sequence and we can place b_2 to the right of any term, there are 5 ways to place b_2. Similarly, there are 6 ways to place the third book, and so on. Therefore, the total number of ways to place the books is $4 \cdot 5 \cdot 6 \cdots 15 = P(15, 12) = \frac{15!}{3!} = 217945728000$.

Section 4.6

12. (a) In 246531, we have a_1 is the number of integers less than 2 which follow 2, which is 1. Similarly, a_2 is the number of integers less than 3 which follow 3, which is 1; $a_3 = 2$; $a_4 = 2$; and $a_5 = 3$. Thus, the integer corresponding to 246531 is $a_1 \cdot 1! + a_2 \cdot 2! + \cdots + a_5 \cdot 5! = 3 \cdot 5! + 2 \cdot 4! + 2 \cdot 3! + 1 \cdot 2! + 1 \cdot 1! = 423$.

    (b) For each n, a_n is the number of integers less than $n+1$ which follow $n+1$; however, since 12345 is increasing, this means each $a_n = 0$ for $n = 1, \ldots, 4$. Therefore, the integer corresponding to 12345 is $0 \cdot 4! + 0 \cdot 3! + 0 \cdot 2! + 0 \cdot 1! = 0$.

    (c) In this case, every integer less than $n+1$ follows $n+1$; therefore, $a_n = n$ for $n = 1, \ldots, 5$. Therefore, the integer corresponding to 654321 is $5 \cdot 5! + 4 \cdot 4! + 3 \cdot 3! + 2 \cdot 2! + 1 \cdot 1! = 719$.