Math 55: Final Exam, 11 December 2003, 12:30-3:30 pm

Write your name, your student ID number, your section time and number, a six-problem grading grid (see right), and your GSI's name on the cover of your blue book. Books, notes, calculators, scratch paper and/or collaboration are not allowed. Remain in your seat and hand in your exam book *to your GSI* at 3:30 pm; solutions will be available after all exams have been collected. When you finish, check over your work — do not leave early!

1	
2	
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Total	

Problem 1: (a) Compute 1155^{55} mod 17.
(b) Find the smallest positive inverse of 10 mod 17.
(c) State the Chinese Remainder Theorem.
(d) Use the procedure of the Chinese Remainder Theorem to compute 1155^{55} mod 170.
(e) State Fermat's little Theorem.
(f) Use Fermat's little Theorem to find the smallest prime number which divides $1155^{55} - 1$.

Problem 2: Choose a *positive* integer solution $(x_1 > 0, x_2 > 0, x_3 > 0)$ of

$$x_1 + x_2 + x_3 = 42$$

at random, where each solution has equal probability.
(a) What is the probability of selecting $(14, 14, 14)$?
(b) What is the probability that at least one of the x's is exactly equal to 20?
(c) What is the probability that $x_1 = 10$, given that $x_2 = 14$?

Problem 3: Define the divisibility relation R on $\mathbf{Z}_n = \{1, 2, 3, 4, \ldots, n\}$ by $aRb \leftrightarrow a|b$.
(a) Define a partial order and prove or disprove that R is one.
(b) Let M be the matrix of R. Show that the number $d(j)$ of divisors of any integer $j \in \mathbf{Z}_n$ is given by the sum of the entries in column j of M:

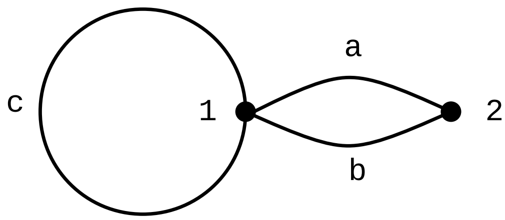
$$d(j) = \sum_{i=1}^{n} M_{ij}.$$

(c) Evaluate $d(pq)$ for any primes p and q such that $pq \in \mathbf{Z}_n$.
(d) Consider the experiment of selecting an integer j from $\mathbf{Z}_n$ at random, with equal probabilities. Show that

$$E(d) \leq \sum_{k=1}^{n} \frac{1}{k}.$$

(e) Prove by induction that $E(d) = O(\log n)$.

Problem 4: Consider the following undirected pseudograph G:



(a) Write down the adjacency matrix A of G.
(b) Find all paths of length 3 from 1 to 2; for example, one is `1c1c1a2`, where the path loops twice at 1 then goes from 1 to 2 via edge `a`.
(c) Use the adjacency matrix A to find a recurrence relation for the number b_n of paths from 1 to 2 of any length $n \geq 1$. You should get

$$b_n = b_{n-1} + 4b_{n-2}.$$

(d) Use the recurrence relation to find a closed form for the generating function $G(x)$ of the sequence b_n.
(e) Express $G(x)$ as the sum of two partial fractions.
(f) Use the geometric series formula to find a closed form for b_n and verify your result for $n = 1$ and 2.

Problem 5: **(a)** List all equivalence relations on $\{1, 2, 3\}$.
(b) Let E be the set of partitions of $\{1, 2, 3\}$ into disjoint nonempty subsets. Let the partial order $\preceq$ on E be defined by

$$\forall p \in E \ \forall q \in E \ \left(p \preceq q \longleftrightarrow \forall A \in p \ \exists B \in q \ (A \subseteq B) \right).$$

Draw the digraph and the Hasse diagram of the relation $\preceq$.
(c) List E in a topologically sorted order.