Week 4 Homework Solutions

Chapter 3

Section 1

12. The quadratic mean of a and b is always greater than or equal to the arithmatical mean of a and b. To prove we start by observing (a-b)2 S 0 and then by algebra that
    $a^2 - 2ab + b^2 \geq 0$
    $a^2 + b^2 \geq 2ab$
    $2a^2 + 2b^2 \geq a^2 + 2ab + b^2$
    $(a^2 + b^2)/2 \geq [(a + b)^2]/4$
    $\text{sqrt}((a^2 + b^2)/2) \geq (a + b)/2$ as desired.

20. We wish to show $x^2 \bmod 5 = 1$ or 4 when $x \bmod 5 \neq 0$. We consider four cases:
    Case 1: $x \bmod 5 = 1$. Then $x^2 \bmod 5 = 1$.
    Case 2: $x \bmod 5 = 2$. Then $x^2 \bmod 5 = 4$.
    Case 3: $x \bmod 5 = 3$. Then $x^2 = 9$ so $x^2 \bmod 5 = 4$
    Case 4: $x \bmod 5 = 4$. Then $x^2 = 16$ so $x^2 \bmod 5 = 1$
    In all cases, we have shown $x^2 \bmod 5 = 1$ or 4.

30. The product of primes of the form 4k+3 is a number of the form 4k+1:
    $(4m + 3)(4n + 3) = 16mn + 12m + 12n + 9 = 4k + 1$ where $k = 4mn + 3m + 3n + 2$

38. Let n be such that the sum of its divisors is n + 1. We note 1 and n are divisors of n and these sum to n + 1. Hence there are no other divisors of n so n is prime.

44. Let $s = (a_1)(a_2) \ldots (a_m)$ and $t = (b_1)(b_2) \ldots (b_n)$ be prime factorizations of s and t. Since gcf (s,t) = 1, $a_i \neq b_j$ for all i, j. The sum of the factors of s is the sum of all possible combinations of the a_i and similarly for t and the b_i. The product of these sums would then be the sum of all possible combinations of the a_i and b_j considered as a single group. Since the lists don't intersect, this is the same as the sum of the factors of st.

Section 2:

6. A. 10, 7, 4, 1, -2, -5, -8, -11, -14, -17
   B. 1, 3, 6, 10, 15, 21, 28, 36, 45, 55
   C. 1, 5, 19, 65, 211, 665, 2059, 6305, 19171, 58025
   D. 1, 1, 1, 2, 2, 2, 2, 2, 3, 3
   E. 1, 2, 3, 5, 8, 13, 21, 34, 55, 89
   F. 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023
   G. 1, 2, 2, 4, 8, 11, 33, 37, 148, 153
   H. 1, 2, 2, 2, 2, 3, 3, 3, 3, 3

20.$\Sigma^{n}_{k=1}$ 1/[k(k+1)] = $\Sigma^{n}_{k=1}$ (1/k - 1/(k+1)) = 1/k - 1/(n+1)

32.A. Countable. Let f(n) = $(-1)^{n}$ [floor((n+1)/2) + floor(floor((n+1)/2)/3)]
B. Countable. Let f(n) = $(-1)^{n}$ [5 (floor((n+1)/2) + floor(floor((n+1)/2)/7))]
C. Countable. Let g(n) = $\Sigma^{n}_{k=0}$ 10^{k} h(n) = $\Sigma^{n}_{k=1}$ 10^{-k}
j(a,b) = g(a-2) + h(b-1) k(n) = floor((n+1)/2)
p,q the first and second coordinate functions of the cannonical bijection between the natural numbers and the natural numbers squared.
f(n) = $(-1)^{n}$ j(p(k(n+1)),q(k(n+1)))
D. Uncountable

36.Let A and B be countable sets and have f and g witness that A and B are countable, respectively. Then let
h(n) = [[($(-1)^{n}$ + 1)/2] f(floor((n+1)/2)] + [[($(-1)^{n}$ -1)/(-2)] g(floor((n+1)/2)]
h works by going through B on the odd numbers and A on the even ones.

38.The same as example 19 on page 234 except we map to (p,q) instead of p/q and we need not worry about repetitions.

