Discrete Mathematics - Math 55 - Spring 1997 - Second Midterm

Instructions: This is a closed book, closed notes, closed calculator, closed computer, closed network, open brain exam. All five questions have equal weight (one question is on the back of this sheet). Write all your answers in a blue book. Put your name *and your TA's name and section number* on the blue book.

Question 1. Among a group of 3 cats (Tabby, Leo, and Garfield) and 4 dogs (Snoopy, Odie, Fido, and Bowser) we wish to arrange a photograph of 5 of them. How many ways are there to arrange 5 of the animals in a photograph under the following conditions (each set of conditions is independent of the others):

1. No conditions; any 5 animals may appear in any order.
2. There must be exactly 3 dogs in the photograph.
3. A cat must be in either the leftmost position or the rightmost position. Hint: Use inclusion-exclusion.

Question 2. What is the least positive integer n such that $n \bmod 5 = 3$ and $n \bmod 7 = 4$?

Question 3. Consider the sequence defined by: $A(0) = 1$, $A(1) = 0$, and $A(n) = 4A(n-2) - 3A(n-1)$ for $n \geq 2$.

1. List the first six terms in the sequence.
2. Find a closed formula for the sequence. That is, find a formula for $A(n)$ which does not refer to $A(k)$ for any $k < n$. Hint: begin by considering solutions of the form $A(n) = r^n$ for some constant r, and solve for r.

Question 4. Prove by induction that $10^n + 3 \cdot 4^{n+2} + 5$ is divisible by 9 for all nonnegative integers n.

Question 5. Each part of this problem contains a statement and a "proof" of that statement. For each part, you must

(a) say whether the statement is true or false (and if it is false, provide a counterexample)

(b) say whether the proof is valid or invalid (and if it is invalid, describe where and how it goes wrong).

1. Statement: "For any positive integer n, if $45|n^2$, then $45|n$." (Recall that $a|b$ means "a divides b").

   Here is the proof (with line numbers added for convenience):

   (1) $45|n^2 \to (3|n^2$ and $5|n^2)$, since 45 is a multiple of both 3 and 5.

   (2) $3|n^2 \to 9|n^2$, since the exponent of every prime in the prime factorization of n^2 must be even.

   (3) Now, $45|n \to (9|n$ and $5|n)$, since 45 is a multiple of both 5 and 9; hence $9|n^2$ and $5|n^2$, as required. So $45|n$.

2. Statement: "If p is a prime and greater than 3, then p^2 is congruent to 1 mod 24."

   Here is the proof (with line numbers added for convenience):

   (1) p is a positive prime equal to neither 2 nor 3.

   (2) $p \neq 2$ implies that p is not congruent to 0 mod 6, 2 mod 6, or 4 mod 6, for otherwise p would be divisible by 2.

   (3) $p \neq 3$ implies that p is not congruent to 0 mod 6 or 3 mod 6, for otherwise p would be divisible by 3.

   (4) Thus p is congruent to 1 mod 6 or 5 mod 6.

   (5) Hence p^2 is congruent to 1 mod 6 or 25 mod 6.

   (6) But 25 is congruent to 1 mod 6, so p^2 is congruent to 1 mod 6.

   (7) $p \neq 2$ implies that p is not congruent to 0 mod 4 or 2 mod 4, for otherwise p would be divisible by 2.

   (8) Thus p is congruent to 1 mod 4 or 3 mod 4.

   (9) Hence p^2 is congruent to 1 mod 4 or 9 mod 4.

   (10) But 9 is congruent to 1 mod 4, so p^2 is congruent to 1 mod 4.

   (11) We now apply the Chinese Remainder Theorem: Since p^2 is congruent to 1 mod 6 and p^2 is congruent to 1 mod 4, we conclude that p^2 is congruent to 1 mod (6×4).

   (12) But $(6 \times 4) = 24$, so we conclude that p^2 is congruent to 1 mod 24.