Section 3:

4. Base case: When n=0, (1 - $(-7)^{0+1}$) / 4 = (1+7) / 4 = 2 = 2 $(-7)^{0}$
Inductive case: We are given that
2 + 2(-7) + $2(-7)^{2}$ + . . . + $2(-7)^{n}$ = (1 - $(-7)^{n+1}$) / 4. By algeba
2 + 2(-7) + $2(-7)^{2}$ + . . . +$2(-7)^{n}$ + $2(-7)^{n+1}$ = [(1 - $(-7)^{n+1}$) / 4] + $2(-7)^{n+1}$
2 + 2(-7) + $2(-7)^{2}$ + . . . +$2(-7)^{n}$ + $2(-7)^{n+1}$ = (1 - $(-7)^{n+1}$ + 8 $(-7)^{n+1}$) / 4
2 + 2(-7) + $2(-7)^{2}$ + . . . +$2(-7)^{n}$ + $2(-7)^{n+1}$ = (1 - (-7)$(-7)^{n+1}$) / 4
2 + 2(-7) + $2(-7)^{2}$ + . . . +$2(-7)^{n}$ + $2(-7)^{n+1}$ = (1 - $(-7)^{n+2}$) / 4

6. 1 / [(1)(2)] + 1/[(2)(3)] + . . . + 1/[(n)(n+1)] = 1 - 1/(n+1)
Base case: When n=1 1/[(1)(2)] = ½ = 1 - ½ = 1 - 1/(1+1)
Inductive case: We are given that
1 / [(1)(2)] + 1/[(2)(3)] + . . . + 1/[(n)(n+1)] = 1 - 1/(n+1) By algebra
1 / [(1)(2)] + 1/[(2)(3)] + . . . + 1/[(n)(n+1)] + 1/[(n+1)(n+2)] = 1 - 1/(n+1) +
1/[(n+1)(n+2)]
1 / [(1)(2)] + 1/[(2)(3)] + . . . + 1/[(n)(n+1)] + 1/[(n+1)(n+2)] = 1 -
(n+2-1)/(n+1)(n+2)
1 / [(1)(2)] + 1/[(2)(3)] + . . . + 1/[(n)(n+1)] + 1/[(n+1)(n+2)] = 1 - (n+1)/(n+1)(n+2)
1 / [(1)(2)] + 1/[(2)(3)] + . . . + 1/[(n)(n+1)] + 1/[(n+1)(n+2)] = 1 - 1 / (n+2)

14.Base case: When n=2 2! = 2 < 4 = 2^{2}.

Inductive case: We are given that $n! < n^n$. By algebra
$n!(n+1) < n^n(n+1)$
$(n+1)! < n^n(n+1) < (n+1)^n(n+1) < (n+1)^{n+1}$.

32. 10 cents, 20 cents, and any number of cents divisible by 5 above 20 cents
Base case: 10 cents = 1 dime, 20 cents = 2 dimes, 25 cents = 1 quarter
Strong inductive case: let n be divisible by 5, n at least 30 cents. By our inductive hypothesis we can make n-5 cents so we add a dime to get n+5 cents, completing the induction.

48. Base case: NOT (p1 OR p2) = NOTp1 AND NOTp2 by DeMorgan
Inductive case: We are given
NOT (p1 OR . . . OR pn) = NOTp1 AND . . . AND NOTpn. Using this fact we note
NOT (p1 OR . . . OR pn OR pn+1) = NOT ((p1 OR . . . OR pn) OR pn+1) =
NOT (p1 OR . . . OR pn) AND NOTpn+1= [DeMorgan]
NOT p1 AND . . . AND NOTpn AND NOTpn+1 [Inductive Hypothesis]

54. The first use of the inductive step (k=0 so k+1=1) involves the assumption that $a^{k-1} = 1$ but k-1 = -1 and the base case started with k=0. In fact, $a^{-1} = 1/a$ which need not be 1.

66. A. Since a and b are positive integers, a + b is a positive integer. Letting s = 1 and t = 1 we see that a + b is an element of S. Hence S is not empty.
B. S is a nonempty set of positive integers so by the well ordering property, S has a least element, c.
C. Suppose d is a divisor of a and b. Let m and n be such that md = a and nd = b. Since c is an element of S, let s and t be such that sa + tb = c. We then observe that c = sa + tb = mds + ndt = d(ms + nt) so d is a divisor of c.
D. By symmetry it suffices to show c divides a. Suppose not. Then a = qc + r for some nonnegative integer q and some r such that $0 < r < c$. By algebra r = a - qc so r = a - q(sa + tb) = a - qsa - qtb = (1 - qs) a + (-qt) b and since $r > 0$ we have that r is and element of S. But $r < c$, contradicting c being the least element of S. We conclude c divides a and c divides b.
E. Part D shows that c is a divisor of both a and b and part C shows c is the greatest common divisor of a and b and that c is unique.

Section 4:

6. A. Valid. $f(n) = (-1)^n$.
Base case: When n=0, $f(0) = 1 = (-1)^0 = (-1)^n$.
Inductive case: We are given that $f(n) = (-1)^n$. Using this we have $f(n+1) = -f(n) = (-1)(-1)^n = (-1)^{n+1}$.

B. Valid. $f(n) = 2^{n \text{ div } 3}$ (g(n mod 3)) where g(0)=1, g(1)=0, g(2)=2.
Base cases: When n=0, $f(0) = 1 = 2^0(1) = 2^{0\text{div}3}$ (g(0 mod 3)). The cases for n=1 and n=2 are similar.
Strong inductive case: We are given that $f(n-2) = 2^{(n-2) \text{ div } 3}$ (g((n-2) mod 3)). We note that (n+1) div 3 = ((n-2) div 3) + 1 and that (n+1) mod 3 = (n-2) mod 3. Hence f(n+1) = 2 f(n-2) = (2) $2^{(n-2) \text{ div } 3}$ (g((n-2) mod 3)) =
$2^{(n+1) \text{ div } 3}$ (g((n+1) mod 3)).

C. Not valid.
D. Not valid.
E. Valid. We have $f(n) = 2^{\text{floor}(n/2) + 1}$.
Base cases: When n=0 we have $f(0) = 2 = 2^1 = 2^{0 + 1} = 2^{\text{floor}(0/2) + 1} = 2^{\text{floor}(n/2) + 1}$. The case for n=1 is similar.
Strong inductive case: We are given that f(n-1) = 2floor((n-1)/2) + 1. Hence
$f(n+1) = 2 f(n-1) = (2)\, 2^{\text{floor}((n-1)/2) + 1} = 2^{\text{floor}((n-1)/2) + 2} = 2^{\text{floor}((n+1)/2) + 1}$.

8. A. $a_1 = 2$, $a_{n+1} = a_n + 4$
B. $a_1 = 3$, $a_{n+1} = a_n + 2$
C. $a_1= 10$, $a_{n+1} = 10(a_n)$
D. $a_1 = 1$, $a_{n+1} = [\text{sqrt}(a_n)+1]^2$.

14.Base case: When n=1 $(f_2)(f_0) - (f_1)^2 = 1(0) - 1^2 = -1 = -1^n$.
Inductive case: We are given that $(f_{n-1})(f_{n+1}) - (f_n)^2 = (-1)^n$. Using this
$(f_n)(f_{n+2}) - (f_{n+1})^2 = (f_n)(f_n + f_{n+1}) - (f_{n-1} + f_n)(f_{n+1}) = (f_n)^2 - (f_{n-1})(f_{n+1}) =$
$(-1)[(f_{n-1})(f_{n+1}) - (f_n)^2] = (-1)(-1)^n = (-1)^{n+1}$ completing the induction.

18.Base case: When n=1, $A^1 = A = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}f_2 & f_1\\ f_1 & f_0\end{bmatrix} = \begin{bmatrix}f_{n+1} & f_n\\ f_n & f_{n-1}\end{bmatrix}$
Inductive case: We are given that $An = \begin{bmatrix}fn+1 & fn\\ fn & fn-1\end{bmatrix}$. Using this
$$A^{n+1} = A^n (A) = \begin{bmatrix}f_{n+1} & f_n\\ f_n & f_{n-1}\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}f_{n+1} + f_n & f_{n+1}\\ f_n + f_{n-1} & f_n\end{bmatrix} = \begin{bmatrix}f_{n+2} & f_{n+1}\\ f_{n+1} & f_n\end{bmatrix}$$

24.A. Basis step: 1 is an odd number
Recursive step: If n is an odd number then n+2 is an odd number
B. Basis step: 3 is in the set
Recursive step: If n is in the set then 3n is in the set
C. Basis step: 1, -1, and x are in the set
Recursive step: If a and b are in the set then a+b and ab are in the set

Section 5:

16.

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Procedure A2N (a: real, n: positive integer)
    If n=0 then A2N := a; end.
    A2N := $[A2N(a,n-1)]^2$.
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18.As suggested in the hint, we use the binary expansion for n (an algorithm for computing the binary expansion can be found on page 171).

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Procedure AN (a: real, n: positive integer in binary form, k: length of n)
product := 1
For i:=0 to k-1
if n(i) = 1 then product := (product) (A2N(a,k))
AN := product
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20.Recursive: f_8 - 1 = 21 - 1 = 20
Iterative: 7 - 1 = 6